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Series representations in the spirit of Ramanujan

Emre Alkan

Department of Mathematics, Koç University, Rumelifeneri Yolu, 34450, Sarıyer, Istanbul, Turkey

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A B S T R A C T

Using an integral transform with a mild singularity, we obtain series representations valid for specific regions in the complex plane involving trigonometric functions and the central binomial coefficient which are analogues of the types of series representations first studied by Ramanujan over certain intervals on the real line. We then study an exponential type series rapidly converging to the special values of *L*-functions and the Riemann zeta function. In this way, a new series converging to Catalan's constant with geometric rate of convergence less than a quarter is deduced. Further evaluations of some series involving hyperbolic functions are also given.

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1. Introduction

Our purpose in this paper is twofold. First we introduce an integral transform having a mild singularity and in particular generalizing the classical representation (see [23])

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^1 \frac{z \log^{n-1} x}{1 - zx}\, dx,$$

for the polylogarithm function of order $n \geqslant 2$, where z is a complex number with $|z| \leqslant 1$. With the help of our transform, we then show how to find analogues of certain series representations involving trigonometric functions and the central binomial coefficient. Such type of series were first studied by Ramanujan [24,25]. We refer the reader to the monograph of Berndt [15] which gives further enlightening discussions on the nature of these representations. Ramanujan proved for example that

$$\sum_{n=0}^{\infty} \frac{(2\sin 2x)^{2n+1}}{\binom{2n}{n}(2n+1)^2} = 2\sum_{n=0}^{\infty} \frac{(\tan x)^{2n+1}}{(2n+1)^2},$$

for any real number x with $|x| \leqslant \frac{\pi}{4}$. In this way, Ramanujan obtained striking rapidly convergent series for the Catalan constant and $\zeta(3)$, where $\zeta(s)$ is the Riemann zeta function. Concerning Catalan's constant, he showed that (see [24,25,15,18])

E-mail address: ealkan@ku.edu.tr.

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$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{\pi}{8}\log(2+\sqrt{3}\,) + \frac{3}{8}\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2}.$$

In recent work, Batır [11] obtained further interesting series related to these constants. An advantage of our approach is that one can even derive such series representations to be valid for specific regions in the complex plane. As a second goal, we study certain exponential type series rapidly converging to the special values of L-functions and the Riemann zeta function. Precisely, our first result is as follows.

Theorem 1. *Let $\mathfrak{F}_1$ be a region in the complex plane defined by the conditions $|\sin 2z| \leqslant 1$ and $|\Re(z)| \leqslant \frac{\pi}{4}$. Then for any $z \in \mathfrak{F}_1$, we have*

$$\sum_{n=1}^{\infty} \frac{(2\sin 2z)^{2n}}{\binom{2n}{n} n^2 (2n+1)^2} = -12 + 8z^2 + 16z\cot 2z + \frac{8}{\sin 2z}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

$$\sum_{n=1}^{\infty} \frac{(2\sin 2z)^{2n}}{\binom{2n}{n} n (2n+1)^2} = 4 - 4z\cot 2z - \frac{4}{\sin 2z}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

$$\sum_{n=1}^{\infty} \frac{\Gamma(\frac{n}{2})^2 (2\sin 2z)^n}{n!(n+1)^2} = -12 + 4\pi z + 8z^2 + 16z\cot 2z - \frac{2\pi}{\sin 2z}\big(2 - \log 2 - 2\cos 2z + \log(1+\cos 2z)\big) + \frac{8}{\sin 2z}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2},$$

where Γ is the Gamma function and the principal branch of the logarithm is used.

Let $\mathfrak{F}_2$ be a region in the complex plane defined by the conditions $|\sin z| \leqslant 1$ and $|\Re(z)| \leqslant \frac{\pi}{2}$. Then for any $z \in \mathfrak{F}_2$, we have

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(\sin z)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2} = z - \frac{1}{\sin z}\big(2 - \log 2 - 2\cos z + \log(1+\cos z)\big),$$

where the principal branch of the logarithm is used.

Let $\mathfrak{F}_3$ be a strip in the complex plane defined by the condition $|\Re(z)| \leqslant \frac{\pi}{4}$. Then for any $z \in \mathfrak{F}_3$, we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} \binom{2n}{n} \frac{(\tan z)^{2n}}{2^{2n+1} n (2n-1)} = -1 + \log 2 + \sec z - \log(1+\sec z),$$

where the principal branch of the logarithm is used.

Let us remark that putting $z = x + iy$, the region $\mathfrak{F}_1$ in the above theorem can be described alternatively as the set of all points (x, y) in the plane satisfying the conditions $|x| \leqslant \frac{\pi}{4}$ and $e^{4y} + e^{-4y} - 2\cos 4x \leqslant 4$. In particular, if $|x| = \frac{\pi}{4}$, then it follows easily that $(-\frac{\pi}{4}, 0)$ and $(\frac{\pi}{4}, 0)$ are the only possible points in this region. A similar description for $\mathfrak{F}_2$ can be given as well. Next let

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_j x^{m-j}$$

be the mth Bernoulli polynomial, where B_j denotes the jth Bernoulli number. For any real number a, $B_m(a)$ can be obtained from the Taylor series expansion

$$\frac{xe^{ax}}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m(a)}{m!} x^m$$

about zero, where the radius of convergence is 2π. Such expansions are often useful for studying generating functions of special values of L-functions. The classical formula

$$\frac{x}{e^x - 1} = 1 + \sum_{m=1}^{\infty} (-1)^{m-1} \zeta(1-m) \frac{x^m}{(m-1)!}$$

furnishes a striking example of this. Inspired by certain mock theta function identities arising from Ramanujan's lost notebook (see [7]), Andrews, Urroz and Ono [8] showed that this phenomenon is indeed quite general and they have found further striking generating functions concerning special values of certain Hecke L-functions (see also [20]). Rapidly convergent series representations of special values of L-functions even go back to the works of Euler who showed (see [9]) in a 1772 paper that

$$\zeta(3) = -\frac{4\pi^2}{7}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{2^{2n}(2n+1)(2n+2)}.$$

Another example is the series of Ramanujan for Catalan's constant that is given above. The formula

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\binom{2n}{n}n^3}$$

served as the starting point of Apéry's proof of the irrationality of $\zeta(3)$. For an infinite family of representations in the case of Catalan's constant, the reader is referred to the works of Bradley [18] and Kanemitsu, Kumagai and Yoshimoto [21]. There has been extensive research by many different authors in the last two decades about rapidly convergent series representations of special values of L-functions and Apéry type formulas involving reciprocals of binomial coefficients. In this connection, one should consult the papers of Batır [11,12], Borwein and Bradley [17], Bailey, Borwein and Bradley [10], Srivastava [26,27], Srivastava and Tsumura [28], Choi and Srivastava [19], Alzer, Karayannakis and Srivastava [6] and the references therein. In the case of the Riemann zeta function, rapidly convergent series representations were first obtained by Ramanujan (see [16]). In particular, he showed that

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}.$$

Generalizations of this formula and other variations were investigated by Berndt [13,14]. The work of Vepštas [29] shed more light on the structure of such representations. In a different direction, subject to a condition called as the parity match, the author [1–5] studied representations of special values of L-functions from various perspectives and showed that they can be written as finite sums involving trigonometric functions.

Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. Assume that $|a_n| = o(r^n)$ for some $0 < r < 1$. Then we define the geometric rate of convergence of this series as the nonnegative number

$$\rho := \inf_{\substack{0<r<1\\ |a_n|=o(r^n)}} r.$$

In the case of Euler's and Apéry's series for $\zeta(3)$ and Ramanujan's series for G, ones sees that $\rho = \frac{1}{4}$. Our next result shows that one can find nontrivial series converging universally to the special values of all L-functions with $\rho < \frac{1}{4}$ regardless of the parity match. Of course a representation such as

$$\zeta(3) = \zeta(3)(1-r)\sum_{n=0}^{\infty} r^n$$

for $0 \leqslant r < 1$ would be trivial.

Theorem 2. *Let χ be a Dirichlet character modulo $q \geqslant 1$ and let*

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

be the corresponding L-function for $\Re(s) > 1$. Then we have

$$L(r,\chi) = \frac{1}{q^r}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\left(\sum_{v=1}^{q}\frac{\chi(v)e^{-\frac{vz}{q}}}{(n+\frac{v}{q})^{r-k}}\right)\frac{z^k}{k!}\right)e^{-nz} + \left(\sum_{v=1}^{q}B_n\left(1-\frac{v}{q}\right)\right)\frac{z^{n+r-1}}{(r-1)!n!(n+r-1)}$$

for any integer $r \geqslant 2$ and $0 \leqslant z < 2\pi$. Consequently, there is a series representation of $L(r,\chi)$ of this type with geometric rate of convergence $\rho \leqslant e^{-\lambda_0} < \frac{1}{4}$, where λ_0 is the unique positive solution of the equation $\lambda e^{\lambda} = 2\pi$. For $0 < a \leqslant 1$ and $\Re(s) > 1$, let

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$$

be the Hurwitz zeta function. Then for any integer $r \geqslant 3$ and $0 \leqslant z < \pi$, we have

$$\frac{\zeta(r,a)}{2^{r+1}}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{n+1}{(n+2a)^r}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+a)^{r-k}}\right)e^{-2(n+a)z}-\frac{1}{2}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2a)^{r-k}}\right)(n+1)e^{-(n+2a)z}$$
$$-\left(\sum_{j+s=n}\frac{2^{s-1}B_jB_s(1-a)}{j!s!}\right)\frac{z^{n+r-2}}{(r-1)!(n+r-2)}.$$

Consequently, we have

$$\zeta(r)-\zeta(r-1)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+1)^{r-k}}\right)e^{-2(n+1)z}$$
$$+\frac{1}{2}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(2n+1)^{r-k}}\right)e^{-(2n+1)z}-\frac{1}{2}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2)^{r-k}}\right)(n+1)e^{-(n+2)z}$$
$$-\frac{1}{2}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+1)^{r-k}}\right)(n+1)e^{-(n+1)z}-\left(\sum_{j+s=n}\frac{B_jB_s}{j!s!}\right)\frac{z^{n+r-2}}{(r-1)!(n+r-2)}$$

for any $r \geqslant 3$ and $0 \leqslant z < \pi$.

An important special case of this theorem concerns the Riemann zeta function. Indeed taking $q = 1 = z$, the formula

$$\zeta(r)=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{r-1}\frac{1}{n^{r-k}k!}\right)e^{-n}+\frac{B_{n-1}}{(r-1)!(n-1)!(n+r-2)}$$

holds for any $r \geqslant 2$. As a further bonus we derive four new series representing the Catalan constant, where although the first three are reminiscent of a series deceleration result

$$G=\frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n}}{\binom{2n}{n}(2n+1)^2}$$

which was known to Ramanujan (see [15] and [18]), they are indeed slight series acceleration results and the last one is a series acceleration result for a specific value of the parameter z with geometric rate of convergence less than a quarter. But perhaps the most interesting outcomes are exact evaluations of certain series involving hyperbolic functions which becomes possible as a result of our flexibility of working in complex regions. In this regard, we especially recommend papers of Berndt [13] and Zucker [30] for many other interesting transformations between such series.

Corollary 1. *Let χ_{-1} be the odd Dirichlet character modulo 4. Then for Catalan's constant $G = L(2,\chi_{-1})$, we have*

$$G=1-\frac{1}{4}\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}n(2n+1)^2}=\frac{3}{2}-\frac{\pi^2}{16}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}n^2(2n+1)^2}$$
$$=\frac{3}{2}-\frac{3\pi^2}{16}+\frac{\pi}{4}(2-\log 2)+\frac{1}{8}\sum_{n=1}^{\infty}\frac{2^n\Gamma(\frac{n}{2})^2}{n!(n+1)^2}.$$

Moreover, for any $0 \leqslant z < 2\pi$, we have

$$G=\frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{e^{-\frac{z}{4}}}{(n+\frac{1}{4})^2}-\frac{e^{-\frac{3z}{4}}}{(n+\frac{3}{4})^2}+\frac{ze^{-\frac{z}{4}}}{n+\frac{1}{4}}-\frac{ze^{-\frac{3z}{4}}}{n+\frac{3}{4}}\right)e^{-nz}+\left(\sum_{v=1}^{4}B_n\left(1-\frac{v}{q}\right)\right)\frac{z^{n+1}}{(n+1)!}.$$

Consequently, there is a series representation of G of this type with geometric rate of convergence $\rho \leqslant e^{-\lambda_0} < \frac{1}{4}$, where λ_0 is the unique positive solution of the equation $\lambda e^{\lambda} = 2\pi$. We also have the evaluations

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n+2}(2n+1)(n+1)^2}=\frac{\pi}{2}-2+\log 2,$$
$$\sum_{n=1}^{\infty}(-1)^{n-1}\binom{2n}{n}\frac{1}{2^{2n+1}n(2n-1)}=\sqrt{2}-1+\log 2-\log(1+\sqrt{2}),$$
$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{(\sinh y)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2}=y+\frac{1}{\sinh y}\big(2-\log 2-2\cosh y+\log(1+\cosh y)\big)$$

when $\cosh 2y \leqslant 3$ *and*

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{(\tanh y)^{2n}}{2^{2n+1} n(2n-1)} = 1 - \log 2 - \frac{1}{\cosh y} + \log\left(1 + \frac{1}{\cosh y}\right)$$

for any real number y.

2. A preliminary result

We first give the integral transform that was promised above.

Theorem 3. *Let* $f(z)$ *be an analytic function represented by the power series*

$$\sum_{m=0}^{\infty} a_m z^m$$

with real coefficients a_m *whose radius of convergence is* 1. *If* z *is a complex number with* $|z| < 1$, *then*

$$\int_0^1 (\log x)^{n-1} f(zx)\, dx = (-1)^{n-1}(n-1)! \sum_{m=0}^{\infty} \frac{a_m}{(m+1)^n} z^m$$

holds for any integer $n \geqslant 2$.

To prove this result, fix a complex number z with $|z| < 1$. Then $|zx| \leqslant |z| < 1$ and we have

$$f(zx) = \sum_{m=0}^{\infty} a_m z^m x^m, \tag{2.1}$$

where the convergence is uniform for $0 \leqslant x \leqslant 1$. Taking into account the singularity of $\log x$ at $x = 0$, we may write, using (2.1), that

$$\int_0^1 (\log x)^{n-1} f(zx)\, dx = \lim_{\eta \to 0+} \int_\eta^1 (\log x)^{n-1} \left(\sum_{m=0}^{\infty} a_m z^m x^m \right) dx = \lim_{\eta \to 0+} \sum_{m=0}^{\infty} a_m z^m \left(\int_\eta^1 (\log x)^{n-1} x^m\, dx \right), \tag{2.2}$$

where the interchange of summation with the integral is justified by uniform convergence. Integrating by parts repeatedly, one obtains

$$\int_\eta^1 (\log x)^{n-1} x^m\, dx = (-1)^{n-1} \frac{(n-1)!}{(m+1)^n} + \eta^{m+1}(n-1)! \sum_{j=0}^{n-1} \frac{(-1)^{n-j}(\log \eta)^j}{j!(m+1)^{n-j}}. \tag{2.3}$$

Using (2.2) and (2.3), it suffices to compute limits of form

$$\lim_{\eta \to 0+} \eta (\log \eta)^j \sum_{m=0}^{\infty} \frac{a_m}{(m+1)^{n-j}} (\eta z)^m. \tag{2.4}$$

Note that the radius of convergence of the power series

$$\sum_{m=0}^{\infty} \frac{a_m}{(m+1)^{n-j}} z^m$$

is also 1. As $\eta z \to 0$ when $\eta \to 0+$, we see that

$$\lim_{\eta \to 0+} \sum_{m=0}^{\infty} \frac{a_m}{(m+1)^{n-j}} (\eta z)^m = a_0. \tag{2.5}$$

It follows from (2.5) that the limit in (2.4) is zero. Thus combining (2.2)–(2.5), we complete the proof.

3. Proof of Theorem 1

Note that the principal branch of $\arcsin w$ with real part in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is a complex analytic function for $|w| < 1$. If the real part is $\pm\frac{\pi}{2}$, then the principal branch picks up the unique complex number with a nonnegative imaginary part. It is also known that (see [22])

$$2\left(\arcsin\left(\frac{wx}{2}\right)\right)^2 = \sum_{m=1}^{\infty} \frac{w^{2m}x^{2m}}{\binom{2m}{m}m^2} \tag{3.1}$$

whenever $|wx| < 2$. Using now Theorem 3 and (3.1), we see that the formula

$$2\int_0^1 \log x\left(\arcsin\left(\frac{wx}{2}\right)\right)^2 dx = -\sum_{n=1}^{\infty} \frac{w^{2n}}{\binom{2n}{n}n^2(2n+1)^2} \tag{3.2}$$

holds for any complex number w with $|w| < 2$. To work out the left-hand side of (3.2) as an indefinite integral, let us temporarily assume that w is real and $w \neq 0$. Using the change of variable $x = 2t$, one obtains

$$\begin{aligned}\int \log x\left(\arcsin\left(\frac{wx}{2}\right)\right)^2 dx &= 2\int \log 2t\,(\arcsin wt)^2\,dt \\ &= 2\log 2\int (\arcsin wt)^2\,dt + 2\int \log t(\arcsin wt)^2\,dt.\end{aligned} \tag{3.3}$$

Integrating by parts and using the change of variable $u = wt$, we have

$$\begin{aligned}\int (\arcsin wt)^2\,dt &= t(\arcsin wt)^2 - 2w\int \frac{t\arcsin wt}{\sqrt{1-w^2t^2}}\,dt = t(\arcsin wt)^2 - \frac{2}{w}\int \frac{u\arcsin u}{\sqrt{1-u^2}}\,du \\ &= t(\arcsin wt)^2 - 2t + \frac{2}{w}\sqrt{1-w^2t^2}\arcsin wt + C.\end{aligned} \tag{3.4}$$

Similarly, we may obtain

$$\begin{aligned}&\int \log t\,(\arcsin wt)^2\,dt \\ &\quad= (t\log t - t)(\arcsin wt)^2 - 2w\int \frac{(t\log t - t)\arcsin wt}{\sqrt{1-w^2t^2}}\,dt \\ &\quad= (t\log t - t)(\arcsin wt)^2 - 2w\int \frac{t\log t\,\arcsin wt}{\sqrt{1-w^2t^2}}\,dt + 2w\int \frac{t\arcsin wt}{\sqrt{1-w^2t^2}}\,dt \\ &\quad= (t\log t - t)(\arcsin wt)^2 + 2t - \frac{2}{w}\sqrt{1-w^2t^2}\arcsin wt - 2w\int \frac{t\log t\,\arcsin wt}{\sqrt{1-w^2t^2}}\,dt.\end{aligned} \tag{3.5}$$

Again integrating by parts, one further gets

$$\begin{aligned}\int \log t\frac{wt\arcsin wt}{\sqrt{1-w^2t^2}}\,dt &= \log t\left(t - \frac{\sqrt{1-w^2t^2}}{w}\arcsin wt\right) - \int \frac{1}{t}\left(t - \frac{\sqrt{1-w^2t^2}}{w}\arcsin wt\right)dt \\ &= \log t\left(t - \frac{\sqrt{1-w^2t^2}}{w}\arcsin wt\right) - t + \int \frac{\sqrt{1-w^2t^2}}{wt}\arcsin wt\,dt.\end{aligned} \tag{3.6}$$

Using successively the change of variables $s = wt$ and $s = \sin y$, one may deduce that

$$\begin{aligned}\int \frac{\sqrt{1-w^2t^2}}{wt}\arcsin wt\,dt &= \frac{1}{w}\int \frac{\sqrt{1-s^2}}{s}\arcsin s\,ds = \frac{1}{w}\int \frac{y\cos^2 y}{\sin y}\,dy \\ &= \frac{1}{w}\left(\int \frac{y}{\sin y}\,dy - \int y\sin y\,dy\right) = \frac{1}{w}\left(y\cos y - \sin y + \int \frac{y}{\sin y}\,dy\right) \\ &= \frac{\sqrt{1-w^2t^2}}{w}\arcsin wt - t + \frac{1}{w}\int \frac{y}{\sin y}\,dy.\end{aligned} \tag{3.7}$$

Let us remark that integrals of form

$$\int \frac{y}{\sin y}\,dy, \qquad \int \frac{y^2}{\sin y}\,dy$$

were studied by Ramanujan over the interval $[0, \frac{\pi}{2}]$ for finding representations of Catalan's constant and $\zeta(3)$ (see [15]). Indeed using the change of variable $v = \tan\frac{y}{2}$, one has

$$\int \frac{y}{\sin y}\,dy = 2\int \frac{\arctan v}{v}\,dv \tag{3.8}$$

which is another integral studied by Ramanujan (see [24]). Gathering (3.2)–(3.8) and computing the definite integral on the left-hand side of (3.2) by taking limits as $\eta \to 0+$, we deduce that

$$\sum_{n=1}^{\infty} \frac{w^{2n}}{\binom{2n}{n} n^2 (2n+1)^2} = -12 + 2\left(\arcsin\left(\frac{w}{2}\right)\right)^2 + \frac{16}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right) + \frac{16}{w}\int_0^{g(w)} \frac{\arctan v}{v}\,dv \tag{3.9}$$

for any real number $w \neq 0$ with $-2 < w < 2$, where $g(w) = \tan(\frac{1}{2}\arcsin(\frac{w}{2}))$. Note that $-1 < g(w) < 1$ and using

$$\frac{\arctan v}{v} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} v^{2n-2}}{2n-1}$$

for $|v| < 1$ and integrating termwise by uniform convergence, we obtain that

$$\int_0^{g(w)} \frac{\arctan v}{v}\,dv = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2}. \tag{3.10}$$

Consequently, from (3.9) and (3.10), the formula

$$\sum_{n=1}^{\infty} \frac{w^{2n}}{\binom{2n}{n} n^2 (2n+1)^2} = -12 + 2\left(\arcsin\left(\frac{w}{2}\right)\right)^2 + \frac{16}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right) + \frac{16}{w}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} \tag{3.11}$$

follows for any real number $w \neq 0$ with $-2 < w < 2$. Moreover, the left-hand side of (3.11) vanishes at $w = 0$ and the right-hand side of (3.11) is easily seen to have a removable singularity at $w = 0$ with limiting value zero as w approaches to zero. Therefore, (3.11) holds for any real number w with $-2 < w < 2$. By Weierstrass M-test, the left-hand side of (3.11) is a uniformly convergent series for $-2 \leqslant w \leqslant 2$ so that it defines a continuous function for $-2 \leqslant w \leqslant 2$ and in particular for $w = \pm 2$. The functions

$$\left(\arcsin\left(\frac{w}{2}\right)\right)^2, \quad \frac{16}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right)$$

are both continuous as well at $w = \pm 2$. Using continuity of g at $w = \pm 2$ and writing

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} = \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} + o(1)$$

as $N \to \infty$ together with

$$\lim_{w \to \pm 2} \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} = \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(\pm 2))^{2n-1}}{(2n-1)^2}$$

shows that

$$\frac{16}{w}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2}$$

is also continuous at $w = \pm 2$. Therefore, (3.11) holds for any real number w with $-2 \leqslant w \leqslant 2$. Next we extend (3.11) to complex values of w. Note that the left-hand side of (3.11) is an analytic function for $|w| < 2$ and continuous for $|w| \leqslant 2$. Moreover, if the principal branch of the square-root where it is positive on the positive real axis and the principal branch of the arcsin where the real part is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ are used, then the complex valued functions

$$\left(\arcsin\left(\frac{w}{2}\right)\right)^2, \quad \left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right)$$

are also analytic for $|w| < 2$. Furthermore, the branch cuts of these functions start at the points $w = \pm 2$ and consequently we see that they are continuous for $|w| \leqslant 2$ except possibly the points $w = \pm 2$ and this is harmless as (3.11) is already shown to hold at these values. Since $\Re(\frac{1}{2}\arcsin(\frac{w}{2})) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, we may put $\frac{1}{2}\arcsin(\frac{w}{2}) = a + ib$ with $-\frac{\pi}{4} \leqslant a \leqslant \frac{\pi}{4}$ to see that

$$\left|g(w)\right|^2 = \frac{e^{2b} + e^{-2b} - 2\cos 2a}{e^{2b} + e^{-2b} + 2\cos 2a} \leqslant 1. \tag{3.12}$$

Thus (3.12) shows that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} \tag{3.13}$$

is uniformly convergent on compact subsets of $|w| < 2$ and represents an analytic function for $|w| < 2$. It is also clear that $g(w)$ is continuous for $|w| \leqslant 2$ except possibly the points $w = \pm 2$, where the branch cuts begin. Fixing $w_0 \neq \pm 2$ with $|w_0| = 2$, let w approach to w_0 radially with $|w| < 2$. Then $g(w)$ approaches to $g(w_0)$ by continuity and writing

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} = \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} + o(1)$$

as $N \to \infty$ together with

$$\lim_{w \to w_0} \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} = \sum_{n \leqslant N} \frac{(-1)^{n-1}(g(w_0))^{2n-1}}{(2n-1)^2}$$

shows that (3.13) is continuous at w_0. By analytic continuation and taking into account the removable singularity at $w = 0$, (3.11) holds for all complex numbers w with $|w| < 2$. Also using continuity of both sides on the boundary, (3.11) holds for $|w| \leqslant 2$. Finally, by the change of variable $w = 2\sin 2z$, the conditions on w give that $|\sin 2z| \leqslant 1$ and $|\Re(z)| \leqslant \frac{\pi}{4}$ which are the defining conditions of the region $\mathfrak{F}_1$ (note that there is no ambiguity when $|\Re(z)| = \frac{\pi}{4}$ since, as mentioned in the introduction, only $z = \pm\frac{\pi}{4}$ can satisfy the condition $|\sin 2z| \leqslant 1$). Therefore, for any $z \in \mathfrak{F}_1$, the desired formula

$$\sum_{n=1}^{\infty} \frac{(2\sin 2z)^{2n}}{\binom{2n}{n} n^2 (2n+1)^2} = -12 + 8z^2 + 16z\cot 2z + \frac{8}{\sin 2z}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2}$$

follows from (3.11). Next note that

$$\arcsin wx = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{w^{2m+1}x^{2m+1}}{2^{2m}(2m+1)} \tag{3.14}$$

whenever $|wx| < 1$. As a result of Theorem 3 and (3.14), we infer that

$$\int_0^1 \log x \arcsin wx \, dx = -\sum_{n=0}^{\infty} \binom{2n}{n} \frac{w^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2}, \tag{3.15}$$

where w is any complex number with $|w| < 1$. Let us assume that w is a real number with $0 < w < 1$. To compute the left-hand side of (3.15) as an indefinite integral, we may integrate by parts to get

$$\int \log x \arcsin wx \, dx = (x\log x - x)\arcsin wx + w\int \frac{(x - x\log x)}{\sqrt{1-w^2x^2}}\, dx. \tag{3.16}$$

First we have

$$\int \frac{wx}{\sqrt{1-w^2x^2}}\, dx = -\frac{\sqrt{1-w^2x^2}}{w} + C. \tag{3.17}$$

A further integration by parts gives

$$\int \frac{wx\log x}{\sqrt{1-w^2x^2}}\, dx = -\frac{\sqrt{1-w^2x^2}\log x}{w} + \int \frac{\sqrt{1-w^2x^2}}{wx}\, dx. \tag{3.18}$$

Moreover, using the change of variables $t = wx$ and $t = \sin y$, we see that

$$\int \frac{\sqrt{1-w^2x^2}}{wx}\,dx = \frac{1}{w}\int \frac{\sqrt{1-t^2}}{t}\,dt = \frac{1}{w}\int (\csc y - \sin y)\,dy = \log|\csc y - \cot y| + \cos y + C. \tag{3.19}$$

Thus combining (3.18) and (3.19), one gets

$$\int \frac{wx\log x}{\sqrt{1-w^2x^2}}\,dx = -\frac{\sqrt{1-w^2x^2}\log x}{w} + \frac{1}{w}\left(\log\left|\frac{1-\sqrt{1-w^2x^2}}{wx}\right| + \sqrt{1-w^2x^2}\right) + C. \tag{3.20}$$

Gathering (3.16), (3.17) and (3.20) and using the fact that w is positive, the formula

$$\int \log x \arcsin wx\,dx = (x\log x - x)\arcsin wx - \frac{2\sqrt{1-w^2x^2}}{w} + \frac{\sqrt{1-w^2x^2}\log x}{w} - \frac{1}{w}\left(\log\left(1-\sqrt{1-w^2x^2}\right) - \log wx\right) + C \tag{3.21}$$

follows. Thus taking the limit of the antiderivative as we approach to zero from right, one may deduce from (3.21) that

$$\int_0^1 \log x\ \arcsin wx\,dx = -\arcsin w - \frac{2\sqrt{1-w^2}}{w} + \frac{\log(1+\sqrt{1-w^2})}{w} + \frac{2-\log 2}{w} \tag{3.22}$$

for $0 < w < 1$. From (3.15) and (3.22), we get

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{w^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2} = \arcsin w - \frac{1}{w}\left(2-\log 2 - 2\sqrt{1-w^2} + \log\left(1+\sqrt{1-w^2}\right)\right) \tag{3.23}$$

for $0 < w < 1$. Since both sides of (3.23) are odd functions, (3.23) holds for $-1 < w < 0$ as well. Moreover, using continuity of both sides at $w = \pm 1$ and the removable singularity of right-hand side at $w = 0$ with limiting value zero as w tends to zero, (3.23) holds for $-1 \leqslant w \leqslant 1$. Using analytic continuation and continuity arguments on the boundary of the unit circle, we can see as above that (3.23) holds for all complex numbers w with $|w| \leqslant 1$. Putting $w = \sin z$ gives $|\sin z| \leqslant 1$ and $|\Re(z)| \leqslant \frac{\pi}{2}$ and the desired formula

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(\sin z)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2} = z - \frac{1}{\sin z}\left(2-\log 2 - 2\cos z + \log(1+\cos z)\right)$$

follows at once for any $z \in \mathfrak{F}_2$, where the principal branch of the logarithm is used. It is known that (see [22])

$$\frac{wx\arcsin(\frac{wx}{2})}{\sqrt{1-\frac{w^2x^2}{4}}} = \sum_{m=1}^{\infty}\frac{w^{2m}x^{2m}}{\binom{2m}{m}m} \tag{3.24}$$

for $|wx| < 2$. Using Theorem 3, (3.24) gives

$$\int_0^1 \log x \frac{wx\arcsin(\frac{wx}{2})}{\sqrt{1-\frac{w^2x^2}{4}}}\,dx = -\sum_{n=1}^{\infty}\frac{w^{2n}}{\binom{2n}{n}n(2n+1)^2} \tag{3.25}$$

for any complex number w with $|w| < 2$. The definite integral on the left-hand side of (3.25) can be handled similarly as above and one obtains that

$$\sum_{n=1}^{\infty}\frac{w^{2n}}{\binom{2n}{n}n(2n+1)^2} = -\int_0^1 \log x\frac{wx\arcsin(\frac{wx}{2})}{\sqrt{1-\frac{w^2x^2}{4}}}\,dx = 4 - \frac{4}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right) - \frac{8}{w}\int_0^{g(w)}\frac{\arctan v}{v}\,dv \tag{3.26}$$

for $-2 \leqslant w \leqslant 2$. By similar analytic continuation and continuity arguments, (3.26) holds for all complex numbers w with $|w| \leqslant 2$. Finally, putting $w = 2\sin 2z$, the desired formula

$$\sum_{n=1}^{\infty}\frac{(2\sin 2z)^{2n}}{\binom{2n}{n}n(2n+1)^2} = 4 - 4z\cot 2z - \frac{4}{\sin 2z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2}$$

holds for any $z \in \mathfrak{F}_1$. Batır showed that [11]

$$2\left(\arcsin\left(\frac{wx}{2}\right)\right)^2+2\pi\arcsin\left(\frac{wx}{2}\right)=\sum_{m=1}^{\infty}\frac{\Gamma(\frac{m}{2})^2w^mx^m}{m!} \tag{3.27}$$

for $|wx|<2$. Using now (3.27) and Theorem 3, we see that

$$\int_0^1\log x\left(2\left(\arcsin\left(\frac{wx}{2}\right)\right)^2+2\pi\arcsin\left(\frac{wx}{2}\right)\right)dx=-\sum_{n=1}^{\infty}\frac{\Gamma(\frac{n}{2})^2w^n}{n!(n+1)^2} \tag{3.28}$$

for any complex number w with $|w|<2$. From (3.9) we know that

$$2\int_0^1\log x\left(\arcsin\left(\frac{wx}{2}\right)\right)^2dx$$
$$=12-2\left(\arcsin\left(\frac{w}{2}\right)\right)^2-\frac{16}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right)-\frac{16}{w}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} \tag{3.29}$$

for $-2<w<2$. Replacing w with $\frac{w}{2}$ in (3.22), one further obtains

$$2\pi\int_0^1\log x\arcsin\left(\frac{wx}{2}\right)dx$$
$$=2\pi\left(-\arcsin\left(\frac{w}{2}\right)+\frac{2}{w}\left(2-\log 2-2\sqrt{1-\frac{w^2}{4}}+\log\left(1+\sqrt{1-\frac{w^2}{4}}\right)\right)\right) \tag{3.30}$$

for $-2<w<2$. Gathering (3.28)–(3.30), the formula

$$\sum_{n=1}^{\infty}\frac{\Gamma(\frac{n}{2})^2w^n}{n!(n+1)^2}=-12+2\left(\arcsin\left(\frac{w}{2}\right)\right)^2+2\pi\arcsin\left(\frac{w}{2}\right)$$
$$+\frac{16}{w}\left(\sqrt{1-\frac{w^2}{4}}\right)\arcsin\left(\frac{w}{2}\right)-\frac{4\pi}{w}\left(2-\log 2-2\sqrt{1-\frac{w^2}{4}}+\log\left(1+\sqrt{1-\frac{w^2}{4}}\right)\right)$$
$$+\frac{16}{w}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(g(w))^{2n-1}}{(2n-1)^2} \tag{3.31}$$

follows when $-2<w<2$. By similar analytic continuation and continuity arguments, we can show that (3.31) is valid for any complex number w with $|w|\leqslant 2$. To this end, let us see that the series on the left-hand side of (3.31) is analytic for $|w|<2$ and continuous for $|w|\leqslant 2$. It suffices to separate even and odd values of n. For even values of n, the corresponding series

$$\sum_{n=1}^{\infty}\frac{\Gamma(n)^2w^{2n}}{(2n)!(2n+1)^2}=\sum_{n=1}^{\infty}\frac{w^{2n}}{\binom{2n}{n}n^2(2n+1)^2}$$

is clearly analytic for $|w|<2$ and continuous for $|w|\leqslant 2$. For odd values of n, using the well-known formula

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!},$$

the corresponding series

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})^2w^{2n+1}}{(2n+1)!(2n+2)^2}=\pi\sum_{n=0}^{\infty}\binom{2n}{n}\frac{w^{2n+1}}{2^{4n}(2n+1)(2n+2)^2}$$

is easily seen to be analytic for $|w|<2$ and continuous for $|w|\leqslant 2$. Using the change of variable $w=2\sin 2z$ in (3.31), the desired formula

$$\sum_{n=1}^{\infty}\frac{\Gamma(\frac{n}{2})^2(2\sin 2z)^n}{n!(n+1)^2}=-12+4\pi z+8z^2+16z\cot 2z-\frac{2\pi}{\sin 2z}\big(2-\log 2-2\cos 2z+\log(1+\cos 2z)\big)$$
$$+\frac{8}{\sin 2z}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\tan z)^{2n-1}}{(2n-1)^2}$$

follows for any $z \in \mathfrak{F}_1$, where the principal branch of the logarithm is used. To prove the remaining formula, first by partial integration we have

$$\int \log x \frac{d}{dx}\left(\sqrt{1+w^2x^2}\right) dx = \sqrt{1+w^2x^2} \log x - \int \frac{\sqrt{1+w^2x^2}}{x}\, dx. \tag{3.32}$$

Using the change of variables $t = wx$, $t = \tan y$, $u = \cos y$ and then using partial fractions, we see that

$$\begin{aligned}\int \frac{\sqrt{1+w^2x^2}}{x}\, dx &= \int \frac{\sqrt{1+t^2}}{t}\, dt = \int \frac{1}{\cos^2 y \sin y}\, dy \\ &= \int \frac{1}{u^2(u^2-1)}\, du = \frac{1}{u} + \frac{1}{2}\log|u-1| - \frac{1}{2}\log|u+1| + C.\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33), the evaluation

$$\begin{aligned}\int_0^1 \log x \frac{d}{dx}\left(\sqrt{1+w^2x^2}\right) dx &= 1 - \sqrt{1+w^2} - \frac{1}{2}\log\left(\sqrt{1+w^2}-1\right) + \frac{1}{2}\log\left(\sqrt{1+w^2}+1\right) \\ &\quad - \frac{\log 2}{2} - \lim_{x\to 0+}\left(\sqrt{1+w^2x^2}\log x - \frac{1}{2}\log\left(\sqrt{1+w^2x^2}-1\right)\right) \\ &= 1 - \sqrt{1+w^2} - \log 2 + \log\left(\sqrt{1+w^2}+1\right)\end{aligned} \tag{3.34}$$

follows for any real number w. Next by the binomial series, one has

$$\sqrt{1+w^2x^2} = \sum_{m=0}^{\infty} \binom{1/2}{m} w^{2m} x^{2m} \tag{3.35}$$

for $|wx| < 1$. Differentiating termwise, (3.35) gives

$$\frac{d}{dx}\left(\sqrt{1+w^2x^2}\right) = \sum_{m=1}^{\infty} 2m \binom{1/2}{m} w^{2m} x^{2m-1}. \tag{3.36}$$

Using now Theorem 3 together with (3.34), (3.36) and the easily checked formula

$$\binom{1/2}{m} = \frac{(-1)^{m-1}}{2^{2m}(2m-1)} \binom{2m}{m},$$

for $m \geqslant 1$, one may deduce that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \binom{2n}{n} \frac{w^{2n}}{2^{2n+1} n(2n-1)} = -1 + \log 2 + \sqrt{1+w^2} - \log\left(\sqrt{1+w^2}+1\right) \tag{3.37}$$

for $-1 < w < 1$. Both sides of (3.37) are analytic functions for $|w| < 1$, where the principal branches of the logarithm and the square-root are used. Therefore, (3.37) holds for any complex number w with $|w| < 1$. Moreover, the series on the left-hand side of (3.37) is continuous for $|w| \leqslant 1$ and both of $\sqrt{1+w^2}$ and $\log(\sqrt{1+w^2}+1)$ are also continuous for $|w| \leqslant 1$ except possibly the points $w = \pm i$, where the branch cuts of these functions begin. Consequently, (3.37) holds for $|w| \leqslant 1$ except possibly the points $w = \pm i$. But using

$$\int \log x \frac{d}{dx}\left(\sqrt{1-w^2x^2}\right) dx = \sqrt{1-w^2x^2} \log x - \int \frac{\sqrt{1-w^2x^2}}{x}\, dx$$

and

$$\frac{d}{dx}\left(\sqrt{1-w^2x^2}\right) = \sum_{m=1}^{\infty} (-1)^m 2m \binom{1/2}{m} w^{2m} x^{2m-1}$$

for $|wx| < 1$, we may similarly derive that

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{w^{2n}}{2^{2n+1} n(2n-1)} = 1 - \log 2 - \sqrt{1-w^2} + \log\left(\sqrt{1-w^2}+1\right) \tag{3.38}$$

for $-1 < w < 1$. Continuity of both sides of (3.38) shows that (3.38) holds for $w = \pm 1$. Thus (3.37) holds for $w = \pm i$ as well. Using the change of variable $w = \tan z$, we may consider the conditions $|\tan z| \leqslant 1$ and $|\Re(z)| \leqslant \frac{\pi}{2}$ which reduce to the single condition $|\Re(z)| \leqslant \frac{\pi}{4}$. Therefore, the desired formula

$$\sum_{n=1}^{\infty}(-1)^{n-1}\binom{2n}{n}\frac{(\tan z)^{2n}}{2^{2n+1}n(2n-1)} = -1 + \log 2 + \sec z - \log(1+\sec z)$$

holds for any $z \in \mathfrak{F}_3$, where the principal branch of the logarithm is used. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let us begin by observing that for any integer $r \geqslant 2$, $0 < z < 2\pi$ and $0 < a \leqslant 1$, the function

$$\frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}\log x}{e^{zx}-1}\right)$$

is integrable on $[0,1]$ since $x^{r-1}\log x$ is integrable on $[0,1]$. It follows that

$$\int_0^1 \frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}\log x}{e^{zx}-1}\right) dx = 0. \tag{4.1}$$

Rewriting (4.1), we see that

$$\int_0^1 \log x \frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1}\right) dx = -\int_0^1 \frac{z^r x^{r-1} e^{(1-a)zx}}{e^{zx}-1}\, dx. \tag{4.2}$$

Note that for $0 < x \leqslant 1$,

$$\frac{1}{e^{zx}-1} = \frac{e^{-zx}}{1-e^{-zx}} = \sum_{n=0}^{\infty} e^{-(n+1)zx} \tag{4.3}$$

and we may treat the integral on the right-hand side of (4.2) as a Lebesgue integral. Since by (4.3), the series $\sum_{n=0}^{\infty} e^{-(n+1)zx}$ converges almost everywhere on $[0,1]$ to the function $\frac{1}{e^{zx}-1}$, one has

$$\int_0^1 \frac{z^r x^{r-1} e^{(1-a)zx}}{e^{zx}-1}\, dx = \int_0^1 z^r x^{r-1}\left(\sum_{n=0}^{\infty} e^{-(n+a)zx}\right) dx. \tag{4.4}$$

Since the integrand on the right-hand side of (4.4) is nonnegative, we may use Levi's monotone convergence theorem to interchange the order of summation with the integral so that the right-hand side of (4.4) becomes

$$\sum_{n=0}^{\infty} z^r \int_0^1 x^{r-1} e^{-(n+a)zx}\, dx. \tag{4.5}$$

Integrating by parts repeatedly and using (4.2), (4.4) and (4.5), we have

$$\int_0^1 \log x \frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1}\right) dx = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{(r-1)!z^k}{k!(n+a)^{r-k}}\right)e^{-(n+a)z} - \sum_{n=0}^{\infty}\frac{(r-1)!}{(n+a)^r}. \tag{4.6}$$

Recall that the Hurwitz zeta function is defined as

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$$

for $\Re(s) > 1$ and $0 < a \leqslant 1$. Rearranging (4.6) in terms of the Hurwitz zeta function, one obtains

$$\zeta(r,a) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+a)^{r-k}}\right)e^{-(n+a)z} - \frac{1}{(r-1)!}\int_0^1 \log x \frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1}\right) dx. \tag{4.7}$$

Moreover, the Taylor series expansion

$$\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1} = (zx)^{r-1}\sum_{m=0}^{\infty}\frac{B_m(1-a)}{m!}z^m x^m = \sum_{m=0}^{\infty}\frac{B_m(1-a)}{m!}z^{m+r-1}x^{m+r-1} \tag{4.8}$$

is valid for $0 \leqslant zx < 2\pi$. Differentiating (4.8) termwise, one gets

$$\frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1}\right) = \sum_{m=0}^{\infty}\frac{(m+r-1)B_m(1-a)}{m!}z^{m+r-1}x^{m+r-2} = z\sum_{m=r-2}^{\infty}\frac{(m+1)B_{m-r+2}(1-a)}{(m-r+2)!}z^m x^m \tag{4.9}$$

for $0 \leqslant zx < 2\pi$. Integrating termwise by uniform convergence and using (4.9), we see that

$$\int_0^1 \log x\,\frac{d}{dx}\left(\frac{(zx)^r e^{(1-a)zx}}{e^{zx}-1}\right)dx = -\sum_{m=r-2}^{\infty}\frac{B_{m-r+2}(1-a)}{(m+1)(m-r+2)!}z^{m+1}. \tag{4.10}$$

Combining (4.7) and (4.10) now gives

$$\zeta(r,a) = \sum_{n=0}^{\infty}\left(\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+a)^{r-k}}\right)e^{-(n+a)z} + \frac{B_n(1-a)z^{n+r-1}}{(r-1)!n!(n+r-1)}\right) \tag{4.11}$$

for $r \geqslant 2$ and $0 < a \leqslant 1$. Using the well-known decomposition of L-functions as a linear combination of Hurwitz zeta functions, we may write

$$L(r,\chi) = \frac{1}{q^r}\sum_{v=1}^{q}\chi(v)\zeta\left(r,\frac{v}{q}\right). \tag{4.12}$$

Combining (4.11), (4.12) and taking $a = \frac{v}{q}$ each time, the desired series representation follows. One can directly check that this representation is also valid for $z = 0$, since

$$L(r,\chi) = \frac{1}{q^r}\sum_{n=0}^{\infty}\left(\sum_{v=1}^{q}\frac{\chi(v)}{(n+\frac{v}{q})^r}\right)$$

holds. By known bounds on the size of Bernoulli polynomials, we have

$$\left|B_n\left(1-\frac{v}{q}\right)\right| \leqslant 2^{1-n}\pi^{-n}\zeta(n)n!$$

for $n \geqslant 2$, so that

$$\left|\sum_{v=1}^{q}B_n\left(1-\frac{v}{q}\right)\right| \ll_q \frac{n!}{(2\pi)^n}$$

follows. Consequently, both the series

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\left(\sum_{v=1}^{q}\frac{\chi(v)e^{-\frac{vz}{q}}}{(n+\frac{v}{q})^{r-k}}\right)\frac{z^k}{k!}\right)e^{-nz}, \qquad \sum_{n=0}^{\infty}\left(\sum_{v=1}^{q}B_n\left(1-\frac{v}{q}\right)\right)\frac{z^{n+r-1}}{(r-1)!n!(n+r-1)}$$

converge geometrically with rates e^{-z} and $\frac{z}{2\pi}$. Thus, if λ_0 is the unique positive solution of $\lambda e^{\lambda} = 2\pi$, then the geometric rate of convergence of the sum of these series is at most $e^{-\lambda_0}$ and by an easy numerical approximation to λ_0, we see that $e^{-\lambda_0} < \frac{1}{4}$. In conclusion, there exists such a series representation of $L(r,\chi)$ with geometric rate of convergence $\rho < \frac{1}{4}$. To prove the other claims, note that for $r \geqslant 3$, $0 < z < \pi$ and $0 < a \leqslant 1$, the function

$$\frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}\log x}{(e^{zx}-1)(e^{2zx}-1)}\right)$$

is integrable on $[0,1]$ so that

$$\int_0^1 \frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}\log x}{(e^{zx}-1)(e^{2zx}-1)}\right)dx = 0 \tag{4.13}$$

follows. As a result of (4.13), one obtains

$$\int_0^1 \log x \frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}\right)dx = -\int_0^1 \frac{z^r x^{r-1} e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}\,dx$$
$$= -\frac{1}{2}\int_0^1 \frac{z^r x^{r-1} e^{2(1-a)zx}}{(e^{zx}-1)^2}\,dx + \frac{1}{2}\int_0^1 \frac{z^r x^{r-1} e^{2(1-a)zx}}{e^{2zx}-1}\,dx. \tag{4.14}$$

Next we have

$$\frac{e^{2(1-a)zx}}{e^{2zx}-1} = \frac{e^{-2azx}}{1-e^{-2zx}} = \sum_{n=0}^{\infty} e^{-2(n+a)zx} \tag{4.15}$$

when $0 < x \leqslant 1$. Using (4.15) and a similar argument as above, we may deduce that

$$\frac{1}{2}\int_0^1 \frac{z^r x^{r-1} e^{2(1-a)zx}}{e^{2zx}-1}\,dx = \frac{(r-1)!}{2^{r+1}}\zeta(r,a) - (r-1)!\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+a)^{r-k}}\right)e^{-2(n+a)z}. \tag{4.16}$$

Similarly using

$$\frac{e^{2(1-a)zx}}{(e^{zx}-1)^2} = \sum_{n=1}^{\infty} n e^{-(n+2a-1)zx}$$

for $0 < x \leqslant 1$, we see that

$$-\frac{1}{2}\int_0^1 \frac{z^r x^{r-1} e^{2(1-a)zx}}{(e^{zx}-1)^2}\,dx = \frac{(r-1)!}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2a)^{r-k}}\right)(n+1)e^{-(n+2a)z} - \frac{(r-1)!}{2}\sum_{n=0}^{\infty}\frac{n+1}{(n+2a)^r}. \tag{4.17}$$

From (4.14), (4.15) and (4.17), one has

$$\frac{1}{(r-1)!}\int_0^1 \log x \frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}\right)dx$$
$$= \frac{\zeta(r,a)}{2^{r+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{n+1}{(n+2a)^r} - \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+a)^{r-k}}\right)e^{-2(n+a)z}$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2a)^{r-k}}\right)(n+1)e^{-(n+2a)z}. \tag{4.18}$$

Next we find the Taylor series expansion of the function

$$\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}$$

about zero. To this end, recall that

$$\frac{zx}{e^{zx}-1} = \sum_{j=0}^{\infty}\frac{B_j}{j!}z^j x^j$$

when $|zx| < 2\pi$ and

$$\frac{2zxe^{2(1-a)zx}}{e^{2zx}-1} = \sum_{s=0}^{\infty}\frac{B_s(1-a)}{s!}2^s z^s x^s$$

when $|zx| < \pi$. Multiplying these series, one obtains

$$\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)} = (zx)^{r-2}\left(\sum_{j=0}^{\infty}\frac{B_j}{j!}z^j x^j\right)\left(\sum_{s=0}^{\infty}\frac{2^{s-1}B_s(1-a)}{s!}z^s x^s\right)$$
$$= \sum_{m=0}^{\infty}\left(\sum_{j+s=m}\frac{2^{s-1}B_j B_s(1-a)}{j!s!}\right)z^{m+r-2}x^{m+r-2} \tag{4.19}$$

for $|zx| < \pi$. From (4.19) we get

$$\frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}\right) = z\sum_{m=0}^{\infty}\left(\sum_{j+s=m}\frac{2^{s-1}B_j B_s(1-a)}{j!s!}\right)(m+r-2)z^{m+r-3}x^{m+r-3}$$
$$= z\sum_{m=r-3}^{\infty}\left(\sum_{j+s=m-r+3}\frac{2^{s-1}B_j B_s(1-a)}{j!s!}\right)(m+1)z^m x^m \tag{4.20}$$

for $|zx| < \pi$. Using (4.20), we may integrate termwise to get

$$\int_0^1 \log x\,\frac{d}{dx}\left(\frac{(zx)^r e^{2(1-a)zx}}{(e^{zx}-1)(e^{2zx}-1)}\right)dx = -\sum_{n=0}^{\infty}\left(\sum_{j+s=n}\frac{2^{s-1}B_j B_s(1-a)}{j!s!}\right)\frac{z^{n+r-2}}{(n+r-2)}.$$

Combining this with (4.18), we finally get the desired representation

$$\frac{\zeta(r,a)}{2^{r+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{n+1}{(n+2a)^r} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+a)^{r-k}}\right)e^{-2(n+a)z} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2a)^{r-k}}\right)(n+1)e^{-(n+2a)z}$$
$$- \frac{1}{(r-1)!}\sum_{n=0}^{\infty}\left(\sum_{j+s=n}\frac{2^{s-1}B_j B_s(1-a)}{j!s!}\right)\frac{z^{n+r-2}}{(n+r-2)} \tag{4.21}$$

for $0 < a \leqslant 1$ and $0 \leqslant z < \pi$, since (4.21) is trivial to check when $z = 0$. Note that when $a = 1$, we have $B_s(1-a) = B_s(0) = B_s$ and (4.21) gives

$$\left(\frac{1}{2}+\frac{1}{2^{r+1}}\right)\zeta(r) - \frac{1}{2}\zeta(r-1)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{2^{r-k+1}k!(n+1)^{r-k}}\right)e^{-2(n+1)z} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+2)^{r-k}}\right)(n+1)e^{-(n+2)z}$$
$$- \frac{1}{(r-1)!}\sum_{n=0}^{\infty}\left(\sum_{j+s=n}\frac{2^{s-1}B_j B_s}{j!s!}\right)\frac{z^{n+r-2}}{(n+r-2)} \tag{4.22}$$

for any integer $r \geqslant 3$ and $0 \leqslant z < \pi$. Moreover, when $a = \frac{1}{2}$, we have $B_s(1-a) = B_s(\frac{1}{2}) = -(1-2^{1-s})B_s$ and (4.21) gives

$$\left(\frac{1}{2}-\frac{1}{2^{r+1}}\right)\zeta(r) - \frac{1}{2}\zeta(r-1)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(2n+1)^{r-k}}\right)e^{-(2n+1)z} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r-1}\frac{z^k}{k!(n+1)^{r-k}}\right)(n+1)e^{-(n+1)z}$$
$$+ \frac{1}{(r-1)!}\sum_{n=0}^{\infty}\left(\sum_{j+s=n}\frac{(2^{s-1}-1)B_j B_s}{j!s!}\right)\frac{z^{n+r-2}}{(n+r-2)} \tag{4.23}$$

for any integer $r \geqslant 3$ and $0 \leqslant z < \pi$. Adding (4.22) and (4.23), one reaches to the desired representation of $\zeta(r) - \zeta(r-1)$ for $r \geqslant 3$. This completes the proof of Theorem 2.

5. Proof of Corollary 1

To obtain the first three representations of G, we simply take $z = \frac{\pi}{4}$ in the first three formulas of Theorem 1. To obtain the last representation of G, we apply Theorem 2, where $q = 4$, $r = 2$ and $\chi = \chi_{-1}$ is the odd character modulo 4. Moreover, for any $z \in \mathfrak{F}_2$, we know from Theorem 1 that

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(\sin z)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2}=z-\frac{1}{\sin z}\big(2-\log 2-2\cos z+\log(1+\cos z)\big).$$

First taking $z=\frac{\pi}{2}$ in this formula, we have one of the desired evaluations. Note that (x,y) belongs to $\mathfrak{F}_2$ if and only if $e^{2y}+e^{-2y}-2\cos 2x\leqslant 4$ and $|x|\leqslant\frac{\pi}{2}$. Taking $z=iy$ in this region, we see that y satisfies $\cosh 2y\leqslant 3$. Using $\sin iy=i\sinh y$ and $\cos iy=\cosh y$, we easily get

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{(\sinh y)^{2n+1}}{2^{2n+2}(2n+1)(n+1)^2}=y+\frac{1}{\sinh y}\big(2-\log 2-2\cosh y+\log(1+\cosh y)\big)$$

when $\cosh 2y\leqslant 3$. The remaining evaluations can be derived similarly by using (3.37), $\tan iy=i\tanh y$ and noting that there is no condition on y since $\mathfrak{F}_3$ is a vertical strip.

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