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ScienceDirect
Acta Mathematica Scientia 2011,**31B**(4):1643–1652
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

SOME INEQUALITIES OF HERMITE–HADAMARD TYPE FOR s-CONVEX FUNCTIONS*

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Abstract In this paper several inequalities of the left-hand side of Hermite-Hadamard's inequality are obtained for s-convex functions.

Key words convex function; s-convex function; Hadamard's inequality

2000 MR Subject Classification 26D15; 26D10

1 Introduction

A function $f : I \subseteq \mathbf{R}_+ \to \mathbf{R}_+$, where $\mathbf{R} = [0, \infty)$ is said to be s-convex on I if the inequality

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in I$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions is usually denoted by K_s^2 (see [13]).

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

One of the most famous inequality for the class of convex functions is so called Hermite-Hadamard inequality, which states that:

For a convex mapping $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and a, $b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

*Received April 17, 2009; revised May 14, 2010. The first author the financial support of Universiti Kebangsaan Malaysia, Faculty of Science and Technology (UKM–GUP–TMK–07–02–107).

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For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities see [1–8] and [10–19].

In [8], Dragomir and Fitzpatrick proved a variant of Hermite–Hadamard inequality which holds for the s-convex functions.

Theorem 1.1 Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[0,1]$, then the following inequalities hold

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le\frac{f(a)+f(b)}{s+1} \tag{2}$$

the constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (6). The above inequalities are sharp.For recent results and generalizations concerning s-convex functions see [4, 7–9, 13, 14].

Along this paper we consider a real interval $I\subset\mathbf{R}$, and we denote that I° is the interior of I.

In [14], Kirmaci proved some inequalities of Hermite–Hadamard type for differentiable convex mappings, the main result in [14] is stated as follows:

Theorem 1.2 Let $f:I\subset\mathbf{R}\to\mathbf{R}$ be differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|\le\frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right]. \tag{3}$$

One more general results related to (3) was established in [15–19].

In [18], Pearce and Pečarić proved the following theorem.

Theorem 1.3 Let $f:I\subset\mathbf{R}\to\mathbf{R}$ be differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, for some $q\ge1$, then the following inequality holds

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left[\frac{|f(a)|^q+|f(b)|^q}{2}\right]^{1/q}. \tag{4}$$

If $|f|^q$ is concave on $[a,b]$ for some $q\ge1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{5}$$

In this paper, we establish a new inequalities of Hadamard's type for the class of s-convex functions in the second sense.

2 Hadamard's Type Inequalities for s-convex Functions

To prove our main result(s) we consider the following lemma (see [4]):

Lemma 2.1 Let $f:I\subset\mathbf{R}\to\mathbf{R}$ be a differentiable mapping on I° where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\\=&\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)\mathrm{d}t+\int_0^1(t-1)f'\left(tb+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right].\end{aligned}$$

Proof We note that

$$\begin{aligned} I_1 &= \int_0^1 t f'\left(t\frac{a+b}{2} + (1-t)\,a\right) \mathrm{d}t \\ &= \frac{2}{b-a} t f\left(t\frac{a+b}{2} + (1-t)\,a\right)\bigg|_0^1 - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)\,a\right)\mathrm{d}t \\ &= \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_0^1 f\left(t\frac{a+b}{2} + (1-t)\,a\right)\mathrm{d}t. \end{aligned}$$

Setting $x = t\frac{a+b}{2} + (1-t)\,a$, and $\mathrm{d}x = \frac{b-a}{2}\mathrm{d}t$, which gives

$$I_1 = \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 (t-1)\, f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\mathrm{d}t \\ &= \frac{2}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(x)\,\mathrm{d}x, \end{aligned}$$

and therefore,

$$\begin{aligned} I &= \frac{b-a}{4}\left[I_1 + I_2\right] = \frac{b-a}{4}\left[\frac{4}{b-a} f\left(\frac{a+b}{2}\right) - \frac{4}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\right] \\ &= f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x, \end{aligned}$$

which completes the proof.

Theorem 2.2 Let $f : I \subset [0,\infty) \to \mathbf{R}$ be a differentiable mapping on I°, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \right| \\ &\leq \frac{(b-a)}{4\,(s+1)\,(s+2)}\left[\left|f'(a)\right| + 2\,(s+1)\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'(b)\right|\right] \qquad (6) \\ &\leq \frac{\left(2^{2-s}+1\right)(b-a)}{4\,(s+1)\,(s+2)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \qquad (7) \end{aligned}$$

Proof From Lemma 2.1, we have

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \right| \\ &\leq \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)\,a\right)\right|\mathrm{d}t + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\,\frac{a+b}{2}\right)\right|\mathrm{d}t\right] \\ &\leq \frac{b-a}{4}\int_0^1 t\left[t^s\left|f'\left(\frac{a+b}{2}\right)\right| + (1-t)^s\left|f'(a)\right|\right]\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\frac{b-a}{4}\int_0^1 (1-t)\left[t^s\left|f'(b)\right|+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\\
&=\frac{b-a}{4}\left[\frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{(s+1)(s+2)}\left|f'(a)\right|\right]\\
&\quad+\frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\left|f'(b)\right|+\frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|\right]\\
&=\frac{(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right],
\end{aligned}
\tag{8}
$$

which proves inequality (6). To prove (7), and since $|f'|$ is s-convex on $[a,b]$, for any $t\in[0,1]$, then by (2) we have

$$
2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|\le\frac{\left|f'(a)\right|+\left|f'(b)\right|}{s+1}. \tag{9}
$$

A combination of (8) and (9), we get

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right]\\
&\le\frac{(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+2(s+1)\,2^{1-s}\frac{\left|f'(a)\right|+\left|f'(b)\right|}{s+1}+\left|f'(b)\right|\right]\\
&=\frac{\left(2^{2-s}+1\right)(b-a)}{4(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],
\end{aligned}
$$

which proves inequality (7), and thus the proof is completed.

Next theorem gives a new upper bound of the left-Hadamard inequality for s-convex mappings.

Theorem 2.3 Let $f:I\subset[0,\infty)\to\mathbf{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)}$, $(p>1)$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds:

$$
\begin{aligned}
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le&\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\
&\left[\left(\left(2^{1-s}+s+1\right)\left|f'(a)\right|^q+2^{1-s}\left|f'(b)\right|^q\right)^{1/q}\right.\\
&\left.+\left(2^{1-s}\left|f'(a)\right|^q+\left(2^{1-s}+s+1\right)\left|f'(b)\right|^q\right)^{1/q}\right],
\end{aligned}
\tag{10}
$$

where p is the conjugate of q, $q=p/(p-1)$.

Proof Suppose that $p>1$. From Lemma 2.1 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|\mathrm{d}t+\int_0^1(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]\\
&\le\frac{b-a}{4}\left(\int_0^1 t^p\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q\mathrm{d}t\right)^{1/q}
\end{aligned}
$$

$$+\frac{b-a}{4}\left(\int_0^1 (1-t)^p \,\mathrm{d}t\right)^{1/p}\left(\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right)^{1/q}.$$

Because $|f'|^q$ is s-convex, we have

$$\begin{aligned}\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right|^q \mathrm{d}t &\le \int_0^1 \left[t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]\mathrm{d}t\\ &= \frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{s+1}\left|f'(a)\right|^q\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t &\le \int_0^1 \left[t^s\left|f'(b)\right|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\mathrm{d}t\\ &= \frac{1}{s+1}\left|f'(b)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|\\ &\le\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{1/q}\right.\\ &\quad\left.+\left(\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right].\end{aligned}\tag{11}$$

Now, since $|f'|^q$ is s-convex on $[a,b]$, for any $t\in[0,1]$, then by (2) we have

$$2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|\le\frac{|f'(a)|+|f'(b)|}{s+1}.\tag{12}$$

A combination of (11) and (12), we get

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\\ &\quad\times\left[\left(\frac{2^{1-s}}{s+1}\left(|f'(a)|^q+|f'(b)|^q\right)+|f'(a)|^q\right)^{1/q}\right.\\ &\quad\left.+\left(|f'(b)|^q+\frac{2^{1-s}}{s+1}\left(|f'(a)|^q+|f'(b)|^q\right)\right)^{1/q}\right]\\ &\le\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\ &\quad\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right)^{1/q}\right.\\ &\quad\left.+\left(2^{1-s}|f'(a)|^q+\left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right],\end{aligned}$$

where $1/p+1/q=1$, which is required.

Theorem 2.3 may be extended to be as follows:

Corollary 2.1 Let $f : I \subset [0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$, $(p > 1)$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\left\{2^{(1-s)/q}+\left(2^{1-s}+s+1\right)^{1/q}\right\}\left(|f'(a)|+|f'(b)|\right), \quad (13)$$

where $q = p/(p-1)$.

Proof We consider inequality (10), and we let $a_1 = \left(2^{1-s}+s+1\right)|f'(a)|^q$, $b_1 = 2^{1-s}|f'(b)|^q$, $a_2 = 2^{1-s}|f'(a)|^q$ and $b_2 = \left(2^{1-s}+s+1\right)|f'(b)|^q$. Here, $0 < 1/q < 1$, for $q > 1$. Using the fact

$$\sum_{i=1}^n (a_i+b_i)^r \leq \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$$

for $0 < r < 1$, $a_1, a_2, \cdots, a_n \geq 0$ and $b_1, b_2, \cdots, b_n \geq 0$, we obtain that, the inequalities

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\ &\quad\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right)^{1/q}\right.\\ &\quad\left.+\left(2^{1-s}|f'(a)|^q+\left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right]\\ &\leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\\ &\quad\times\left\{2^{(1-s)/q}+\left(2^{1-s}+s+1\right)^{1/q}\right\}\left(|f'(a)|+|f'(b)|\right).\end{aligned}$$

The following theorem holds.

Theorem 2.4 Let $f : I \subset [0,\infty) \to \mathbf{R}$ be a differentiable mapping on I°, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$, $q \geq 1$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left\{\left(2^{1-s}+1\right)|f'(a)|^q+2^{1-s}|f'(b)|^q\right\}^{1/q}\right.\\ &\quad\left.+\left\{\left(2^{1-s}+1\right)|f'(b)|^q+2^{1-s}|f'(a)|^q\right\}^{1/q}\right]. \qquad (14)\end{aligned}$$

Proof Suppose that $p \geq 1$. From Lemma 2.1 and using the power mean inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\leq \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|\mathrm{d}t+\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]\end{aligned}$$

$$\leq \frac{b-a}{4}\left(\int_0^1 t\mathrm{d}t\right)^{1-1/q}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right|^q dt\right)^{1/q}$$
$$+\frac{b-a}{4}\left(\int_0^1 (1-t)\,\mathrm{d}t\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\,\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right)^{1/q}.$$

Because $|f'|^q$ is s-convex, we have

$$\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right|^q \mathrm{d}t$$
$$\leq \int_0^1 \left[t^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q + t\,(1-t)^s\,|f'(a)|^q\right]\mathrm{d}t$$
$$= \frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(s+1)\,(s+2)}\,|f'(a)|^q$$

and

$$\int_0^1 (1-t)\left|f'\left(tb+(1-t)\,\frac{a+b}{2}\right)\right|^q \mathrm{d}t$$
$$\leq \int_0^1 \left[(1-t)\,t^s\,|f'(b)|^q + (1-t)^{s+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]\mathrm{d}t$$
$$= \frac{1}{s+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(s+1)\,(s+2)}\,|f'(b)|^q.$$

Therefore, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)\,(s+2)}\right)^{1/q}\left[\left((s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right)^{1/q}\right.$$
$$\left.+\left(|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]. \tag{15}$$

Now, since $|f'|^q$ is s-convex on $[a,b]$, for any $t\in[0,1]$, then by (2) we have

$$2^{s-1}\left|f'\left(\frac{a+b}{2}\right)\right|^q \leq \frac{|f'(a)|^q + |f'(b)|^q}{s+1}. \tag{16}$$

A combination of (15) and (16), we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)\,(s+2)}\right)^{1/q}\left[\left((s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q\right)^{1/q}\right.$$
$$\left.+\left(|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]$$
$$\leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)\,(s+2)}\right)^{1/q}\left[\left(2^{1-s}\left(|f'(a)|^q + |f'(b)|^q\right) + |f'(a)|^q\right)^{1/q}\right.$$

$$+\left(\left|f'(b)\right|^q+2^{1-s}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)\right)^{1/q}\Big]$$
$$=\left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left\{\left(2^{1-s}+1\right)\left|f'(a)\right|^q+2^{1-s}\left|f'(b)\right|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\left(2^{1-s}+1\right)\left|f'(b)\right|^q+2^{1-s}\left|f'(a)\right|^q\right\}^{1/q}\right].$$

Corollary 2.2 Let f be as in Theorem 2.4, then the following inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq\left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left\{2^{(1-s)/q}+\left(2^{1-s}+1\right)^{1/q}\right\}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{17}$$

holds, where $q\geq 1$.

Proof Using the technique in the proof of Corollary 2.1, by considering inequality (14), with $a_1=\left(2^{1-s}+1\right)\left|f'(a)\right|^q$, $b_1=2^{1-s}\left|f'(b)\right|^q$, $a_2=2^{1-s}\left|f'(a)\right|^q$ and $b_2=\left(2^{1-s}+1\right)\left|f'(b)\right|^q$. However, the details are left to the reader.

Now, we give the following Hadamard-type inequality for concave mappings.

Theorem 2.5 Let $f: I\subset[0,\infty)\to\mathbf{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$, $q\geq 1$ is concave on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq\left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \tag{18}$$

Proof From Lemma 2.1 and using the Hölder inequality for $q>1$, and $p=\frac{q}{q-1}$, we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leq\frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|\mathrm{d}t+\int_0^1(1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]$$
$$\leq\frac{b-a}{4}\left(\int_0^1 t^p\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q\mathrm{d}t\right)^{1/q}$$
$$+\frac{b-a}{4}\left(\int_0^1(1-t)^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right)^{1/q},$$

where p is the conjugate of q. We note that, since $|f'|^q$ is concave on $[a,b]$, and using the power mean inequality, we have

$$\left|f'(\lambda x+(1-\lambda)y)\right|^q\geq\lambda\left|f'(x)\right|^q+(1-\lambda)\left|f'(y)\right|^q$$
$$\geq\left(\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(y)\right|\right)^q,$$

$\forall x,y\in[a,b]$. Hence,

$$\left|f'(\lambda x+(1-\lambda)y)\right|\geq\lambda\left|f'(x)\right|+(1-\lambda)\left|f'(y)\right|,$$

so $|f'|$ is also concave.

By the Jensen integral inequality, we have

$$\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)\,a\right)\right| \mathrm{d}t \le \left(\int_0^1 t^0 \mathrm{d}t\right) \left|f'\left(\frac{\int_0^1 \left(t\frac{a+b}{2}+(1-t)\,a\right)}{\int_0^1 t^0 \mathrm{d}t}\right)\right|^q$$

$$\le \left|f'\left(\frac{3a+b}{4}\right)\right|^q,$$

and analogously,

$$\int_0^1 \left|f'\left(tb+(1-t)\,\frac{a+b}{2}\right)\right| \mathrm{d}t \le \left|f'\left(\frac{a+3b}{4}\right)\right|^q.$$

Combining all obtained inequalities, we get the required result.

Remark 2.1 i) The above results hold for log-convex (concave) mappings. Simply set $s=1$, and apply the results for $f' = \log g$, where $g > 0$.

ii) We note that, inequality (6) with $s=1$ gives a new refinement for inequality (3), and (7) with s = 1 improves the bound of (3). Also, (14) improves (4) and (10), (13) and (17) improves the bound of (5).

3 Applications to Special Means

In [13], the following example is given.

Let $s \in (0,1)$ and $a,b,c \in \mathbf{R}$. We define function $f:[0,\infty) \to \mathbf{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

If $b \ge 0$ and $0 \le c \le a$, then $f \in K_s^2$. Hence, for $a=c=0$, $b=1$, we have $f:[0,1] \to [0,1]$, $f(t) = t^s$, $f \in K_s^2$.

Now, using the results of Section 2, we give some applications to special means of real numbers.

We shall consider the means for arbitrary real numbers α, β ($\alpha \ne \beta$). We take

(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbf{R}.$$

(2) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \ n \in \mathbf{Z}\backslash\{-1,0\}, \ \alpha,\beta \in \mathbf{R}, \ \alpha \ne \beta.$$

Therefore, by applying the s-convex mapping $f:[0,1] \to [0,1]$, $f(x) = x^s$, the following inequalities hold:

Proposition 3.1 Let $a,b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, we have

$$|L_s^s(a,b) - A^s(a,b)| \le s\left(\frac{b-a}{4(s+1)(s+2)}\right)\left(|a|^{s-1} + 2(s+1)\left|\frac{a+b}{2}\right|^{s-1} + |b|^{s-1}\right)$$

$$\le s\frac{\left(2^{2-s}+1\right)(b-a)}{4(s+1)(s+2)}\left(|a|^{s-1}+|b|^{s-1}\right).$$

Proposition 3.2 Let $a, b \in I^{\circ}$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have

$$\begin{aligned}&\left|L_s^s(a,b) - A^s(a,b)\right| \\ &\le s\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\left[2^{(1-s)/q} + \left(2^{1-s}+s+1\right)^{1/q}\right]\left(|a|^{s-1} + |b|^{s-1}\right).\end{aligned}$$

Proposition 3.3 Let $a, b \in I^{\circ}$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have

$$\begin{aligned}&\left|L_s^s(a,b) - A^s(a,b)\right| \\ &\le \frac{s}{2^{1/p}}\left(\frac{b-a}{4}\right)\left(\frac{1}{(s+1)(s+2)}\right)^{1/q}\left[2^{(1-s)/q} + \left(2^{1-s}+1\right)^{1/q}\right]\left(|a|^{s-1} + |b|^{s-1}\right).\end{aligned}$$

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