NEIGHBOURHOODS OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH PARABOLA

ROSIHAN M. ALI, K. G. SUBRAMANIAN, V. RAVICHANDRAN, AND OM P. AHUJA

Abstract. Let f be a normalized analytic function defined on the unit disk and $f_\lambda(z) := (1-\lambda)z + \lambda f(z)$ for $0 < \lambda \leq 1$. For $\alpha > 0$, a function $f \in \mathcal{SP}(\alpha, \lambda)$ if $zf'(z)/f_\lambda(z)$ lies in the parabolic region $\Omega := \{w : |w - \alpha| < Re\ w + \alpha\}$. Let $\mathcal{CP}(\alpha, \lambda)$ be the corresponding class consisting of functions f such that $(zf'(z))'/f'_\lambda(z)$ lies in the region Ω. For an appropriate $\delta > 0$, the δ-neighbourhood of a function $f \in \mathcal{CP}(\alpha, \lambda)$ is shown to consist of functions in the class $\mathcal{SP}(\alpha, \lambda)$.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ defined on the open unit disk $\Delta := \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{ST}$ and $\mathcal{CV}$ be the well-known subclasses of $\mathcal{S}$ respectively consisting of starlike and convex functions. Given $\delta \geq 0$, Ruscheweyh [24] defined the *δ-neighbourhood* $N_\delta(f)$ of a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$$

to be the set

$$N_\delta(f) := \left\{ g(z) : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta \right\}.$$

Ruscheweyh [24] proved among other results that $N_{1/4}(f) \subset \mathcal{ST}$ for $f \in \mathcal{CV}$. Sheil-Small and Silvia [28] introduced more general notions of neighbourhood of an analytic function. These included non-coefficient neighbourhoods as well. Problems related to the neighbourhoods of analytic functions were considered by many others, for example, see [1, 2, 3, 4, 10, 11, 12, 17, 18, 31].

An analytic function $f(z) \in \mathcal{S}$ is *uniformly convex* [8] if for every circular arc γ contained in $\triangle$ with center $\zeta \in \triangle$, the image arc $f(\gamma)$ is convex. Denote the class of all uniformly

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.

Key words and phrases. Analytic functions, univalent functions, parabolic starlike functions, parabolic convex function, neighbourhoods.

The first two authors acknowledge support from the USM's RU grant, and the fourth author acknowledges support from the University Research Council of Kent State University and Universiti Sains Malaysia. This work was completed while the third and fourth authors were visiting USM.

convex functions by $\mathcal{UCV}$. In [13, 20], it was shown that a function $f(z)$ is uniformly convex if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\Delta).$$

The class $\mathcal{S}_p$ of functions $zf'(z)$ with $f(z)$ in $\mathcal{UCV}$ was introduced in [20] and clearly $f(z)$ is in $\mathcal{S}_p$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)}-1\right| \quad (z\in\Delta).$$

The class $\mathcal{UCV}$ of uniformly convex functions and the class $\mathcal{S}_p$ of parabolic starlike functions were investigated in [7, 19, 22, 26, 27]. A survey of these functions can be found in [21].

Let $\alpha>0$ and $0<\lambda\leq 1$. The class $\mathcal{SP}(\alpha,\lambda)$ consists of functions $f\in\mathcal{S}$ satisfying

$$Re\left\{\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}\right\}+\alpha > \left|\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-\alpha\right| \quad (z\in\Delta). \tag{1.1}$$

By writing $f_\lambda(z):=(1-\lambda)z+\lambda f(z)$, the inequality in (1.1) can be written as

$$Re\left\{\frac{zf'(z)}{f_\lambda(z)}\right\}+\alpha > \left|\frac{zf'(z)}{f_\lambda(z)}-\alpha\right|.$$

Observe that (1.1) defines a parabolic region. More explicitly, $f\in\mathcal{SP}(\alpha,\lambda)$ if and only if the values of the functional $zf'(z)/f_\lambda(z)$ lie in the parabolic region Ω where

$$\Omega:=\{w:|w-\alpha|<Re\ w+\alpha\}=\{w=u+iv: v^2<4\alpha u\}.$$

The geometric properties of the function f_λ when f belongs to certain classes of starlike and convex functions were investigated by several authors [5, 6, 9, 16, 23, 30]; in particular, we recall the following result:

Theorem 1.1. [16] *Let $f\in\mathcal{CV}$. Then*

(1) $f_\lambda(z):=(1-\lambda)z+\lambda f(z)\in\mathcal{ST}$ *if and only if* $\lambda\in\mathbb{C}$ *and* $|\lambda-1|\leq 1/3$;
(2) *if* $f''(0)=0$, *then* $f_\lambda\in\mathcal{ST}$ *for* $\lambda\in[0,1]$.

For $\alpha>0$ and $0<\lambda\leq 1$, the class $\mathcal{CP}(\alpha,\lambda)$ consists of functions $f\in S$ satisfying

$$Re\left\{\frac{(zf'(z))'}{f'_\lambda(z)}\right\}+\alpha > \left|\frac{(zf'(z))'}{f'_\lambda(z)}-\alpha\right| \quad (z\in\Delta).$$

When $\lambda=1$, the classes $\mathcal{SP}(\alpha,\lambda)$ and $\mathcal{CP}(\alpha,\lambda)$ reduce respectively to the classes introduced in [29] and [33]. Besides several other properties, the authors in [29] and [33] also gave geometric interpretations, respectively, of the classes $\mathcal{SP}(\alpha):=\mathcal{SP}(\alpha,1)$ and $\mathcal{CP}(\alpha):=\mathcal{CP}(\alpha,1)$.

In this paper, the neighbourhood $N_\delta(f)$ for functions $f\in\mathcal{CP}(\alpha,\lambda)$ is investigated. It is shown that all functions $g\in N_\delta(f)$ are in the class $\mathcal{SP}(\alpha,\lambda)$ for a certain $\delta>0$. It is of interest to note that the conditions on δ obtained here coincide with those in [32] for corresponding results in the classes $\mathcal{CP}(\alpha)$ and $\mathcal{SP}(\alpha)$.

2. Main Results

In order to obtain the main results, a characterization of the class $\mathcal{SP}(\alpha,\lambda)$ in terms of the functions in another class $\mathcal{SP}'(\alpha,\lambda)$ is needed. For a fixed $\alpha>0$, $0<\lambda\leq 1$, and $t\geq 0$, a function $H_{t,\ \lambda}$ is said to be in the class $\mathcal{SP}'(\alpha,\lambda)$ if the function $H_{t,\ \lambda}$ is of the form

$$H_{t,\ \lambda}(z) := \frac{1}{1-(t\pm 2\sqrt{\alpha t}\ i)}\left[\frac{z}{(1-z)^2} - \frac{[z-(1-\lambda)z^2]}{1-z}(t\pm 2\sqrt{\alpha t}\ i)\right], \quad (z\in\Delta). \quad (2.1)$$

Recall that for any two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g*f)(z).$$

Lemma 2.1. *Let $\alpha>0$ and $0<\lambda\leq 1$. A function f is in the class $\mathcal{SP}(\alpha,\lambda)$ if and only if*

$$\frac{1}{z}(f*H_{t,\ \lambda})(z)\neq 0 \quad (z\in\Delta)$$

for all $H_{t,\ \lambda}\in\mathcal{SP}'(\alpha,\lambda)$.

Proof. Let $f\in\mathcal{SP}(\alpha,\lambda)$. Then the image of Δ under $w=zf'(z)/f_\lambda(z)$ lies in the parabolic region $\Omega(\alpha,\lambda)=\{w:|w-\alpha|<Re\ w+\alpha\}$ so that

$$\frac{zf'(z)}{f_\lambda(z)}\neq t\pm 2\sqrt{\alpha t}\ i, \quad (z\in\Delta, t\geq 0).$$

Thus $f\in\mathcal{SP}(\alpha,\lambda)$ if and only if

$$\frac{zf'(z)-[t\pm 2\sqrt{\alpha t}\ i]f_\lambda(z)}{z(1-[t\pm 2\sqrt{\alpha t}\ i])} \neq 0, \quad (z\in\Delta, t\geq 0), \quad (2.2)$$

or equivalently

$$\frac{1}{z}(f*H_{t,\ \lambda})(z)\neq 0, \quad (z\in\Delta, t\geq 0)$$

for all $H_{t,\ \lambda}\in\mathcal{SP}'(\alpha,\lambda)$. □

Lemma 2.2. *Let $\alpha>0$ and $0<\lambda\leq 1$. If*

$$H_{t,\ \lambda}(z) := z + \sum_{k=2}^{\infty} h_{k,\ \lambda}(t) z^k \in\mathcal{SP}'(\alpha,\lambda),$$

then

$$|h_{k,\lambda}(t)| \leq \begin{cases} \frac{k}{2\sqrt{\alpha(1-\alpha)}}, & 0 < \alpha < 1/2, \\ k, & \alpha \geq 1/2 \end{cases}$$

for all $t \geq 0$.

Proof. Writing $H_{t,\ \lambda}(z) = z + \sum_{k=2}^{\infty} h_{k,\ \lambda}(t) z^k$, and on comparing coefficients of z^k in (2.1), one obtains

$$h_{k,\ \lambda}(t) = \frac{k - \lambda(t \pm 2\sqrt{\alpha t}\ i)}{1 - (t \pm 2\sqrt{\alpha t}\ i)}.$$

Thus, for $t \geq 0$ and $0 < \lambda \leq 1$,

$$\begin{aligned} |h_{k,\ \lambda}(t)|^2 &= \left| \frac{k - \lambda(t \pm 2\sqrt{\alpha t}\ i)}{1 - (t \pm 2\sqrt{\alpha t}\ i)} \right|^2 \\ &= \frac{(k - \lambda t)^2 + 4\lambda^2 \alpha t}{(1-t)^2 + 4\alpha t} \\ &= \lambda^2 + \frac{(k-\lambda)(k + \lambda - 2\lambda t)}{(1-t)^2 + 4\alpha t} \\ &\leq \lambda^2 + \frac{(k^2 - \lambda^2)}{(1-t)^2 + 4\alpha t}. \end{aligned}$$

It is easy to see that

$$(1-t)^2 + 4\alpha t \geq \begin{cases} 4\alpha(1-\alpha), & 0 < \alpha < 1/2, \\ 1, & \alpha \geq 1/2\ . \end{cases}$$

Hence, for $0 < \alpha < 1/2$, and $0 < \lambda \leq 1$, we have

$$|h_{k,\ \lambda}(t)|^2 \leq \lambda^2 + \frac{(k^2 - \lambda^2)}{4\alpha(1-\alpha)} \leq \frac{k^2}{4\alpha(1-\alpha)},$$

and, for $\alpha \geq 1/2$,

$$|h_{k,\ \lambda}(t)|^2 \leq \lambda^2 + k^2 - \lambda^2 = k^2.$$

□

Lemma 2.3. *For each complex number* ϵ *and* $f \in \mathcal{A}$, *define the function* F_ϵ *by*

$$F_\epsilon(z) := \frac{f(z) + \epsilon z}{1 + \epsilon}. \tag{2.3}$$

Let $\alpha > 0$, $0 < \lambda \leq 1$, *and* $F_\epsilon \in \mathcal{SP}(\alpha,\ \lambda)$ *for* $|\epsilon| < \delta$ *for some* $\delta > 0$. *Then*

$$\left| \frac{1}{z}(f * H_{t,\ \lambda})(z) \right| \geq \delta, \quad (z \in \Delta)$$

for every $H_{t,\ \lambda} \in \mathcal{SP}'(\alpha,\ \lambda)$.

Proof. If $F_\epsilon \in \mathcal{SP}(\alpha, \lambda)$ for $|\epsilon| < \delta$, where $\delta > 0$ is fixed, then by Lemma 2.1, for all $H_{t,\ \lambda} \in \mathcal{SP}'(\alpha,\ \lambda)$, it follows that

$$\frac{1}{z}(F_\epsilon * H_{t,\ \lambda})(z) \neq 0, \quad (z \in \Delta)$$

or equivalently

$$\frac{(f * H_{t,\ \lambda})(z) + \epsilon z}{(1+\epsilon)z} \neq 0.$$

Since $|\epsilon| < \delta$, it easily follows that

$$\left|\frac{1}{z}(f * H_{t,\ \lambda})(z)\right| \geq \delta. \qquad \square$$

Theorem 2.1. *Let $\alpha > 0$ and $0 < \lambda \leq 1$. Let $f \in \mathcal{A}$ and $\delta > 0$. For a complex number ϵ with $|\epsilon| < \delta$, let the function F_ϵ, defined by (2.3), be in $\mathcal{SP}(\alpha,\ \lambda)$. Then $N_{\delta'}(f) \subset \mathcal{SP}(\alpha,\ \lambda)$, for*

$$\delta' := \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)}, & 0 < \alpha < 1/2, \\ \delta, & \alpha \geq 1/2. \end{cases}$$

Proof. Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta'}(f)$. For any $H_{t,\ \lambda} \in \mathcal{SP}'(\alpha,\ \lambda)$,

$$\begin{aligned} \left|\frac{1}{z}(g * H_{t,\ \lambda})(z)\right| &= \left|\frac{1}{z}(f * H_{t,\ \lambda})(z) + \frac{1}{z}((g-f) * H_{t,\ \lambda})(z)\right| \\ &\geq \left|\frac{1}{z}(f * H_{t,\ \lambda})(z)\right| - \left|\frac{1}{z}((g-f) * H_{t,\ \lambda})(z)\right|. \end{aligned}$$

Using Lemma 2.3, it follows that

$$\begin{aligned} \left|\frac{1}{z}(g * H_{t,\ \lambda})(z)\right| &\geq \delta - \left|\sum_{k=2}^{\infty} \frac{(b_k - a_k)h_{k,\ \lambda}(t)z^k}{z}\right| \\ &\geq \delta - \sum_{k=2}^{\infty} |b_k - a_k||h_{k,\ \lambda}(t)|. \end{aligned}$$

Using Lemma 2.2 and noting that $g \in N_{\delta'}(f)$, and whence $\sum_{k=2}^{\infty} k|b_k - a_k| < \delta'$, thus

$$\left|\frac{1}{z}(g * H_{t,\ \lambda})(z)\right| \geq \begin{cases} \delta - \frac{\delta'}{2\sqrt{\alpha(1-\alpha)}}, & 0 < \alpha < 1/2, \\ \delta - \delta', & \alpha \geq 1/2. \end{cases}$$

Therefore $|\frac{1}{z}(g * H_{t,\ \lambda})(z)| \neq 0$ in Δ for all $H_{t,\ \lambda} \in \mathcal{SP}(\alpha,\ \lambda)$ if

$$\delta' = \begin{cases} 2\delta\sqrt{\alpha(1-\alpha)}, & 0 < \alpha < 1/2, \\ \delta, & \alpha \geq 1/2. \end{cases}$$

By Lemma 2.1, this means that $g \in \mathcal{SP}(\alpha,\ \lambda)$. This proves that $N_{\delta'}(f) \subset \mathcal{SP}(\alpha,\ \lambda)$. $\square$

We need the following well-known result in [25] concerning convolution of functions.

Lemma 2.4. [25] *Let $f \in \mathcal{CV}$, $g \in \mathcal{ST}$. Then for any analytic function F defined on Δ, we have*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \subset \overline{co}F(\Delta), \quad (z \in \Delta)$$

where $\overline{co}$ stands for the closed convex hull.

Lemma 2.5. *If $f \in \mathcal{CV}$, $g \in \mathcal{SP}(\alpha, \lambda)$ and $g_\lambda \in \mathcal{ST}$, then $f * g \in \mathcal{SP}(\alpha,\ \lambda)$.*

Proof. The conclusion $f * g \in \mathcal{SP}(\alpha,\ \lambda)$ is a consequence of Lemma 2.4 on noting that

$$\frac{z(f(z) * g(z))'}{(f(z) * g(z))_\lambda} = \frac{f(z) * zg'(z)}{f(z) * g_\lambda(z)} = \frac{f(z) * g_\lambda(z)\frac{zg'(z)}{g_\lambda(z)}}{f(z) * g_\lambda(z)} \subset \overline{co}\left\{\frac{zg'(z)}{g_\lambda(z)} : z \in \Delta\right\}. \qquad \square$$

Theorem 2.2. *Let $\alpha > 0$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{CP}(\alpha,\ \lambda)$ and $f_\lambda \in \mathcal{CV}$, then the function F_ϵ defined by (2.3) belongs to $\mathcal{SP}(\alpha,\ \lambda)$ for $|\epsilon| < 1/4$.*

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{CP}(\alpha,\ \lambda)$. Then

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1 + \epsilon} = (f * h)(z)$$

where

$$h(z) := \frac{z - \frac{\epsilon}{1+\epsilon} z^2}{1 - z} = \frac{z - \rho z^2}{1 - z} \quad (z \in \Delta)$$

and $\rho := \epsilon/(1 + \epsilon)$. Note that

$$Re \frac{zh'(z)}{h(z)} \geq \frac{1}{2} - \frac{|\rho|}{1 - |\rho|} > 0 \quad (z \in \Delta)$$

if $|\rho| \leq 1/3$. This clearly holds for $|\epsilon| < 1/4$. Thus the function $h(z)$ is starlike for $|\epsilon| < 1/4$ and whence the function

$$\int_0^z \frac{h(t)}{t} dt = h(z) * \log \frac{1}{1 - z} \quad (z \in \Delta)$$

is in $\mathcal{CV}$. Since $f(z) \in \mathcal{CP}(\alpha,\ \lambda)$, the function $zf'(z) \in \mathcal{SP}(\alpha,\ \lambda)$. Also $f_\lambda(z) \in \mathcal{CV}$ implies that $(zf'(z))_\lambda \in \mathcal{ST}$. By Lemma 2.5,

$$F_\epsilon(z) = (f * h)(z) = zf'(z) * \left(h(z) * \log \frac{1}{1 - z}\right) \in \mathcal{SP}(\alpha,\ \lambda)$$

for $|\epsilon| < 1/4$. $\square$

Theorem 2.3. *Let $\alpha > 0$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{CP}(\alpha,\ \lambda)$ and $f_\lambda \in \mathcal{CV}$, then $N_{\delta'}(f) \subset \mathcal{SP}(\alpha,\ \lambda,)$ where*

$$\delta' := \begin{cases} \frac{1}{2}\sqrt{\alpha(1 - \alpha)}, & 0 < \alpha < 1/2, \\ 1/4, & \alpha \geq 1/2. \end{cases}$$

Proof. The result follows from Theorem 2.1 and Theorem 2.2 on taking $\delta = 1/4$ in Theorem 2.1. $\square$

Remark 2.1. It is interesting to note that the values of δ' in Theorem 2.1 and Theorem 2.3 are independent of λ. In fact, the conclusion of Theorem 2.1, Theorem 2.2, and Theorem 2.3 are the same as found in [33] for the subclasses $\mathcal{SP}(\alpha)$ and $\mathcal{CP}(\alpha)$.

To prove our next result, we need the following results.

Lemma 2.6. [15] *Let* Ω *be a set in the complex plane* $\mathbb{C}$ *and suppose that the mapping* $\Phi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ *satisfies* $\Phi(i\rho, \sigma; z) \notin \Omega$ *for* $z \in \Delta$*, and for all real* ρ*,* σ *such that* $\sigma \leq -n(1+\rho^2)/2$*. If the function* $p(z) = 1 + c_n z^n + \cdots$ *is analytic in* Δ *and* $\Phi(p(z), zp'(z); z) \in \Omega$ *for all* $z \in \Delta$*, then Re* $p(z) > 0$*.*

Lemma 2.7. *Let* $0 \leq \lambda \leq \frac{1}{3}$*. If* $p(z) = 1 + cz + \cdots$ *is analytic in* Δ *and*

$$Re \left\{ \frac{p(z) + zp'(z)}{(1-\lambda) + \lambda p(z)} \right\} > 0, \tag{2.4}$$

then Re $p(z) > 0$*.*

Proof. Let $\Omega := \{w : Re\ w > 0\}$ and

$$\psi(r, s) := \frac{r + s}{(1-\lambda) + \lambda r}.$$

Then the given inequality (2.4) can be written as $\psi(p(z), zp'(z); z) \in \Omega$. Since

$$Re\ \psi(i\rho, \sigma; z) = \frac{\lambda\rho^2 + \sigma(1-\lambda)}{(1-\lambda)^2 + \lambda^2\rho^2} \leq \frac{(3\lambda - 1)\rho^2 - (1-\lambda)}{2[(1-\lambda)^2 + \lambda^2\rho^2]} \leq 0$$

when $\rho \in \Re$ and $\sigma \leq -\frac{1+\rho^2}{2}$, the condition of Lemma 2.6 is satisfied. Thus $Re\ p(z) > 0$.

□

Theorem 2.4. *Let* $0 \leq \lambda \leq \frac{1}{3}$*. If* $f \in \mathcal{SP}(\alpha, \lambda)$*, then* $f_\lambda \in \mathcal{ST}$*.*

Proof. If $f \in \mathcal{SP}(\alpha, \lambda)$, then

$$Re \left\{ \frac{zf'(z)}{f_\lambda(z)} \right\} + \alpha > \left| \frac{zf'(z)}{f_\lambda(z)} - \alpha \right|$$

and hence

$$Re \frac{zf'(z)}{f_\lambda(z)} > 0. \tag{2.5}$$

Let the analytic function $p(z)$ be defined by

$$p(z) = \frac{f(z)}{z} \quad (z \in U).$$

Computations show that

$$Re \frac{p(z) + zp'(z)}{(1-\lambda) + \lambda p(z)} = Re \frac{zf'(z)}{f_\lambda(z)} > 0.$$

By Lemma 2.7, we see that $Re\ p(z) > 0$ or $Re\ \frac{f(z)}{z} > 0$ in U.

In view of (2.5), it follows from $Re\ \frac{f(z)}{z} > 0$ and

$$\frac{zf'_\lambda(z)}{f_\lambda(z)} = \frac{1-\lambda}{1 - \lambda + \lambda\frac{f(z)}{z}} + \lambda \frac{zf'(z)}{f_\lambda(z)}$$

that

$$Re\frac{zf'_\lambda(z)}{f_\lambda(z)} > 0,$$

or equivalently $f_\lambda \in \mathcal{ST}$. □

As an immediate consequence, we have

Corollary 2.1. *Let* $0 \leq \lambda \leq \frac{1}{3}$. *If* $f \in \mathcal{CP}(\alpha, \lambda)$, *then* $f_\lambda \in \mathcal{CV}$.

In view of this corollary, the statement that $f_\lambda \in \mathcal{CV}$ can be omitted from Theorem 2.2 and Theorem 2.3 if $0 \leq \lambda \leq 1/3$. Also clearly that $f \in \mathcal{CP}(\alpha, 1)$ implies $f_1 = f \in \mathcal{CV}$. Thus Theorem 2.3 reduces to the corresponding result in [32] for $\lambda = 1$.

References

[1] O. P. Ahuja, Hadamard products and neighbourhoods of spirallike functions, *Yokohama Math. J.* **40** (1993), no. 2, 95–103.

[2] O. P. Ahuja, M. Jahangiri, H. Silverman, E. M. Silvia, Perturbations of classes of functions and their related dual spaces, *J. Math. Anal. Appl.* **173** (1993), no. 2, 542–549.

[3] U. Bednarz, S. Kanas, Stability of the integral convolution of k-uniformly convex and k-starlike functions, *J. Appl. Anal.* **10** (2004), no. 1, 105115.

[4] U. Bednarz, Stability of the Hadamard product of k-uniformly convex and k-starlike functions in certain neighbourhood, *Demonstratio Math.* **38** (2005), no. 4, 837–845.

[5] P. N. Chichra, R. Singh, Convex sum of univalent functions. *J. Austral. Math. Soc.* **14** (1972), 503–507.

[6] P. N. Chichra, Convex sum of regular functions. *J. Indian Math. Soc. (N.S.)* **39** (1975), 299–304 (1976).

[7] A. Gangadharan, V. Ravichandran and T. N. Shanmugam, Radii of convexity and strong starlikeness for some classes of analytic functions, *J. Math. Anal. Appl.*, **211**(1997), 301–313.

[8] A. W. Goodman, On uniformly convex functions, *Annal. Polon. Math.*, **56**(1991), 87–92.

[9] R. S. Gupta, Radius of convexity of convex sum of univalent functions. *Publ. Math. Debrecen* **19** (1972), 39–42 (1973).

[10] R. Fournier, A note on neighbourhoods of univalent functions, *Proc. Amer. Math. Soc.* **87** (1983), no. 1, 117–120.

[11] R. Fournier, On neighbourhoods of univalent starlike functions, *Ann. Polon. Math.* **47** (1986), no. 2, 189–202.

[12] R. Fournier, On neighbourhoods of univalent convex functions, *Rocky Mountain J. Math.* **16** (1986), no. 3, 579–589.

[13] W. Ma and D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, **57**(1992), 165–175.

[14] W. Ma and D. Minda, A Unified treatment of some special classes of univalent functions, in: *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press (1994), 157–169.

[15] S. S. Miller, P. T. Mocanu, Differential subordinations and inequalities in the complex plane, *J. Differ. Equations*, **67** (1987), 199–211.

[16] E. P. Merkes, On the convex sum of certain univalent functions and the identity function, *Rev. Colombiana Mat.* **21** (1987), no. 1, 5–11.

[17] R. Parvatham, M. Premabai, On the neighbourhood of Pascu class of α-convex functions, *Yokohama Math. J.* **43** (1995), no. 2, 89–93.

[18] Q. I. Rahman, J. Stankiewicz, On the Hadamard products of schlicht functions, *Math. Nachr.* **106** (1982), 7–16.

[19] V. Ravichandran, F. Rønning and T. N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions, *Complex Variables*, **33**(1997), 265–280.

[20] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**(1993), 189–196.
[21] F. Rønning, A survey on uniformly convex and uniformly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska, Sect A*, **47**(1993), 123–134.
[22] F. Rønning, Some radius results for univalent functions, *J. Math. Anal. Appl.*, **194**(1995), 319–327.
[23] St. Ruscheweyh, K. J. Wirths, Convex sums of convex univalent functions, *Indian J. Pure Appl. Math.* **7** (1976), no. 1, 49–52.
[24] St. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.* **81** (1981), no. 4, 521–527.
[25] S. Ruscheweyh, *Convolutions in geometric function theory*, Presses Univ. Montréal, Montreal, Que., 1982.
[26] T. N. Shanmugam and V. Ravichandran, Certain properties of uniformly convex functions, In: *Computational Methods and Function Theory 1994*, (R. M. Ali, St. Ruscheweyh & E. B. Saff (Eds.)), World Scientific Publ. Co., Singapore, 1995, pp. 319–324.
[27] T. N. Shanmugam and V. Ravichandran, Radius problems for analytic functions, *Chinese J. Math.*, **23**(1995), 343–351.
[28] T. Sheil-Small, E. M. Silvia, Neighborhoods of analytic functions, *J. Analyse Math.* **52** (1989), 210–240.
[29] J. Sokół and A. Wiśniowska, On some classes of starlike functions related with parabola, *Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz.* **18** (1993), 35–42.
[30] S. Y. Trimble, The convex sum of convex functions. *Math. Z.* **109** 1969 112–114.
[31] J. B. Walker, Anote on neighborhoods of analytic functions having positive real part, *Internat. J. Math. Math. Sci.* **13** (1990), no. 3, 425429.
[32] A. Wiśniowska, Neighbourhoods of convex functions related with parabola, Demonstratio Math. **30** (1997), no. 1, 109–114.
[33] A. Wiśniowska, On convex functions related with parabola, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* **18** (1995), 49–55.

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

E-mail address: rosihan@cs.usm.my

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

E-mail address: kgs@usm.my

Department of Mathematics, university of Delhi, Delhi 110 007, India

E-mail address: vravi@maths.du.ac.in

Department of Mathematics, Kent State University Burton, Ohio, 44021-9500 USA

E-mail address: oahuja@kent.edu