# Uniformly Convex and Uniformly Starlike Functions

Rosihan M. Ali and V. Ravichandran

**Abstract**

A normalized univalent function is uniformly convex if it maps every circular arc contained in the open unit disk with center in it into a convex curve. This article surveys recent results on the class of uniformly convex functions and on an analogous class of uniformly starlike functions.

## 1. Introduction

One of the cornerstones in geometric function theory is the proof of the coefficient conjecture of Bieberbach (1916) by Louis de Branges [**13**] in the year 1985. The conjecture asserts that the coefficient of a univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfies $|a_n| \leq n$ with strict inequality unless $f$ is a rotation of the Koebe function

$$k(z) = \frac{z}{(1-z)^2}.$$

In fact, de Branges proved the Milin conjecture (1971) on logarithmic coefficients, which in turn implied the Robertson conjecture (1936) on odd univalent functions, the Rogosinski conjecture (1943) on subordinate functions, and finally the Bieberbach conjecture. Milin's conjecture asserts that the logarithmic coefficients $\gamma_n$ of a univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined by

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n$$

satisfy the inequality

$$\sum_{k=1}^{n} (n+1-k)\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0, \quad n = 1, 2, \cdots.$$

The logarithmic coefficients of the Koebe function are $\gamma_n = 1/n$ and trivially satisfy the Milin's conjecture. The Robertson conjecture asserts that the inequality

$$1 + |c_3|^2 + \cdots + |c_{2n-1}|^2 \leq n$$

is satisfied by every odd univalent function of the form $g(z) = z + c_3 z^3 + c_5 z^5 + \cdots$. Rogosinski conjecture will be stated shortly. The proof that Milin conjecture implies the other conjectures can be found in the books on univalent functions, see for example, [**14**].

The long quest for the proof of the conjecture lead to many profound contributions in geometric function theory, particularly the development of various tools for its resolution. These include Loewner's parametric method, the area method and Grunsky inequalities, Milin's and FitzGerald's methods of exponentiating the Grunsky inequalities, Baernstein's method of maximal functions, and variational methods. Several subclasses of univalent functions were also introduced from geometric considerations and investigated in an attempt to settle the conjecture. Certain subclasses are described below.

Let $\mathcal{A}$ be the class of all analytic functions in $\mathbb{D}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. A domain $D$ is starlike with respect to a point $a \in D$ if every line segment joining the point $a$ to any other point in $D$ lies completely inside $D$. A domain starlike with respect to the origin is simply called starlike. A domain $D$ is convex if every line segment joining any two points in $D$ lies completely inside $D$; in other words, the domain $D$ is convex if and only if it is starlike with respect to every point in $D$. A function $f \in \mathcal{S}$ is starlike if $f(\mathbb{D})$ is starlike (with respect to the origin) while it is convex if $f(\mathbb{D})$ is convex. The classes of all starlike and convex functions are respectively denoted by $\mathcal{S}^*$ and $\mathcal{C}$. Analytically, these classes are characterized by the inequalities

$$f \in \mathcal{S}^* \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,$$

and

$$f \in \mathcal{C} \Leftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

More generally, for $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of $\mathcal{S}$ consisting of respectively starlike functions of order $\alpha$, and convex functions of order $\alpha$. These classes are defined analytically by the inequalities

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,$$

and

$$f \in \mathcal{C}(\alpha) \Leftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

Another generalization of the class of starlike functions is the class $\mathcal{S}^*_\gamma$ of strongly starlike functions of order $\gamma$, $0 < \gamma \le 1$, consisting of $f \in \mathcal{S}$ satisfying the inequality

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad z \in \mathbb{D}.$$

Another related class is the class of close-to-convex functions. A function $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \quad 0 \le \alpha < 1,$$

for some (not necessarily normalized) convex univalent function $g$, is called close-to-convex of order $\alpha$. The class of all such functions is denoted by $\mathcal{K}(\alpha)$. Close-to-convex functions of order 0 are simply called close-to-convex functions. Using the fact that a function $f \in \mathcal{A}$ with

$$\operatorname{Re}(f'(z)) > 0$$

is in $\mathcal{S}$, close-to-convex functions can be shown to be univalent. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points of order $\alpha$ if

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > \alpha, \quad 0 \le \alpha < 1.$$

These functions are also univalent and the class of all such functions is denoted by $\mathcal{S}^*_s(\alpha)$. When $\alpha = 0$, this class is denoted by $\mathcal{S}^*_s$. Coefficient estimates for functions in all these classes can be obtained from the coefficient estimates for functions with positive real part.

Starlikeness and convexity are hereditary properties in the sense that every starlike (convex) function maps each disk $|z| < r < 1$ onto a starlike (convex) domain. However, Brown [**12**] showed it is not always true that $f \in \mathcal{S}^*$ maps each disk $|z - z_0| < \rho < 1 - |z_0|$ onto a domain starlike with respect to $f(z_0)$. He did prove that the result is true for each $f \in \mathcal{S}$ and for all sufficiently small disks in $\mathbb{D}$. This motivates the definition of uniformly starlike functions, though it was introduced independently of the work of Brown [**12**]. For this purpose, the notion of starlikeness and convexity of curves is needed. Let $\gamma$ be a curve in $\mathbb{D}$. Then the curve $\gamma$ is starlike with respect to $w_0$ if $\arg(\gamma(t) - w_0)$ is a non-decreasing function of $t$. The arc $\gamma$ is convex if the argument of the tangent to $\gamma(t)$ is a non-decreasing function of $t$.

**Definition 1.1** ([**16**, Definition 1, p. 364], [**15**, Definition 1, p. 87])**.** A function $f \in \mathcal{S}$ is **uniformly starlike** if $f$ maps every circular arc $\gamma$ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ onto a starlike arc with respect to $f(\zeta)$. The function $f \in \mathcal{S}$ is **uniformly convex** if $f$ maps every circular arc $\gamma$ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ onto a convex arc. The classes of uniformly starlike functions and uniformly convex functions are denoted respectively by $\mathcal{UST}$ and $\mathcal{UCV}$.

This article surveys results on uniformly starlike and uniformly convex functions. While there is quite a bit of literature on uniformly convex functions, not much is known about uniformly starlike functions. The survey by Rønning [**49**] provides a summary of early works on uniformly starlike and uniformly convex functions.

## 2. Uniformly Starlike Functions

**2.1. Analytic characterization and basic properties.** The following two-variable analytic characterization of the class $\mathcal{UST}$ is important for obtaining information about functions in the class $\mathcal{UST}$.

**Theorem 2.1.** [**16**, Theorem 1, p. 365] *The function $f$ is in $\mathcal{UST}$ if and only if*

$$\operatorname{Re}\left(\frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)}\right) \ge 0, \quad z, \zeta \in \mathbb{D}. \tag{2.1}$$

By taking $\zeta = -z$ in the above theorem, evidently the class $\mathcal{UST} \subset \mathcal{S}^*_s$ and hence $|a_n| \le 1$ for $f \in \mathcal{UST}$. A better bound $|a_n| \le 2/n$ for $f \in \mathcal{UST}$, proved by Charles Horowitz, was also reported in Goodman [**16**, Theorem 4, p. 368]. The proof involved showing $\mathcal{UST}$ is a subclass of $\mathcal{UST}^*$ consisting of functions $f \in \mathcal{A}$ for which $e^{i\alpha}f'(z)$ have positive real part for some real number $\alpha$.

**Open Problem 2.1.** Determine the sharp coefficient estimates for functions in the class $\mathcal{UST}$ of uniformly starlike functions.

Using Theorem 2.1, Goodman [**16**] showed that the function

$$F_1(z) = \frac{z}{1 - Az} \in \mathcal{UST} \Leftrightarrow |A| \le \frac{1}{\sqrt{2}}.$$

Similarly, if $F_2(z) = z + Az^n$, $n > 1$, and

$$|A| \le \frac{\sqrt{2}}{2n},$$

then he showed that $F_2$ is in $\mathcal{UST}$. Merkes and Salamasi [**32**] improved the bound to be

$$|A| \le \sqrt{\frac{n+1}{2n^3}}.$$

For $n \ne 2$, the bound need not be sharp. The sharp upper bound was obtained by Nezhmetdinov [**33**, Corollary 4, p. 47]. The class $\mathcal{UST}$ can also be seen to be preserved under the transformations $e^{-i\alpha} f(e^{i\alpha}z)$ and $f(tz)/t$, where $\alpha \in \mathbb{R}$ and $0 < t \le 1$. For a given locally univalent analytic function $f \in \mathcal{A}$, the disk automorphism is the function $\Lambda_f : \mathbb{D} \to \mathbb{C}$ given by

$$\Lambda_f(z) := \frac{f(\varphi(z)) - f(\lambda)}{(1-|\lambda|^2)f'(\lambda)}, \quad \varphi(z) = \frac{z+\lambda}{1+\overline{\lambda}z}.$$

A family $\mathcal{F}$ is linearly invariant if $\Lambda_f \in \mathcal{F}$ whenever $f \in \mathcal{F}$. The families $\mathcal{S}$ of univalent functions and $\mathcal{C}$ of convex functions are linearly invariant families. The disk automorphism of the function $F_1$ with $A = 1/2$ is not in $\mathcal{UST}$. This shows that the class $\mathcal{UST}$ is not a linearly invariant family.

To provide another application of the above theorem, expand the function

$$\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}$$

in its Taylors series in powers of $z$ and $\zeta$ respectively. Use of the inequality $|c_n| \le 2\,\mathrm{Re}\, c_0$ for a function $p(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ with positive real part in $\mathbb{D}$ yields the following result:

**Theorem 2.2.** [**16**, Lemma 1, p. 365] *Let $f \in \mathcal{UST}$, and define $p_0, p_1, q_0, q_1$ by*

$$p_0(\zeta) = \frac{f(\zeta)}{\zeta}, \qquad p_1(z) = \frac{f(\zeta)(1-2a_2\zeta)-\zeta}{\zeta^2},$$
$$q_0(\zeta) = \frac{f(z)}{zf'(z)}, \quad q_1(z) = \frac{f(z)-z}{z^2 f'(z)}.$$

*Then*

$$|p_1(\zeta)| \le 2\,\mathrm{Re}(p_0(\zeta)), \quad \textit{and} \quad q_1(z)| \le 2\,\mathrm{Re}(q_0(z)).$$

Theorem 2.2 and the coefficient estimate $|a_n| \le 2/n$ for $f \in \mathcal{UST}$ yield the growth inequality for $\mathcal{UST}$:

$$\frac{r}{1+2r} \le |f(z)| \le -r + 2\ln\frac{1}{1-r}, \quad |z| = r < 1.$$

This inequality provides the lower bound for the Koebe constant for the family $\mathcal{UST}$:

$$\frac{1}{3} \le K(\mathcal{UST}) \le 1 - \frac{\sqrt{3}}{4}.$$

The upper bound follows from the function $f$ given by $f(z) = z + \sqrt{3}z^2/4$.

**Open Problem 2.2.** Determine the sharp growth, distortion and rotation estimates, as well as the Koebe constant for the class $\mathcal{UST}$.

Another application of Theorem 2.1 follows from the simple identity

$$\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} = \int_0^1 \frac{f'(tz+(1-t)\zeta)}{f'(z)}dt.$$

Using this identity, Merkes and Salamasi [**32**, Theorem 4, p. 451] showed that

$$f \in \mathcal{UST} \quad \text{if} \quad \mathrm{Re}\left(\frac{f'(w)}{f'(z)}\right) > 0, \quad z, w \in \mathbb{D}.$$

If $f \in \mathcal{UST}$, they also showed that

$$\mathrm{Re}\left(\frac{f'(w)}{f'(z)}\right)^{1/2} > 0, \quad z, w \in \mathbb{D},$$

and the exponent $1/2$ is best possible.

**2.2. Convolution and Radius Problems.** The convolution (or Hadamard product) of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is the analytic function

$$(f*g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The term "convolution" is used since

$$(f*g)(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} f\left(\frac{z}{\zeta}\right) g(\zeta)\frac{d\zeta}{\zeta}, \quad |z| < \rho < 1.$$

The classes of starlike, convex and close-to-convex functions are closed under convolution with convex functions. This was conjectured by Pólya and Schoenberg [**38**] and proved by Ruscheweyh and Sheil-Small [**58**]. Ruscheweyh's monograph [**57**] gives a comprehensive survey on convolutions. To make use of this theory in the investigation of the class $\mathcal{UST}$, Merkes and Salamasi [**32**] proved the following result.

**Theorem 2.3** ([**32**, Theorem 1, p. 450])**.** *Let $f \in \mathcal{A}$. Then $f \in \mathcal{UST}$ if and only if for all complex numbers $\alpha$, $\beta$ with $|\alpha| < 1$ and $|\beta| < 1$,*

$$\mathrm{Re}\left(\frac{f(z) * \frac{z}{(1-\alpha z)(1-\beta z)}}{f(z) * \frac{z}{(1-\alpha z)^2}}\right) \ge 0, \quad z \in \mathbb{D}.$$

The following result of Rønning [**51**] is also useful in using convolution technique to investigate $\mathcal{UST}$.

**Theorem 2.4.** [**51**, Lemma 3.3, p. 236] *The function* $f \in \mathcal{UST}$ *if and only if*

$$\text{Re}\left(\frac{f(z)-f(xz)}{(1-x)zf'(z)}\right) \geq 0, \quad z \in \mathbb{D}, |x| = 1. \tag{2.2}$$

Let $\mathcal{G}$ denote the subset of $\mathcal{A}$ having the property $\mathcal{P}$. If, for every $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for $r \leq R$, and $R$ is the largest number for which this holds, then $R$ is the $\mathcal{G}$-radius (or the radius of the property $\mathcal{P}$) in $\mathcal{F}$. Thus, the radius of a property $\mathcal{P}$ in the set $\mathcal{F}$ is the largest number $R$ such that every function in the set $\mathcal{F}$ has the property $\mathcal{P}$ in each disk $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ for every $r < R$. For example, a starlike function need not be convex; however, every starlike function maps the disk $|z| < 2-\sqrt{3}$ onto a convex domain and hence the radius of convexity of the class $\mathcal{S}^*$ of starlike functions is $2-\sqrt{3}$.

Merkes and Salamasi [**32**] (using Theorem 2.3) and Rønning [**53**] (using Theorem 2.4) independently showed that the $\mathcal{UST}$-radius of the class $\mathcal{C}$ of convex functions is $1/\sqrt{2}$. Merkes and Salamasi [**32**, Theorem 5, p. 451] also obtained a lower bound for the $\mathcal{UST}$-radius for the class of pre-starlike functions. For $\alpha \leq 1$, the class $\mathcal{R}_\alpha$ of *prestarlike* functions of order $\alpha$ consists of functions $f \in \mathcal{A}$ satisfying

$$\begin{cases} f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha), & \alpha < 1, \\ \text{Re}\,\frac{f(z)}{z} > \frac{1}{2}, & \alpha = 1. \end{cases}$$

Note that $\mathcal{R}_0 = \mathcal{C}$ and $\mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$. The known radius results are recorded in the following theorem.

**Theorem 2.5.**

(1) *The* $\mathcal{UST}$*-radius for the class of univalent functions* $\mathcal{S}$ *is* $r_0 \approx 0.3691$.

(2) *The* $\mathcal{UST}$*-radius* $r_0^*$ *for the class* $\mathcal{S}^*$ *satisfies*

$$0.369 < r_0^* \leq 1/\sqrt{7}.$$

(3) *The* $\mathcal{UST}$*-radius for the class of convex functions* $\mathcal{C}$ *is* $1/\sqrt{2}$.

(4) *The* $\mathcal{UST}$*-radius for the class of pre-starlike functions is at least* $(1+\alpha)/(1-\alpha)$ *for*

$$\frac{\sqrt{2}-1}{\sqrt{2}+1} \leq \alpha < 1.$$

The exact value of the $\mathcal{UST}$-radius $r_0$ of $\mathcal{S}$ is obtained as the unique root of $\varphi(t) = \pi/2$ in the interval $[0,1]$ where $\varphi(t)$ is the expression in [**53**, Equation (2.1), p. 320].

**Open Problem 2.3.** Determine the (exact) $\mathcal{UST}$-radius $r_0^*$ of the class $\mathcal{S}^*$ and the exact $\mathcal{UST}$-radius of the class of pre-starlike functions. Determine whether the class $\mathcal{UST}$ is closed under convolution with convex functions.

For a given subset $\mathcal{V} \subset \mathcal{A}$, its dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* := \left\{ g \in \mathcal{A} : \frac{(f*g)(z)}{z} \neq 0 \text{ for all } f \in \mathcal{V} \right\}.$$

Nezhmetdinov [**33**, Theorem 2, p. 43] showed that the the dual set of the class $\mathcal{UST}$ is the subset of $\mathcal{A}$ consisting of functions $h : \mathbb{D} \to \mathbb{C}$ given by

$$h(z) = \frac{z\left(1 - \frac{(w+i\alpha)}{1+i\alpha}z\right)}{(1-wz)(1-z)^2}, \quad \alpha \in \mathbb{R},\ |w| = 1.$$

He determined the uniform estimate $|a_n(h)| \leq dn$ for the $n$-th Taylor coefficient of $h$ in the dual set of $\mathcal{UST}$ with a sharp constant $d = \sqrt{M} \approx 1.2557$, where $M \approx 1.5770$ is the maximum value of a certain trigonometric expression. Using this, he showed that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{\sqrt{M}} \Rightarrow f \in \mathcal{UST}.$$

The bound $1/\sqrt{M}$ is sharp.

**Open Problem 2.4.** Rønning [**53**] proved that $\mathcal{UST} \not\subset \mathcal{S}^*(1/2)$ and posed the problem of determining the largest $\alpha$ such that $\mathcal{UST} \subset \mathcal{S}^*(\alpha)$. Nezhmetdinov [**34**] showed that $\mathcal{UST} \not\subset \mathcal{S}^*(\alpha_0)$ for some $\alpha_0 \approx 0.1483$. Determine the largest $\alpha$ such that $\mathcal{UST} \subset \mathcal{S}^*(\alpha)$.

## 3. Uniformly Convex Functions

### 3.1. Analytic characterizations and parabolic starlike functions.

Recall that a univalent function $f$ is in the class $\mathcal{UCV}$ of uniformly convex functions if for every circular arc $\gamma$ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ the image arc $f(\gamma)$ is convex. From this definition, the following theorem is readily obtained.

**Theorem 3.1** ([**15**, Theorem 1, p. 88])**.** *The function* $f$ *belongs to* $\mathcal{UCV}$ *if and only if*

$$\text{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0, \quad z, \zeta \in \mathbb{D}. \tag{3.1}$$

Though the class $\mathcal{C}$ is a linear invariant family, the class $\mathcal{UCV}$ is not. This was proved by Goodman [**15**, Theorem 5, p. 90] by using the function

$$F(z) = \frac{z}{1-Az}.$$

This function $F \in \mathcal{UCV}$ if and only if $|A| \leq 1/3$.

From the geometric definition or from Theorem 3.1, it is evident that $\mathcal{UCV} \subset \mathcal{CV}$. However, by taking $\zeta = -z$

in Theorem 3.1, it is evident that $\mathcal{UCV} \subset \mathcal{C}(1/2)$. In view of this inclusion and the coefficient estimate for functions in $\mathcal{C}(1/2)$, the Taylor coefficients $a_n$ of $f \in \mathcal{UCV}$ satisfy $|a_n| \leq 1/n$. Unlike the uniformly starlike functions, uniformly convex functions admit a one-variable characterization, and this readily yields several important properties of functions in $\mathcal{UCV}$. This one-variable characterization, obtained independently by Rønning [**50**, Theorem 1, p. 190], and Ma and Minda [**29**, Theorem 2, p. 162], is the following result.

**Theorem 3.2.** *Let $f \in \mathcal{A}$. Then $f \in \mathcal{UCV}$ if and only if*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \; z \in \mathbb{D}. \tag{3.2}$$

If $f \in \mathcal{UCV}$, then equation (3.2) follows from (3.1) for a suitable choice of $\zeta$. For the converse, the minimum principle for harmonic function is used to restrict $\zeta$ and $z$ to $|\zeta| < |z| < 1$. With this restriction, (3.1) immediately follows from (3.2). To give a nice geometric interpretation of (3.2), let

$$\Omega_p := \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|\}.$$

The set $\Omega_p$ is the interior of the parabola

$$(\operatorname{Im} w)^2 = 2\operatorname{Re} w - 1$$

and it is therefore symmetric with respect to the real axis and has $(1/2, 0)$ as its vertex. Then $f \in \mathcal{UCV}$ if and only if

$$1+\frac{zf''(z)}{f'(z)} \in \Omega_p.$$

A class closely related to the class $\mathcal{UCV}$ is the class of parabolic starlike functions defined below.

**Definition 3.1.** [**50**] The class $\mathcal{S_P}$ of parabolic starlike functions consists of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right|, \quad z \in \mathbb{D}.$$

In other words, the class $\mathcal{S_P}$ consists of function $f = zF'$ where $F \in \mathcal{UCV}$.

Since the parabolic region $\Omega_p$ is contained in the half-plane

$$\{w : \operatorname{Re} w > 1/2\}$$

and the sector

$$\{w : |\arg w| < \pi/4\},$$

Rønning [**50**] noted that

$$\mathcal{S_P} \subset \mathcal{S}^*(1/2) \cap \mathcal{S}^*_{1/2}.$$

The class $\mathcal{C}$ of convex functions and the class $\mathcal{S}^*$ of starlike functions are connected by the Alexander result that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Such a question between the classes $\mathcal{UST}$ and $\mathcal{UCV}$ is in fact a question of equality between $\mathcal{UST}$ and $\mathcal{S_P}$. It turns out (see [**15, 51**]) that that there is no inclusion between them:

$$\mathcal{UST} \not\subset \mathcal{S_P} \quad \text{and} \quad \mathcal{S_P} \not\subset \mathcal{UST}.$$

**3.2. Examples.** To give some examples of functions in $\mathcal{UCV}$ and $\mathcal{S_P}$, note [**40**] that

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2} \Rightarrow f \in \mathcal{UCV} \tag{3.3}$$

and

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{1}{2} \Rightarrow f \in \mathcal{S_P}. \tag{3.4}$$

The proof follows readily from the implication

$$|w| < \frac{1}{2} \Rightarrow |w| < \frac{1}{2} = 1-\frac{1}{2} < 1-|w| < \operatorname{Re}(1+w).$$

A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ with $a_n \geq 0$ is called a function with negative coefficients. For functions with negative coefficients, the above condition is also necessary for a function $f$ to be in $\mathcal{UCV}$ or $\mathcal{S_P}$ (see [**11, 61**]). In terms of the coefficients, the results can be stated as follows:

**Theorem 3.3.** *Let $f$ be a function of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ with $a_n \geq 0$. Then*

$$f \in \mathcal{UCV} \Leftrightarrow \sum_{n=2}^{\infty} n(2n-1)a_n \leq 1$$

*and*

$$f \in \mathcal{S_P} \Leftrightarrow \sum_{n=2}^{\infty} (2n-1)a_n \leq 1.$$

Denote the class of all functions with negative coefficients by $\mathcal{T}$. Define

$$\mathcal{TUCV} := \mathcal{T} \cap \mathcal{UCV}, \qquad \mathcal{TS_P} := \mathcal{T} \cap \mathcal{S_P},$$
$$\mathcal{TS}^* := \mathcal{T} \cap \mathcal{S}^*, \quad \text{and} \quad \mathcal{TC} := \mathcal{T} \cap \mathcal{C}.$$

In terms of these classes, the above result can be stated as

$$\mathcal{TUCV} = \mathcal{TC}(1/2) \quad \text{and} \quad \mathcal{TS_P} = \mathcal{TS}^*(1/2).$$

For these and other related results, see [**11, 61**]. Using Theorem 3.3, it can be seen [**50**] that

$$f(z) = z - A_n z^n \in \mathcal{S_P} \Leftrightarrow |A_n| \leq \frac{1}{2n-1},$$

and

$$f \in \mathcal{UCV} \Leftrightarrow |A_n| \leq \frac{1}{n(2n-1)}.$$

Goodman [**15**] showed

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1}{3} \Rightarrow f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{UCV};$$

this easily follows from Theorem 3.3 since

$$\sum_{n=2}^{\infty} n(2n-1)a_n \le 3\sum_{n=2}^{\infty} n(n-1)|a_n| \le 1.$$

The sufficient condition in (3.3) can be extended to a more general circular region. For this purpose, let $a > 1/2$. Then it can be shown that the minimum distance from the point $w = a$ to points on the parabola

$$|w-1| = \operatorname{Re} w$$

is given by

$$R_a = \begin{cases} a - \frac{1}{2}, & if\ \frac{1}{2} < a \le \frac{3}{2} \\ \sqrt{2a-2}, & if\ a \ge \frac{3}{2}. \end{cases}$$

Thus [**59**]

$$\left|1 + \frac{zf''(z)}{f'(z)} - a\right| < R_a \Rightarrow f \in \mathcal{UCV}$$

and

$$\left|\frac{zf'(z)}{f(z)} - a\right| < R_a \Rightarrow f \in \mathcal{S_P}.$$

**3.3. Subordination and its consequences.** Let $f$ and $F$ be analytic functions in $\mathbb{D}$. Then $f$ is said to be *subordinate* to the function $F$, written $f(z) \prec F(z)$, if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ satisfying $w(0) = 0$ such that $f(z) = F(w(z))$. If $p : \mathbb{D} \to \mathbb{C}$, $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, then

$$p(z) \prec \frac{1+z}{1-z}.$$

This follows since the mapping $q(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane $\Omega_H := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. In this light, the classes of starlike and convex functions can be expressed as follows:

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$$

and

$$\mathcal{C} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}.$$

Rønning [**50**] and Ma and Minda [**29**] showed that the function $\varphi_p : \mathbb{D} \to \mathbb{C}$ defined by

(3.5)

$$\begin{aligned} \varphi_p(z) &= 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \\ &= 1 + \frac{8}{\pi^2}\left(z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \frac{44}{105}z^4 + \cdots\right) \end{aligned}$$

maps $\mathbb{C}$ onto the parabolic region

$$\Omega_p := \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|\}.$$

Therefore the classes $\mathcal{UCV}$ and $\mathcal{S_P}$ can be expressed in the form

$$\mathcal{S_P} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_p(z) \right\}$$

and

$$\mathcal{UCV} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_p(z) \right\}.$$

Rønning [**50**, Theorem 6, p. 195] went on to show the sharp inequality

$$|f'(z)| \le \exp\left(\frac{14\zeta(3)}{\pi^2}\right) \approx 5.502$$

for $f \in \mathcal{UCV}$, where $\zeta(t)$ denotes the Riemann zeta function. Ma and Minda [**29**] on the other hand obtained distortion (bounds for $|f'(z)|$), growth (bounds for $|f(z)|$), covering (the radius of the largest disk centered at origin contained in $f(\mathbb{D})$) and rotation (the upper bound for $|\arg(f'(z))|$) estimates for functions in $\mathcal{UCV}$. These results are then proved for more general classes of functions by Ma and Minda [**30**]. For this purpose, let $\phi$ be an analytic function with positive real part in $\mathbb{D}$, normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$, such that $\phi$ maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 that is symmetric with respect to the real axis. They introduced the following classes:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

These functions are called Ma-Minda starlike and convex functions respectively. For special choices of $\varphi$, these classes become well-known classes of starlike and convex functions. For example, for the choice

$$\varphi_{A,B}(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1,$$

the class $\mathcal{S}^*[A,B] := \mathcal{S}^*(\varphi_{A,B})$ is the class of Janowski starlike functions. For the classes of Ma-Minda starlike and convex functions, the following theorem is obtained.

**Theorem 3.4.** [**30**] *If $f \in \mathcal{C}(\varphi)$, then, for $|z| = r$,*

$$\begin{aligned} k'_\varphi(-r) &\le |f'(z)| \le k'_\varphi(r), \\ -k_\varphi(-r) &\le |f(z)| \le k_\varphi(r), \end{aligned}$$

*where $k_\varphi : \mathbb{D} \to \mathbb{C}$ is defined by*

$$1 + \frac{zk''_\varphi(z)}{k'_\varphi(z)} = \varphi(z).$$

*Equality holds for some $z \neq 0$ if and only if $f$ is a rotation of $k_\varphi$. Also either $f$ is a rotation of $k_\varphi$ or $f(\mathbb{D})$ contains the disk $|w| \leq -k_\varphi(-1)$, where*

$$-k_\varphi(-1) = \lim_{r\to 1-}(-k_\varphi(-r)).$$

*Further, for $|z_0| = r < 1$,*

$$|\arg(f'(z_0))| \leq \max_{|z|=r} |\arg k'_\varphi(z)|.$$

The proof relies on the subordination $f'(z) \prec k'_\varphi(z)$ satisfied by functions $f \in \mathcal{C}(\varphi)$. Corresponding results for functions in $\mathcal{S}^*(\varphi)$ were also obtained by Ma and Minda [**30**]. The distortion theorem for $f \in \mathcal{S}^*(\varphi)$ requires some additional assumptions on $\varphi$. Theorem 3.4 contains the corresponding results for uniformly convex functions [**29**] as special cases. Extension of these (and other closely related) results to functions starlike with respect to symmetric points, conjugate points, multivalent starlike functions, and meromorphic functions were investigated in [**43, 6, 5**].

Let $h_\varphi : \mathbb{D} \to \mathbb{C}$ be defined by

$$\frac{zh'_\varphi(z)}{h_\varphi(z)} = \varphi(z).$$

Ma and Minda [**30**] proved that

$$f \in \mathcal{S}^*(\varphi) \Rightarrow \frac{f(z)}{z} \prec \frac{h_\varphi(z)}{z}.$$

In the case when $\varphi$ is a convex univalent function, this result is a special case of the following general result:

**Theorem 3.5** (Ruscheweyh [**55**, Theorem 1, p. 275])**.** *Let $\phi$ be a convex function defined in $\mathbb{D}$ with $\phi(0) = 1$. Define $F$ by*

$$F(z) = z\exp\left(\int_0^z \frac{\phi(x)-1}{x}dx\right).$$

*The function $f$ belongs to $S^*(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$,*

$$\frac{sf(tz)}{tf(sz)} \prec \frac{sF(tz)}{tF(sz)}.$$

**Open Problem 3.1.** Determine the sharp bound of $|f^{(n)}(z)|$ for $f \in \mathcal{C}(\varphi)$ and $f \in \mathcal{S}^*(\varphi)$. For $f \in \mathcal{C}(\varphi)$, the bounds for the cases $n = 0, 1$ are given by Theorem 3.4. Similar bounds for $f \in \mathcal{S}^*(\varphi)$ are also known with some restrictions on $\varphi$.

**3.4. Coefficient Problems.** As noted earlier, the inclusion $\mathcal{UCV} \subset \mathcal{C}(1/2)$ shows that each Taylor coefficient $a_n$ of $f \in \mathcal{UCV}$ satisfies $|a_n| \leq 1/n$. These bounds can be improved. Since the classes $\mathcal{UCV}$ and $\mathcal{S_P}$ are connected by the Alexander relation that $f \in \mathcal{UCV}$ if and only if $zf' \in \mathcal{S_P}$, it suffices to give the coefficient estimate for functions in $\mathcal{S_P}$.

**Theorem 3.6.** [**50**, Theorem 5, p. 194] *Let $f \in \mathcal{S_P}$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$. Then*

(3.6)

$$|a_2| \leq c, \quad \text{and} \quad |a_n| \leq \frac{c}{n-1}\prod_{k=3}^n\left(1+\frac{c}{k-2}\right),$$

*where $c = 8/\pi^2$.*

Let $p(z) = zf'(z)/f(z) = 1 + c_1z + c_2z^2 + \cdots$, and $p(z) \prec \varphi_p(z)$ where $\varphi_p$ is given by (3.5). Rogosinski's theorem states that $|c_k| \leq c$ for any function $p(z) = 1 + c_1z + c_2z^2 + \cdots$ subordinate to the convex univalent function $P(z) = 1 + cz + \cdots$. The coefficients of $f$ and the coefficients of $p$ are related by

$$(n-1)a_n = \sum_{k=1}^{n-1} c_{n-k}a_k.$$

This together with Rogosinksi's theorem yield the desired coefficient bounds. Whenever $\varphi$ is a convex univalent function, the bounds for $|a_n|$ for $f \in \mathcal{S}^*(\varphi)$ is also given by (3.6) where $c := \varphi'(0)$. The estimates given by (3.6) are not sharp in general. However, in the case, $\varphi(z) = (1+z)/(1-z)$, the inequalities in (3.6) give sharp bounds for the coefficients of starlike functions.

The sharp coefficient estimates for functions in $\mathcal{UCV}$ or $\mathcal{S_P}$ is still an open problem. However, the sharp estimates of $|a_n|$ for $f \in \mathcal{UCV}$ were obtained by Ma and Minda [**29, 31**]. They [**29**, Theorem 5, p. 172] also proved the sharp order of growth $|a_n| = O(1/n^2)$ for $f \in \mathcal{UCV}$. The same order of growth holds for $f \in \mathcal{C}(\varphi)$ if $\varphi$ belongs to the Hardy class of analytic functions $\mathcal{H}^2$ (see [**30**]). They [**31**] also found the sharp upper bound for the Fekete-Szegő functional $|\mu a_2^2 - a_3|$ in the class $\mathcal{UCV}$ for all real $\mu$. For the inverse function

$$f^{-1}(w) = w + \sum_{n=2}^\infty d_n w^n,$$

they [**31**] obtained the sharp inequality

$$|d_n| \leq \frac{8}{(n-1)n\pi^2}, \quad n = 2, 3, 4.$$

More generally, the coefficient problem for $f \in \mathcal{C}(\varphi)$ is also open. Estimates for the first two coefficients as well as for the Fekete-Szegő functional for functions in $\mathcal{C}(\varphi)$ were obtained in [**30**]. For several related coefficient problems, see [**7**].

**Theorem 3.7.** *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{C}(\varphi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6}(B_2 - (3/2)\mu B_1^2 + B_1^2) \\ \quad \text{if } 3B_1^2\mu \leq 2(B_2 + B_1^2 - B_1) \\ \frac{B_1}{6} \quad \text{if } 2(B_2 + B_1^2 - B_1) \leq 3B_1^2\mu \\ \quad \leq 2(B_2 + B_1^2 + B_1) \\ \frac{1}{6}(-B_2 + (3/2)\mu B_1^2 - B_1^2) \\ \quad \text{if } 2(B_2 + B_1^2 + B_1) \leq 3B_1^2\mu \end{cases}$$

*The result is sharp.*

To see an outline of the proof, first express the coefficient of $f$ in terms of the coefficients $c_k$ for functions with positive real part. For $f \in \mathcal{C}(\varphi)$, let $p : \mathbb{D} \to \mathbb{C}$ be defined by

$$p(z) := \frac{zf'(z)}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

so that $2a_2 = b_1$ and $6a_3 = b_2 + b_1^2$. Since $\phi$ is univalent and $p(z) \prec \phi(z)$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 + \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in $\mathbb{D}$. Also

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{3.7}$$

and from this equation (3.7), it follows that

$$b_1 = \frac{1}{2}B_1 c_1 \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Therefore

$$a_3 - \mu a_2^2 = \frac{B_1}{12}\left(c_2 - v c_1^2\right), \tag{3.8}$$

where

$$v := \frac{1}{2B_1}\left(B_1 - B_1^2 - B_2 + \frac{3}{2}\mu B_1^2\right).$$

The theorem then follows by an application of the corresponding coefficient results for function with positive real part. Notice that this method is difficult to apply to get bounds for $|a_n|$ for large $n$, as $a_n$ can only be expressed as a non-linear function of the coefficients $c_k$.

**Open Problem 3.2.** Determine the sharp bound for the Taylor coefficients $|a_n|$ $(n \geq 5)$ for $f \in \mathcal{C}(\varphi)$ and $f \in \mathcal{S}^*(\varphi)$. The same problem for the other classes defined by subordination is still open.

**3.5. Convolution.** Recall that the **convolution** of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is the analytic function defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The convolution of two functions in $\mathcal{A}$ is again in $\mathcal{A}$. Since the $n$th coefficient of normalized univalent function is bounded by $n$, the convolution of the Koebe function $k(z) = z/(1-z)^2$ with itself is not univalent. Thus, the convolution of two univalent (or starlike) functions need not be univalent. Pólya and Schoenberg [**38**] conjectured that the class of convex functions $\mathcal{C}$ is preserved under convolution with convex functions:

$$f, g \in \mathcal{C} \Rightarrow f * g \in \mathcal{C}.$$

In 1973, Ruscheweyh and Sheil-Small [**58**] (see also [**57**]) proved the Polya-Schoenberg conjecture. In fact, they also proved that the classes of starlike functions and close-to-convex functions are closed under convolution with convex functions. The proof of these facts follow from the following result which is also used below to show that the classes $\mathcal{UCV}$ and $\mathcal{S}_{\mathcal{P}}$ are closed under convolution with starlike functions of order 1/2.

**Theorem 3.8.** [**58**, Theorem 2.4, p. 54] *Let $\alpha \leq 1$, $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{S}^*(\alpha)$. Then, for any analytic function $H \in \mathcal{H}(\mathbb{D})$,*

$$\frac{f * Hg}{f * g}(\mathbb{D}) \subset \overline{co}(H(\mathbb{D})),$$

*where $\overline{co}(H(\mathbb{D}))$ denote the closed convex hull of $H(U)$.*

**Theorem 3.9.** [**49**, Theorem 3.6, p. 131] *Let $\varphi$ be a convex function with* $\operatorname{Re}\varphi(z) \geq \alpha$, $\alpha < 1$. *If $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{S}^*(\varphi)$, then $f * g \in \mathcal{S}^*(\varphi)$.*

The proof of this theorem follows readily from Theorem 3.8 by putting $H(z) = zg'(z)/g(z)$. In view of the fact that $f \in \mathcal{C}(\varphi)$ if and only if $zf' \in \mathcal{S}^*(\varphi)$, an immediate consequence of the above theorem is the corresponding result for $\mathcal{C}(\varphi)$: if $f \in \mathcal{R}_\alpha$ and $g \in \mathcal{C}(\varphi)$, then $f * g \in \mathcal{C}(\varphi)$ for any convex function $\varphi$ with $\operatorname{Re}\varphi(z) \geq \alpha$. In particular, the classes $\mathcal{UCV}$ and $\mathcal{S}_{\mathcal{P}}$ are closed under convolution with starlike functions of order 1/2. Similar results for several other related classes of functions can be found in [**3, 4, 42**] or references therein.

Goodman remarked that the class $\mathcal{UCV}$ is preserved under the transformation $e^{-i\alpha}f(e^{i\alpha}z)$ and no other transformation seems to be available. However, since $\mathcal{UCV}$ is closed under convolution with starlike functions of order 1/2 and in particular with convex functions, the following result is obtained.

**Corollary 3.1.** [**40**] *Let*

$$\Gamma_1(f(z)) = zf'(z),$$
$$\Gamma_2(f(z)) = \frac{1}{2}[f(z) + zf'(z)]$$
$$\Gamma_3(f(z)) = \frac{k+1}{z^k}\int_0^z \zeta^{k-1}f(\zeta)d\zeta,\ \mathrm{Re}\,k > 0$$
$$\Gamma_4(f(z)) = \int_0^z \frac{f(\zeta) - f(\eta\zeta)}{\zeta - \eta\zeta}d\zeta,\ |\eta| \le 1,\ \eta \ne 1.$$

*Then* $\Gamma_i(f) \in \mathcal{UCV}$ *in* $|z| < r_i$ *whenever* $f \in \mathcal{UCV}$, *where*

$$r_1 = \frac{1}{3},\ r_2 = \frac{\sqrt{17}-3}{2} \approx .56155,\ r_3 = r_4 = 1.$$

**3.6. Gaussian Hypergeometric functions.** For complex numbers $a, b, c \in \mathbb{C}$ with $c \ne 0, -1, -2, \ldots$, the Gaussian hypergeometric function $F(a, b; c; z)$ is defined by the power series

$$F(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}.$$

Here $(a)_0 := 1$ for $a \ne 0$ and if $n$ is a positive integer, then $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$. For $\beta < 1$ and $\eta \in \mathbb{R}$, define the class $R_\eta(\beta)$ by

$$R_\eta(\beta) = \left\{f \in \mathcal{A} \mid \mathrm{Re}(e^{i\eta}(f'(z) - \beta)) > 0 \quad \text{for} \quad z \in \mathbb{D}\right\}.$$

For the Gaussian hypergeometric function $F(a, b, c; z)$, Kim and Ponnusamy [**27**] found conditions which would imply that $zF(a, b; c; z)$ belongs to $\mathcal{UCV}$ or $R_\eta(\beta)$. Further they derived conditions under which $f \in R_\eta(\beta)$ implies

$$zF(a, b; c; z) * f(z) \in \mathcal{UCV}.$$

In fact, by making use of the Gauss summation theorem and Theorem 3.3, they obtained the following sufficient condition for $zF(a, b; c; z) \in \mathcal{UCV}$.

**Theorem 3.10.** [**27**, Theorem 1, p. 768] *Let* $a, b \in \mathbb{C} - \{0\}$ *and* $c > |a| + |b| + 2$. *If*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\times \left(1 + \frac{2(|a|)_2(|b|)_2}{(c-2-|a|-|b|)_2} + \frac{5|ab|}{c-|a|-|b|-1}\right) \le 2,$$

*then* $zF(a, b; c; z) \in \mathcal{UCV}$.

They also obtained a weaker condition on the parameters so that the function $zF(a, \overline{a}; c; z) \in \mathcal{UCV}$. The following result provides a mapping of $R_\eta(\beta)$ into $\mathcal{UCV}$.

**Theorem 3.11.** [**27**, Theorem 4, p. 771] *Let* $a, b \in \mathbb{C} - \{0\}$ *and* $c > |a| + |b| + 1$. *If*

$$2(1-\beta)\cos\eta\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\times \left(1 + \frac{2|ab|}{c-|a|-|b|-1}\right) - 1\right) \le 1,$$

*and* $f \in R_\eta(\beta)$, *then* $zF(a, b; c; z) * f(z) \in \mathcal{UCV}$.

An extension of these results to other related classes can be found, for example, in [**19, 25**].

**3.7. Integral transform.** The classes $\mathcal{UCV}$ and $\mathcal{S_P}$ are closed under several integral operators.

**Theorem 3.12.** [**59**, Theorem 1, p. 320] *Let* $f_i \in \mathcal{UCV}$ *and* $\alpha_i$*'s be real numbers such that* $\alpha_i \ge 0$, *and* $\sum_1^n \alpha_i \le 1$. *Then the function*

$$g(z) = \int_0^z \prod_{i=1}^n [f_i'(\zeta)]^{\alpha_i} d\zeta$$

*belongs to* $\mathcal{UCV}$.

As an immediate consequence of this theorem, the function $g$ defined by

$$g(z) = \int_0^z \prod_1^n (1 - A_i\zeta)^{-2\alpha_i} d\zeta$$

$$(0 \le \alpha_i < 1,\ \sum_1^n \alpha_i \le 1,\ |A_i| \le \frac{1}{3},\ i = 1, 2, \ldots, n)$$

belongs to $\mathcal{UCV}$. The first implication in (3.3) yields the following result.

**Theorem 3.13.** [**59**, Theorem 2, p. 320] *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{4},$$

*then*

$$g(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^2 d\zeta$$

*belongs to* $\mathcal{UCV}$.

**3.8. $k$-Uniformly convex function.** Let $k \ge 0$. A function $f \in \mathcal{S}$ is called $k$-uniformly convex in $\mathbb{D}$ if the image of every circular arc $\gamma$ contained in the unit disk $\mathbb{D}$, with center $\zeta$, $|\zeta| \le k$, is convex. For any fixed $k \ge 0$, the class of all $k$-uniformly convex functions is denoted by $k - \mathcal{UCV}$. The class $k - \mathcal{UCV}$ was introduced and investigated by Kanas and Wisinowska [**22**]. As in the case of uniformly convex functions, the following theorem holds.

**Theorem 3.14** ([**22**]). *Let $f \in \mathcal{S}$. Then the following are equivalent:*

(1) $f \in k-\mathcal{UCV}$,
(2) *the inequality*
$$\operatorname{Re}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0$$
*holds for all $z \in \mathbb{D}$ and for all $|\zeta| \leq k$,*
(3) *the inequality*
$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|$$
*holds for all $z \in \mathbb{D}$.*

Interestingly, the class of $k$-uniformly convex functions unifies the class of convex functions ($k=0$) and the class of uniformly convex functions ($k=1$). Let
$$\Omega_k = \{w : \operatorname{Re} w > k|w-1|\}.$$
Then the region $\Omega_k$ is elliptic for $k>1$, parabolic for $k=1$, and hyperbolic for $0<k<1$. The region $\Omega_0$ is the right-half plane. Several properties of uniformly convex functions extend to $k-\mathcal{UCV}$ functions; these properties are treated in [**22, 23, 17, 19, 20, 21**].

**3.9. Uniformly spirallike functions.** Let $\Gamma_w$ be the image of an arc $\Gamma_z : z = z(t)$, $(a \leq t \leq b)$ under the function $f(z)$ and let $w_0$ be a point not on $\Gamma_w$. Recall that the arc $\Gamma_w$ is starlike with respect to $w_0$ if $\arg(w - w_0)$ is a nondecreasing function of $t$. This condition is equivalent to
$$\operatorname{Im}\frac{f'(z)z'(t)}{f(z)-w_0} \geq 0 \quad (a \leq t \leq b).$$
The arc $\Gamma_w$ is $\alpha$-spirallike with respect to $w_0$ if
$$\arg\frac{z'(t)f'(z)}{f(z)-w_0}$$
lies between $\alpha$ and $\alpha+\pi$ [**14**]. The function $f$ is uniformly $\alpha$-spirallike if the image of every circular arc $\Gamma_z$ with center at $\zeta$ lying in $\mathbb{D}$ is $\alpha$-spirallike with respect to $f(\zeta)$. The class of all uniformly $\alpha$-spirallike functions is denoted by $\mathcal{USP}(\alpha)$. Here is an analytic description of $\mathcal{USP}(\alpha)$ analogous to the class $\mathcal{UST}$.

**Theorem 3.15.** [**46**] *Let $|\alpha| < \frac{\pi}{2}$. A function $f \in \mathcal{A}$ belongs to $\mathcal{USP}(\alpha)$ if and only if*
$$\operatorname{Re}\left(e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right) \geq 0, \quad z \neq \zeta, \quad z,\zeta \in \mathbb{D}.$$

The arc $\Gamma_w$ is convex $\alpha$-spirallike if
$$\arg\left(\frac{z''(t)}{z'(t)}+\frac{z'(t)f''(z)}{f'(z)}\right)$$
lies between $\alpha$ and $\alpha+\pi$. The function $f$ is a uniformly convex $\alpha$-spiral function if the image of every circular arc $\Gamma_z$ with center at $\zeta$ lying in $\mathbb{D}$ is convex $\alpha$-spirallike. The class of all uniformly convex $\alpha$-spiral functions is denoted by $\mathcal{UCSP}(\alpha)$. An analytic description of $\mathcal{UCSP}(\alpha)$ analogous to the class $\mathcal{UCV}$ is the following:

**Theorem 3.16.** [**46**] *Let $f \in \mathcal{A}$. The the following are equivalent.*

(1) $f \in \mathcal{UCSP}(\alpha)$,
(2) *$f$ satisfies the inequality*
$$\operatorname{Re}\left(e^{-i\alpha}\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)\right) \geq 0, \quad z \neq \zeta, \quad z,\zeta \in \mathbb{D},$$
(3) *$f$ satisfies the inequality*
$$\operatorname{Re}\left(e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathbb{D}.$$

For $f \in \mathcal{A}$, define the function $s$ by
$$f'(z) = (s'(z))^{e^{i\alpha}\cos\alpha}.$$
Then $f \in \mathcal{UCSP}(\alpha)$ if and only if $s \in \mathcal{UCV}$. In view of this connection with $\mathcal{UCV}$, properties of functions in $\mathcal{UCSP}$ can be obtained from the corresponding properties of $\mathcal{UCV}$. The classes of uniformly spirallike and uniformly convex spirallike functions were introduced by Ravichandran *et al.* [**46**], and for a generalization of the class, see [**63**].

**3.10. Radius problems.** The determination of the radius of starlikeness or convexity typically requires an estimate for the real part of the quantities
$$Q_{ST} := \frac{zf'(z)}{f(z)} \quad \text{and} \quad Q_{CV} := 1+\frac{zf''(z)}{f'(z)}.$$
This method of estimating the real part of $Q_{ST}$ or $Q_{CV}$ will not work for the radius problems associated with uniformly convex functions, parabolic starlike functions, strongly starlike functions and several other subclasses of starlike/convex functions. In these cases, one need to know the region of values of $Q_{ST}$ or $Q_{CV}$. This idea was first used by Rønning for computing the sharp radius of parabolic starlikeness for univalent functions.

**Theorem 3.17.** *The $\mathcal{S}_{\mathcal{P}}$-radius of the class $\mathcal{S}$ of univalent functions is 0.33217 and the $\mathcal{S}_{\mathcal{P}}$-radius of the class $\mathcal{S}^*$ of starlike functions is $1/3 \approx 0.3333$* [**50**, Corollary 3, Theorem 4, p. 192]. *The $\mathcal{S}_{\mathcal{P}}$-radius of the class $\mathcal{C}$ of* *convex functions is* $1/\sqrt{2} \approx 0.7071$ [**51**, Theorem 3.1 9b, p. 236].

The $\mathcal{S}_{\mathcal{P}}$-radii for the following classes of functions were determined by Shanmugam and Ravichandran [**59**]:

(1) the class of close-to-starlike functions of order $\alpha$; these are functions $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}(f(z)/g(z)) > 0$ for some function $g$ starlike of order $\alpha$.
(2) the class of functions $f(z) = z + a_n z^n + \cdots$ satisfying the condition $\mathrm{Re}(f(z)/z) > 0$.
(3) the class of functions $f \in \mathcal{A}$ satisfying
$$\left|\frac{f(z)}{g(z)} - 1\right| < 1$$
for some function $g$ starlike of order $\alpha$.

Rønning [**53**, Theorem 4, p. 321] showed that the radius of uniform convexity of the classes $\mathcal{S}$ and $\mathcal{S}^*$ is $(4-\sqrt{13})/3 \approx 0.1314$. Let $\mathcal{S}_n^*[A,B]$ consists of functions
$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$$
satisfying
$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$
For the special case $A = 1-2\alpha$, $B = -1$, the class is denoted by $\mathcal{S}_n^*(\alpha)$. Ravichandran, Rønning and Shanmugam [**47**] investigated $\mathcal{S}_n^*(\beta)$-radius and $\mathcal{S}_{\mathcal{P}}$-radius for the class $\mathcal{S}_n^*[A,B]$. They also investigated the radii of convexity and uniform convexity in $\mathcal{S}_n^*(0)$. Additionally they studied the radius problems for functions whose derivatives belong to the Kaplan classes $\mathcal{K}(\alpha,\beta)$; their results, in special cases, yield radius results for various classes of close-to-convex functions and functions of bounded boundary rotation. For $0 \le \alpha \le \beta$, the Kaplan classes $\mathcal{K}(\alpha,\beta)$ can be defined as follows. A function $f' \in \mathcal{K}(\alpha,\beta)$ if and only if there is a function $g \in \mathcal{S}^*((2+\alpha-\beta)/2)$ and a real number $t \in \mathbb{R}$ such that
$$\left|\arg\left(e^{it}\frac{zf'(z)}{g(z)}\right)\right| \le \frac{\alpha\pi}{2}.$$
For the radius of uniform convexity of a closely related class, see [**48**] wherein they investigated $\mathcal{S}^*(\beta)$-radius and $\mathcal{S}_{\mathcal{P}}$-radius of certain integral transforms and Bloch functions. Related radius results can also be found in [**45**].

**3.11. Neighborhood problems.** Given $\delta \ge 0$, Ruscheweyh [**56**] defined the *δ-neighbourhood* $N_\delta(f)$ of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ to be the set
$$N_\delta(f) := \left\{ g : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|a_k - b_k| \le \delta \right\}.$$
Ruscheweyh [**56**] proved among other results that
$$N_{1/4}(f) \subset \mathcal{S}^*$$
for $f \in \mathcal{C}$. For a more general notion of $T$-$\delta$-neighbourhood of an analytic function, see Sheil-Small and Silvia [**60**]. Padmanabhan [**35**] investigated the neighbourhood problem for the class $\mathcal{UCV}$. Since the class $\mathcal{UCV}$ is closed under convolution with starlike functions of order 1/2, it follows that the function $(f(z)+\epsilon z)/(1+\epsilon) \in \mathcal{S}_{\mathcal{P}}$ for $|\epsilon| < 1/4$. Using
$$f \in \mathcal{S}_{\mathcal{P}} \Leftrightarrow \frac{1}{z}(f * h)(z) \ne 0, \quad t \in \mathbb{R}, z \in \mathbb{D},$$
where
$$h(z) := \frac{2}{1-2it-t^2}\left(\frac{z}{(1-z)^2} - \left(\frac{t^2+1}{2} + it\right)\frac{z}{1-z}\right),$$
Padmanabhan proved that $N_\delta(f) \subset \mathcal{S}_{\mathcal{P}}$ whenever
$$\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{S}_{\mathcal{P}}$$
for $|\epsilon| < \delta < 1$. These two assertions together show that
$$N_{1/8}(f) \subset \mathcal{S}_{\mathcal{P}}$$
for $f \in \mathcal{UCV}$. For some related results, see [**17**].

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School of Mathematical Sciences,
Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
*E-mail address*: `rosihan@cs.usm.my`

Department of Mathematics,
University of Delhi, Delhi–110 007, India
*E-mail address*: `vravi68@gmail.com`