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ISOPERIMETRIC PROBLEMS OF THE CALCULUS OF VARIATIONS WITH FRACTIONAL DERIVATIVES*

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Abstract In this article, we study isoperimetric problems of the calculus of variations with left and right Riemann-Liouville fractional derivatives. Both situations when the lower bound of the variational integrals coincide and do not coincide with the lower bound of the fractional derivatives are considered.

Key words Calculus of variations; fractional derivatives; isoperimetric problems

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1 Introduction

Isoperimetric problems consist in maximizing or minimizing a cost functional subject to integral constraints [5]. They found a broad class of important applications throughout the centuries. Areas of application include astronomy, physics, geometry, algebra, and analysis [6, 17]. Concrete isoperimetric problems in engineering were also investigated by a number of authors [18].

The study of isoperimetric problems is nowadays done, in an elegant and rigorously way, by means of the theory of calculus of variations. This is possible through a powerful tool known as the Euler-Lagrange equation [33]. Recently, the theory of the calculus of variations was considered in the fractional context [7, 8, 10–14, 23, 25, 27, 32]. The fractional calculus allows to generalize the ordinary differentiation and integration to an arbitrary (non-integer) order, and provides a powerful tool for modeling and solving various problems in science and

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engineering [28, 29, 31]. The problems considered are more general, and hold for a bigger class of admissible functions which are not necessarily differentiable in the classical sense [30]. Several results were proved for the new calculus of variations. They include Euler-Lagrange equations for fractional variational problems with Riemann-Liouville [1], Riesz [3], Caputo, and (α, β) derivatives [19]; transversality conditions [2]; and Noether's symmetry theorem [21, 22]. For a state of the art of the fractional variational theory, see the recent articles [4, 9, 15, 16, 20, 24] and references therein. In this article, we develop further the theory of the fractional variational calculus by studying isoperimetric problems.

The article is organized as follows. In Section 2, we shortly review the necessary background on fractional calculus. Our results are given in Section 3. In Section 3.1, we introduce the basic fractional isoperimetric problem and prove correspondent necessary optimality conditions, both for normal and abnormal extremizers (Theorems 6 and 7, respectively). In Section 3.2, we generalize our results for functionals where the lower bound of the integral is greater than the lower bound of the Riemann-Liouville derivatives. Finally, in Section 3.3, we present a necessary condition of optimality for the case where the order of the derivative is taken as a free variable.

2 Preliminaries of Fractional Calculus

A fractional derivative is a generalization of the ordinary differentiation, which allows real number powers of the differential operator. There exist numerous applications of fractional derivatives to several fields, like geometry, physics, engineering, etc. In the literature, we may find a great number of definitions for fractional derivatives (see, for example, [28, 29, 31]). In this article, we deal with the left and right Riemann-Liouville fractional derivatives, which are defined in the following way.

Definition 1 Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ are defined, respectively, by

$${}_a\mathcal{D}_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{\mathrm{d}x^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t, \quad x \in (a, b],$$

and

$${}_x\mathcal{D}_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{\mathrm{d}x^n} \int_x^b (t-x)^{n-\alpha-1} f(t)\mathrm{d}t, \quad x \in [a, b),$$

where Γ is the Euler gamma function, α is the order of the derivative, and $n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α.

If $\alpha \geq 1$ is an integer, these fractional derivatives are understood in the sense of usual differentiation, that is,

$${}_a\mathcal{D}_x^\alpha f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha f(x) \quad \text{and} \quad {}_x\mathcal{D}_b^\alpha f(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha f(x).$$

From the physical point of view, if $f(x)$ describes a certain process through time x, then the left derivative is related to the past of this process, while the right derivative belongs to the future.

These operations are linear, in the sense that

$${}_a\mathcal{D}_x^\alpha(\mu f(x) + \nu g(x)) = \mu\, {}_a\mathcal{D}_x^\alpha f(x) + \nu\, {}_a\mathcal{D}_x^\alpha g(x)$$

and

$$ {}_x\mathcal{D}_b^\alpha(\mu f(x)+\nu g(x)) = \mu\, {}_x\mathcal{D}_b^\alpha f(x) + \nu\, {}_x\mathcal{D}_b^\alpha g(x). $$

We now present the formula of integration by parts for fractional derivatives.

Lemma 2 ([31, p.46]) If f and g and the fractional derivatives ${}_a\mathcal{D}_x^\alpha g$ and ${}_x\mathcal{D}_b^\alpha f$ are continuous at every point $x\in[a,b]$, then for $0<\alpha<1$, we have

$$ \int_a^b f(x)_a\mathcal{D}_x^\alpha g(x)\mathrm{d}x = \int_a^b g(x)_x\mathcal{D}_b^\alpha f(x)\mathrm{d}x. \tag{1} $$

Moreover, formula (1) is still valid for $\alpha = 1$ provided f or g are zero at $x=a$ and $x=b$.

3 Main Results

From now on, we fix $\alpha,\beta\in(0,1)$. We consider functionals $\mathcal{J}$ of the form

$$ \mathcal{J}(y) = \int_a^b L(x,y,\, {}_a\mathcal{D}_x^\alpha y,\, {}_x\mathcal{D}_b^\beta y)\mathrm{d}x \tag{2} $$

defined on the set of admissible functions y that have continuous left fractional derivatives of order α and continuous right fractional derivatives of order β in $[a,b]$, and where $(x,y,u,v)\to L(x,y,u,v)$ is a function with continuous first and second partial derivatives with respect to all its arguments such that $\frac{\partial L}{\partial u}(x,y,\, {}_a\mathcal{D}_x^\alpha y,\, {}_x\mathcal{D}_b^\beta y)$ has continuous right fractional derivative of order α for all $x\in[a,b]$ and $\frac{\partial L}{\partial v}(x,y,\, {}_a\mathcal{D}_x^\alpha y,\, {}_x\mathcal{D}_b^\beta y)$ has continuous left fractional derivative of order β in $[a,b]$.

Remark 1 The left Riemann-Liouville fractional derivative is infinite at $x=a$ if $y(a)\neq 0$. If $y(b)\neq 0$, then the right Riemann-Liouville fractional derivative is also not finite at $x=b$ [30]. For this reason, by considering that the admissible functions y have continuous left fractional derivatives, then necessarily $y(a)=0$; by considering that the admissible functions y have continuous right fractional derivatives, then necessarily $y(b)=0$. This fact seems to be neglected in some previous works on the calculus of variations with Riemann-Liouville fractional derivatives. Alternatively, we can consider the general case of boundary conditions, say $y(a)=y_a$ and $y(b)=y_b$, and study functionals of type

$$ \mathcal{J}(y) = \int_a^b L(x,y(x),\, {}_a\mathcal{D}_x^\alpha(y(x)-y_a),\, {}_x\mathcal{D}_b^\beta(y(x)-y_b))\mathrm{d}x. $$

This needs, however, a modified fractional calculus [26].

Definition 3 The functional $\mathcal{J}$ is said to have a local minimum (resp. local maximum) at y if there exists a $\delta>0$ such that $\mathcal{J}(y)\le\mathcal{J}(y_1)$ (resp. $\mathcal{J}(y)\ge\mathcal{J}(y_1)$) for all y_1 satisfying $\|y-y_1\|<\delta$.

In [1], the following problem was addressed: among all curves $y(x)$ satisfying the boundary conditions, find those that maximize or minimize a given functional $\mathcal{J}$. The solution to this problem is given in the next theorem.

Theorem 4 [1] Let $\mathcal{J}$ be a functional as in (2) and y an extremizer of $\mathcal{J}$. Then, y satisfies the following Euler-Lagrange equation:

$$ \frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0. \tag{3} $$

3.1 The fractional isoperimetric problem

We introduce the fractional isoperimetric problem as follows: find the functions y that satisfy boundary conditions

$$y(a) = y_a, \quad y(b) = y_b \tag{4}$$

($y_a = 0$ if left Riemann-Liouville fractional derivatives are presented in (2); $y_b = 0$ if right Riemann-Liouville fractional derivatives are presented in (2), cf. Remark 1), the integral constraint

$$\mathcal{I}(y) = \int_a^b g(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)\mathrm{d}x = l, \tag{5}$$

and give a minimum or a maximum to (2). We assume that l is a specified real constant, functions y have continuous left and right fractional derivatives (if presented in (2)), and $(x, y, u, v) \to g(x, y, u, v)$ is a function with continuous first and second partial derivatives with respect to all its arguments such that $\frac{\partial g}{\partial u}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous right fractional derivative of order α for all $x \in [a, b]$ and $\frac{\partial g}{\partial v}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous left fractional derivative of order β in $[a, b]$. Theorem 4 motivates the following definition.

Definition 5 An admissible function y is an extremal for $\mathcal{I}$ in (5) if it satisfies the equation

$$\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v} = 0$$

for all $x \in [a, b]$.

The following theorem gives a necessary condition for y to be a solution of the fractional isoperimetric problem defined by (2)-(4)-(5) under the assumption that y is not an extremal for $\mathcal{I}$.

Theorem 6 Suppose that $\mathcal{J}$ given by (2) has a local minimum or a local maximum at y subject to the boundary conditions (4) and the isoperimetric constraint (5). Further, suppose that y is not an extremal for the functional $\mathcal{I}$. Then, there exists a constant λ, such that y satisfies the fractional differential equation

$$\frac{\partial F}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial F}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial F}{\partial v} = 0 \tag{6}$$

with $F = L - \lambda g$.

Proof Consider neighboring functions of the form

$$\hat{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \tag{7}$$

where for each $i \in \{1, 2\}$ ϵ_i is a sufficiently small parameter, η_i has continuous left and right fractional derivatives, and $\eta_i(a) = \eta_i(b) = 0$.

First, we will show that (7) has a subset of admissible functions for the fractional isoperimetric problem. Consider the quantity

$$\mathcal{I}(\hat{y}) = \int_a^b g(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, {}_a\mathcal{D}_x^\alpha y + \epsilon_1 {}_a\mathcal{D}_x^\alpha \eta_1 + \epsilon_2 {}_a\mathcal{D}_x^\alpha \eta_2, {}_x\mathcal{D}_b^\beta y + \epsilon_1 {}_x\mathcal{D}_b^\beta \eta_1 + \epsilon_2 {}_x\mathcal{D}_b^\beta \eta_2)\mathrm{d}x.$$

Then, we can regard $\mathcal{I}(\hat{y})$ as a function of ϵ_1 and ϵ_2. Define $\hat{I}(\epsilon_1, \epsilon_2) = \mathcal{I}(\hat{y}) - l$. Thus,

$$\hat{I}(0, 0) = 0. \tag{8}$$

In contrast, we have

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} = \int_a^b \left[\frac{\partial g}{\partial y}\eta_2 + \frac{\partial g}{\partial u}{}_a\mathcal{D}_x^\alpha \eta_2 + \frac{\partial g}{\partial v}{}_x\mathcal{D}_b^\beta \eta_2\right] \mathrm{d}x$$
$$= \int_a^b \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_2 \mathrm{d}x, \tag{9}$$

where (9) follows from (1). As y is not an extremal for $\mathcal{I}$, by the fundamental lemma of the calculus of variations (see, for example, [33, p.32]), there exists a function η_2, such that

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0. \tag{10}$$

Using (8) and (10), the implicit function theorem asserts that there exists a function $\epsilon_2(\cdot)$, defined in a neighborhood of zero, such that $\hat{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. We are now in a position to derive the necessary condition (6). Consider the real function $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. By hypothesis, $\hat{J}$ has minimum (or maximum) at $(0,0)$ subject to the constraint $\hat{I}(0,0) = 0$, and we have proved that $\nabla \hat{I}(0,0) \neq \mathbf{0}$. We can appeal to the Lagrange multiplier rule (see, for example, [33, p.77]) to assert the existence of a number λ such that $\nabla(\hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}$. Repeating the calculations as before, we obtain

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta_1(x) \mathrm{d}x$$

and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_1(x) \mathrm{d}x.$$

Therefore, one has

$$\int_a^b \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - \lambda\left(\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right)\right] \eta_1(x) \mathrm{d}x = 0. \tag{11}$$

As (11) holds for any function η_1, we obtain (6):

$$\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - \lambda\left(\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right) = 0.$$

Remark 2 Theorem 6 holds true in the case where α or β is equal to 1. Indeed, in the proof we imposed the condition $\eta_2(a) = \eta_2(b) = 0$, and formula (1) is valid.

Example 1 Let α be a given number in the interval $(0,1)$. We consider the following fractional isoperimetric problem:

$$\int_0^1 (x^4 + ({}_0\mathcal{D}_x^\alpha y)^2) \mathrm{d}x \longrightarrow \min,$$
$$\int_0^1 x^2 {}_0\mathcal{D}_x^\alpha y \, \mathrm{d}x = \frac{1}{5}, \tag{12}$$
$$y(0) = 0, \quad y(1) = \frac{2}{2\alpha + 3\alpha^2 + \alpha^3}.$$

The augmented Lagrangian is $F(x,y,{}_0\mathcal{D}_x^\alpha y,{}_x\mathcal{D}_1^\beta y) = x^4 + ({}_0\mathcal{D}_x^\alpha y)^2 - \lambda x^2 {}_0\mathcal{D}_x^\alpha y$, and it is a simple exercise to see that

$$y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{t^2}{(x-t)^{1-\alpha}}\mathrm{d}t = \frac{1}{\Gamma(\alpha)}\,\frac{2x^{\alpha+2}}{2\alpha+3\alpha^2+\alpha^3}, \tag{13}$$

(i) it is not an extremal for the isoperimetric functional, (ii) it satisfies ${}_0\mathcal{D}_x^\alpha y = x^2$, (iii) (6) holds for $\lambda = 2$, that is, ${}_x\mathcal{D}_1^\alpha(2\,{}_0\mathcal{D}_x^\alpha y - 2x^2) = 0$. We remark that, for $\alpha = 1$, (13) gives $y(x) = x^3/3$, which coincides to the solution of the associated classical variational problem (Fig.1). Indeed, as $\alpha \to 1$, our fractional problem (12) tends to the classical isoperimetric problem of minimizing the functional $\int_0^1 (x^4 + (y')^2)\mathrm{d}x$ subject to the isoperimetric constraint $\int_0^1 x^2 y'\,\mathrm{d}x = \frac{1}{5}$ and the boundary conditions $y(0) = 0$ and $y(1) = 1/3$. Then, $F = x^4 + (y')^2 - \lambda x^2 y'$ and the classical Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0 \Leftrightarrow -2y'' + 2\lambda x = 0. \tag{14}$$

The solution of (14) subject to $y(0) = 0$, $y(1) = 1/3$, and $\int_0^1 x^2 y'\,\mathrm{d}x = \frac{1}{5}$ is $\lambda = 2$ and $y = x^3/3$.

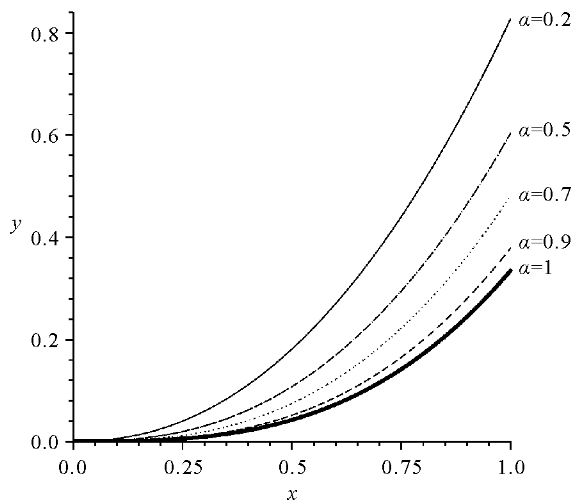

Fig.1 The fractional solution converges to the classical one as $\alpha \to 1$

Introducing a multiplier λ_0 associated with the cost functional (2), we can include in Theorem 6 the situation when the solution of the fractional isoperimetric problem defined by (2)-(4)-(5) is an extremal for the fractional isoperimetric functional. This is done in Theorem 7.

Theorem 7 If y is a local minimizer or a local maximizer of (2) subject to the boundary conditions (4) and the isoperimetric constraint (5), then there exist two constants λ_0 and λ, not both zero, such that

$$\frac{\partial K}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial K}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial K}{\partial v} = 0 \tag{15}$$

with $K = \lambda_0 L - \lambda g$.

Proof Using the same notations as in the proof of Theorem 6, we verify that $(0,0)$ is an extremal of $\hat{J}$ subject to the constraint $\hat{I} = 0$. Then, by the abnormal Lagrange multiplier rule (see, for example, [33, p.82]), there exist two reals λ_0 and λ, not both zero, such that $\nabla(\lambda_0 \hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}$. Therefore,

$$\lambda_0 \left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} - \lambda \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = 0.$$

Applying the same reasoning as in the proof of Theorem 6, we end up with (15).

3.2 An extension

In [9], a fractional functional

$$\mathcal{L}(y) = \int_A^B L(x, y, {}_a\mathcal{D}_x^\alpha y)\mathrm{d}x \tag{16}$$

is considered with $[A, B] \subset [a, b]$, that is, with the lower bound of the integral not coinciding with the lower bound of the fractional derivative. The main result of [9] is a new Euler-Lagrange equation for the functional (16). We now extend the techniques of [9] to prove an Euler-Lagrange equation for functionals containing both left and right Riemann-Liouville fractional derivatives, that is, for fractional functionals of the form

$$\mathcal{J}(y) = \int_A^B L(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)\mathrm{d}x, \tag{17}$$

where the integrand L satisfies the same conditions as before. Let y be a local extremizer of $\mathcal{J}$, such that $y(a) = y_a$ and $y(b) = y_b$, and let $\hat{y} = y + \epsilon\eta$ with $\eta(a) = \eta(b) = 0$. Consider the function $\hat{J}(\epsilon) = \mathcal{J}(y + \epsilon\eta)$. As $\hat{J}(\epsilon)$ has a local extremum at $\epsilon = 0$, then,

$$\begin{aligned}
0 &= \int_A^B \left[\frac{\partial L}{\partial y} \cdot \eta + \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta + \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta\right] \mathrm{d}x \\
&= \int_A^B \frac{\partial L}{\partial y} \cdot \eta \mathrm{d}x + \left[\int_a^B \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta \mathrm{d}x - \int_a^A \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta \mathrm{d}x\right] \\
&\quad + \left[\int_A^b \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta \mathrm{d}x - \int_B^b \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta \mathrm{d}x\right] \\
&= \int_A^B \frac{\partial L}{\partial y} \cdot \eta \mathrm{d}x + \left[\int_a^B \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} \mathrm{d}x - \int_a^A \eta \cdot {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} \mathrm{d}x\right] \\
&\quad + \left[\int_A^b \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} \mathrm{d}x - \int_B^b \eta \cdot {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} \mathrm{d}x\right].
\end{aligned}$$

Continuing in a similar way,

$$\begin{aligned}
0 &= \int_A^B \frac{\partial L}{\partial y} \cdot \eta \mathrm{d}x + \left[\int_a^A \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} \mathrm{d}x + \int_A^B \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} \mathrm{d}x - \int_a^A \eta \cdot {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} \mathrm{d}x\right] \\
&\quad + \left[\int_A^B \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} \mathrm{d}x + \int_B^b \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} \mathrm{d}x - \int_B^b \eta \cdot {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} \mathrm{d}x\right] \\
&= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u}\right] \eta \mathrm{d}x + \int_A^B \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta \mathrm{d}x \\
&\quad + \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta \mathrm{d}x.
\end{aligned}$$

Let $\eta_1 : [a, A] \to \mathbb{R}$ be any function satisfying $\eta_1(a) = 0$, and $\eta(x)$ be given by

$$\eta(x) = \begin{cases} \eta_1(x) & \text{if } x \in [a, A], \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore,

$$0 = \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} \right] \eta_1 \mathrm{d}x.$$

By the arbitrariness of η_1 and the fundamental lemma of calculus of variations,

$${}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} = 0 \text{ for all } x \in [a, A].$$

Analogously, we have

$$\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 \text{ for all } x \in [A, B],$$

and

$${}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 \text{ for all } x \in [B, b].$$

We have just proved the following.

Theorem 8 Let y be a local extremizer of (17). Then, y satisfies the following equations:

$$\begin{cases} \dfrac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \dfrac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \dfrac{\partial L}{\partial v} = 0 & \text{for all } x \in [A, B], \\ {}_x\mathcal{D}_B^\alpha \dfrac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \dfrac{\partial L}{\partial u} = 0 & \text{for all } x \in [a, A], \\ {}_A\mathcal{D}_x^\beta \dfrac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \dfrac{\partial L}{\partial v} = 0 & \text{for all } x \in [B, b]. \end{cases}$$

Remark 3 Theorem 8 simplifies to the result proved in [9] in the case that the Lagrangian L in (17) does not depend on the right Riemann-Liouville fractional derivative ${}_x\mathcal{D}_b^\beta y$.

We will study now the fractional isoperimetric problem for functionals of type (17) subject to an integral constraint

$$\mathcal{I}(y) = \int_A^B g(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y) \mathrm{d}x = l. \tag{18}$$

Definition 9 We say that y is an extremal for functional $\mathcal{I}$ given in (18) if

$$\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} = 0 \quad \text{for all } x \in [A, B].$$

Theorem 10 Let y give a local minimum or a local maximum to the fractional functional (17) subject to the constraint (18). If y is not an extremal for $\mathcal{I}$, then there exists a constant λ, such that

$$\begin{cases} \dfrac{\partial F}{\partial y} + {}_x\mathcal{D}_B^\alpha \dfrac{\partial F}{\partial u} + {}_A\mathcal{D}_x^\beta \dfrac{\partial F}{\partial v} = 0 & \text{for all } x \in [A, B], \\ {}_x\mathcal{D}_B^\alpha \dfrac{\partial F}{\partial u} - {}_x\mathcal{D}_A^\alpha \dfrac{\partial F}{\partial u} = 0 & \text{for all } x \in [a, A], \\ {}_A\mathcal{D}_x^\beta \dfrac{\partial F}{\partial v} - {}_B\mathcal{D}_x^\beta \dfrac{\partial F}{\partial v} = 0 & \text{for all } x \in [B, b], \end{cases} \tag{19}$$

with $F = L - \lambda g$.

Proof Consider a variation $(\epsilon_1, \epsilon_2) \mapsto \hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2$, where $\eta_1(a) = \eta_1(b) = \eta_2(a) = \eta_2(b) = 0$. Let

$$\hat{I}(\epsilon_1, \epsilon_2) = \int_A^B g(x, \hat{y},\ {}_a\mathcal{D}_x^\alpha \hat{y},\ {}_x\mathcal{D}_b^\beta \hat{y})\mathrm{d}x - l.$$

Then, $\hat{I}(0,0) = 0$ and

$$\begin{aligned}\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} &= \int_A^B \left[\frac{\partial g}{\partial y}\eta_2 + \frac{\partial g}{\partial u}{}_a\mathcal{D}_x^\alpha \eta_2 + \frac{\partial g}{\partial v}{}_x\mathcal{D}_b^\beta \eta_2\right]\mathrm{d}x \\ &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial g}{\partial u}\right]\eta_2 \mathrm{d}x + \int_A^B \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right]\eta_2 \mathrm{d}x \\ &\quad + \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right]\eta_2 \mathrm{d}x.\end{aligned}$$

As y is not an extremal for $\mathcal{I}$, there exists a function η_2, such that $\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0$. By the implicit function theorem, there exists a subset of curves $\{y + \epsilon_1\eta_1 + \epsilon_2\eta_2 \,|\, (\epsilon_1, \epsilon_2) \in \mathbb{R}^2\}$ admissible for the fractional isoperimetric problem. Let $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. Then, there exists a real λ, such that $\nabla(\hat{J}(0,0) - \lambda\hat{I}(0,0)) = \mathbf{0}$. Because

$$\begin{aligned}\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u}\right]\eta_1 \mathrm{d}x + \int_A^B \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right]\eta_1 \mathrm{d}x \\ &\quad + \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right]\eta_1 \mathrm{d}x, \\ \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial g}{\partial u}\right]\eta_1 \mathrm{d}x + \int_A^B \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right]\eta_1 \mathrm{d}x \\ &\quad + \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right]\eta_1 \mathrm{d}x,\end{aligned}$$

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} - \lambda \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = 0,$$

and η_1 is an arbitrary function, it follows (19).

Similar as before, we can include in Theorem 10 the situation when the solution y is an extremal for $\mathcal{I}$ (abnormal extremizer). For that, we introduce a new multiplier λ_0 that will be zero when the solution y is an extremal for $\mathcal{I}$ and one otherwise.

Theorem 11 If y is a local minimizer or a local maximizer of (17) subject to the isoperimetric constraint (18), then there exist two constants λ_0 and λ, not both zero, such that

$$\begin{cases} \dfrac{\partial K}{\partial y} + {}_x\mathcal{D}_B^\alpha \dfrac{\partial K}{\partial u} + {}_A\mathcal{D}_x^\beta \dfrac{\partial K}{\partial v} = 0 & \text{for all } x \in [A, B] \\ {}_x\mathcal{D}_B^\alpha \dfrac{\partial K}{\partial u} - {}_x\mathcal{D}_A^\alpha \dfrac{\partial K}{\partial u} = 0 & \text{for all } x \in [a, A] \\ {}_A\mathcal{D}_x^\beta \dfrac{\partial K}{\partial v} - {}_B\mathcal{D}_x^\beta \dfrac{\partial K}{\partial v} = 0 & \text{for all } x \in [B, b] \end{cases}$$

with $K = \lambda_0 L - \lambda g$.

3.3 Dependence on a parameter

Consider the following fractional problem of the calculus of variations: to extremize the functional

$$\Psi(y) = \int_0^1 \left[\frac{x^\alpha}{\Gamma(\alpha+1)}({}_0D_x^\alpha y)^2 - 2\overline{y}\,{}_0D_x^\alpha y\right]^2 \mathrm{d}x$$

when subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Here, $\overline{y} := x^\alpha, x \in [0,1]$. The fractional Euler-Lagrange equation associated to this problem is

$$ {}_xD_1^\alpha \left(2\left[\frac{x^\alpha}{\Gamma(\alpha+1)}({}_0D_x^\alpha y)^2 - 2\overline{y}\,{}_0D_x^\alpha y\right] \cdot \left[\frac{2x^\alpha}{\Gamma(\alpha+1)}{}_0D_x^\alpha y - 2\overline{y}\right]\right) = 0. \tag{20}$$

Replacing y by $\overline{y}$, and as ${}_0D_x^\alpha \overline{y} = \Gamma(\alpha+1)$, we conclude that $\overline{y}$ is a solution of (20).

Consider now the following question: what is the order of the derivative α, such that $\Psi(\overline{y})$ attains a maximum or a minimum? In other words, find the extremizers for $\psi(\alpha) = \Psi(\overline{y})$. Direct computations show that

$$\psi(\alpha) = \int_0^1 \left[x^\alpha \Gamma(\alpha+1)\right]^2 \mathrm{d}x.$$

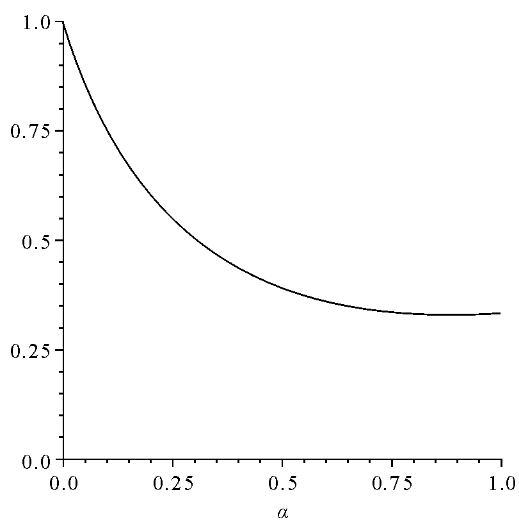

Fig.2 Graph of $\Psi(\overline{y})$ for $\alpha \in [0,1]$

Evaluating its derivative,

$$\begin{aligned}\psi'(\alpha) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[x^\alpha\Gamma(\alpha+1)\right]^2 \mathrm{d}x \\ &= \int_0^1 2x^\alpha\Gamma(\alpha+1)\left[x^\alpha \ln x\,\Gamma(\alpha+1) + x^\alpha \int_0^\infty t^\alpha \ln t\, e^{-t}\mathrm{d}t\right]\mathrm{d}x.\end{aligned}$$

We prove that $\alpha \approx 0.901$ is a solution of equation $\psi'(\alpha) = 0$, and such value is precisely where $\Psi(\overline{y})$ attains a minimum.

More generally, consider the functional

$$\Phi(y,\alpha)=\int_a^b L(x,y(x),\ {}_aD_x^\alpha y(x))\mathrm{d}x. \tag{21}$$

Functional (21) contains the left Riemann-Liouville derivative only, but we can consider functionals containing right Riemann-Liouville derivatives or both in a similar way. Let h be a curve, such that $h(a)=h(b)=0$, δ be a real number, and (y,α) be an extremal for Φ. Then,

$$\begin{aligned}&\Phi(y+h,\alpha+\delta)-\Phi(y,\alpha)\\ =&\int_a^b \frac{\partial L}{\partial y}\cdot h+\frac{\partial L}{\partial u}\cdot {}_aD_x^{\alpha+\delta}h+\frac{\partial L}{\partial u}\cdot({}_aD_x^{\alpha+\delta}y-{}_aD_x^\alpha y)\mathrm{d}x+O|(h,\delta)|^2.\end{aligned}$$

For $\delta=0$, using the formula of fractional integration by parts and the fundamental lemma of the calculus of variations, we obtain the known fractional Euler-Lagrange equation

$$\frac{\partial L}{\partial y}(x,y(x),\ {}_aD_x^\alpha y(x))+{}_xD_b^\alpha\frac{\partial L}{\partial u}(x,y(x),\ {}_aD_x^\alpha y(x))=0.$$

For $h=0$, we obtain the relation

$$\int_a^b\frac{\partial L}{\partial u}(x,y(x),\ {}_aD_x^\alpha y(x))\phi'(\alpha)\mathrm{d}x=0,$$

where $\phi(\alpha)={}_aD_x^\alpha y(x)$. In summary, we have

Theorem 12 If (y,α) is an extremizer of Φ given by (21), satisfying the boundary conditions $y(a)=0$ and $y(b)=y_b$, then, y satisfies the system

$$\begin{cases}\dfrac{\partial L}{\partial y}(x,y(x),\ {}_aD_x^\alpha y(x))+{}_xD_b^\alpha\dfrac{\partial L}{\partial u}(x,y(x),\ {}_aD_x^\alpha y(x))=0,\\ \displaystyle\int_a^b\frac{\partial L}{\partial u}(x,y(x),\ {}_aD_x^\alpha y(x))\phi'(\alpha)\mathrm{d}x=0,\end{cases} \tag{22}$$

where $\phi(\alpha)={}_aD_x^\alpha y(x)$.

In the previous example, the solution obtained satisfies system (22) because

$$\frac{\partial L}{\partial u}(x,\overline{y}(x),\ {}_aD_x^\alpha\overline{y}(x))=0.$$

References

[1] Agrawal O P. Formulation of Euler-Lagrange equations for fractional variational problems. J Math Anal Appl, 2002, **272**(1): 368–379

[2] Agrawal O P. Fractional variational calculus and the transversality conditions. J Phys A, 2006, **39**(33): 10375–10384

[3] Agrawal O P. Fractional variational calculus in terms of Riesz fractional derivatives. J Phys A, 2007, **40**(24): 6287–6303

[4] Almeida R, Malinowska A B, Torres D F M. A fractional calculus of variations for multiple integrals with application to vibrating string. J Math Phys, 2010, **51**(3): 033503, 12 pp.

[5] Almeida R, Torres D F M. Hölderian variational problems subject to integral constraints. J Math Anal Appl, 2009, **359**(2): 674–681

[6] Almeida R, Torres D F M. Isoperimetric problems on time scales with nabla derivatives. J Vib Control, 2009, **15**(6): 951–958

[7] Almeida R, Torres D F M. Calculus of variations with fractional derivatives and fractional integrals. Appl Math Lett, 2009, **22**(12): 1816–1820

[8] Almeida R, Torres D F M. Leitmann's direct method for fractional optimization problems. Appl Math Comput, 2010, **217**(3): 956–962

[9] Atanacković T M, Konjik S, Pilipović S. Variational problems with fractional derivatives: Euler-Lagrange equations. J Phys A, 2008, **41**(9): 095201, 12 pp.

[10] Baleanu D. New applications of fractional variational principles. Rep Math Phys, 2008, **61**(2): 199–206

[11] Baleanu D. Fractional variational principles in action. Phys Scr, 2009, **T136**: 014006

[12] Baleanu D, Maaraba T, Jarad F. Fractional variational principles with delay. J Phys A, 2008, **41**(31): 315403, 8 pp.

[13] Baleanu D, Muslih S I, Rabei E M. On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative. Nonlinear Dynam, 2008, **53**(1/2): 67–74

[14] Baleanu D, Trujillo J J. On exact solutions of a class of fractional Euler-Lagrange equations. Nonlinear Dynam, 2008, **52**(4): 331–335

[15] Baleanu D, Trujillo J I. A new method of finding the fractional Euler-Lagrange and Hamilton equations within Caputo fractional derivatives. Commun Nonlinear Sci Numer Simul, 2010, **15**(5): 1111–1115

[16] Bastos N R O, Ferreira R A C, Torres D F M. Discrete-time fractional variational problems. Signal Process, 2011, **91**(3): 513–524

[17] Blåsjö V. The isoperimetric problem. Amer Math Monthly, 2005, **112**(6): 526–566

[18] Curtis J P. Complementary extremum principles for isoperimetric optimization problems. Optim Eng, 2004, **5**(4): 417–430

[19] El-Nabulsi R A, Torres D F M. Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β). Math Methods Appl Sci, 2007, **30**(15): 1931–1939

[20] El-Nabulsi R A, Torres D F M. Fractional actionlike variational problems. J Math Phys, 2008, **49**(5): 053521, 7 pp.

[21] Frederico G S F, Torres D F M. A formulation of Noether's theorem for fractional problems of the calculus of variations. J Math Anal Appl, 2007, **334**(2): 834–846

[22] Frederico G S F, Torres D F M. Fractional conservation laws in optimal control theory. Nonlinear Dynam, 2008, **53**(3): 215–222

[23] Frederico G S F, Torres D F M. Fractional Noether's theorem in the Riesz-Caputo sense. Appl Math Comput, 2010, **217**(3): 1023–1033

[24] Herzallah M A E, Baleanu D. Fractional-order Euler-Lagrange equations and formulation of Hamiltonian equations. Nonlinear Dynam, 2009, **58**(1/2): 385–391

[25] Jarad F, Maaraba T, Baleanu D. Fractional variational principles with delay within caputo derivatives. Rep Math Phys, 2010, **65**(1): 17–28

[26] Jumarie G. Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Appl Math Lett, 2009, **22**(3): 378–385

[27] Malinowska A B, Torres D F M. Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative. Comput Math Appl, 2010, **59**(9): 3110–3116

[28] Miller K S, Ross B. An introduction to the fractional calculus and fractional differential equations. New York: Wiley, 1993

[29] Podlubny I. Fractional differential equations. San Diego, CA: Academic Press, 1999

[30] Ross B, Samko S G, Love E R. Functions that have no first order derivative might have fractional derivatives of all orders less than one. Real Anal Exchange, 1994/1995, **20**(1): 140–157

[31] Samko S G, Kilbas A A, Marichev O I. Fractional integrals and derivatives Translated from the 1987 Russian original. Yverdon: Gordon and Breach, 1993

[32] Tarasov V E, Zaslavsky G M. Nonholonomic constraints with fractional derivatives. J Phys A, 2006, **39**(31): 9797–9815

[33] van Brunt B. The calculus of variations. New York: Springer, 2004