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Differential subordination and superordination of analytic functions defined by the Dziok–Srivastava linear operator ☆

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Abstract

Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with the Dziok–Srivastava linear operator. These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}_n$ denote the class of all analytic functions of the form

$$f(z) = z^n + \sum_{k=n+1}^{\infty} a_k z^k \quad (z \in U) \tag{1.1}$$

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and let $\mathcal{A}_1 := \mathcal{A}$. Let f and F be members of $\mathcal{H}(U)$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z \in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$. For two functions f given by Eq. (1.1) and $g(z) = z^n + \sum_{k=n+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^n + \sum_{k=n+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2}$$

For $\alpha_j \in \mathbb{C}$ $(j=1,2,\ldots,l)$ and $\beta_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$ $(j=1,2,\ldots,m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$ is defined by the infinite series

$${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!} \quad (l \leq m+1;\ l,m \in \mathbb{N}_0 := \{0,1,2,\ldots\}),$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & (k=0), \\ a(a+1)(a+2)\ldots(a+k-1) & (k \in \mathbb{N} := \{1,2,3\ldots\}). \end{cases}$$

Corresponding to the function $h_n(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) := z^n\, {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$, the Dziok–Srivastava operator [12] (see also [22]) $H_n^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m) : \mathcal{A}_n \to \mathcal{A}_n$ is defined by the Hadamard product

$$\begin{aligned} H_n^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) &:= h_n(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) * f(z) \\ &= z^n + \sum_{k=n+1}^{\infty} \frac{(\alpha_1)_{k-n} \ldots (\alpha_l)_{k-n}}{(\beta_1)_{k-n} \ldots (\beta_m)_{k-n}} \frac{a_k z^k}{(k-n)!}. \end{aligned} \tag{1.3}$$

For brevity, we write

$$H_n^{l,m}[\alpha_1]f(z) := H_n^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z).$$

Special cases of the Dziok–Srivastava linear operator include the Hohlov linear operator [13], the Carlson–Shaffer linear operator [11], the Ruscheweyh derivative operator [21], the generalized Bernardi–Libera–Livingston linear integral operator [10,15,16] and the Srivastava–Owa fractional derivative operator [19,20]. See also [23] for a related operator.

We need the following definitions and theorems.

Definition 1.1 (*Miller and Mocanu [17, Definition 2.2b, p. 21]*). Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{U}\backslash E(q)$ where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0)=a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.2 (*Miller and Mocanu [17, Definition 2.3a, p. 27]*). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega$

whenever $r=q(\zeta)$, $s=k\zeta q'(\zeta)$, and

$$\mathrm{Re}\left\{\frac{t}{s}+1\right\}\geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z\in U$, $\zeta\in\partial U\backslash E(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.

In particular when $q(z)=M(Mz+a)/(M+\overline{a}z)$, with $M>0$ and $|a|<M$, then $q(U)=U_M:=\{w:|w|<M\}$, $q(0)=a$, $E(q)=\emptyset$ and $q\in\mathcal{Q}$. In this case, we set $\Psi_n[\Omega,M,a]:=\Psi_n[\Omega,q]$, and in the special case when the set $\Omega=U_M$, the class is simply denoted by $\Psi_n[M,a]$.

Definition 1.3 (*Miller and Mocanu [18, Definition 3, p. 817]*). Let Ω be a set in $\mathbb{C}$, $q\in\mathcal{H}[a,n]$ with $q'(z)\neq 0$. The class of admissible functions $\Psi'_n[\Omega,q]$ consists of those functions $\psi:\mathbb{C}^3\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\zeta)\in\Omega$ whenever $r=q(z)$, $s=zq'(z)/m$, and

$$\mathrm{Re}\left\{\frac{t}{s}+1\right\}\leq\frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z\in U$, $\zeta\in\partial U$ and $m\geq n\geq 1$. In particular, we write $\Psi'_1[\Omega,q]$ as $\Psi'[\Omega,q]$.

Theorem 1.1 (*Miller and Mocanu [17, Theorem 2.3b, p. 28]*). *Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If the analytic function $p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots$ satisfies*

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega,$$

then $p(z)\prec q(z)$.

Theorem 1.2 (*Miller and Mocanu [18, Theorem 1, p. 818]*). *Let $\psi\in\Psi'_n[\Omega,q]$ with $q(0)=a$. If $p\in\mathcal{Q}(a)$ and $\psi(p(z),zp'(z),z^2p''(z);z)$ is univalent in U, then*

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z);z):z\in U\}$$

implies $q(z)\prec p(z)$.

In the present investigation, the differential subordination result of Miller and Mocanu [17, Theorem 2.3b, p. 28] is extended for functions associated with the Dziok–Srivastava linear operator $H_n^{l,m}$, and we obtain certain other related results. A similar problem for analytic functions was studied by Aghalary et al. [1]. See [2–9,14] for related works. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination results involving the Dziok–Srivastava linear operator

We define the following class of admissible functions that will be required in our first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{Q}_0\cap\mathcal{H}[0,p]$. The class of admissible functions $\Phi_H[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z)\notin\Omega,$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\alpha_1 - n)q(\zeta)}{\alpha_1} \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1),$$

$$\mathrm{Re}\left\{\frac{\alpha_1(\alpha_1+1)w + (n-\alpha_1)(\alpha_1-n+1)u}{\alpha_1 v + (n-\alpha_1)u} - (2(\alpha_1-n)+1)\right\} \geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \backslash E(q)$ and $k \geq n$.

Theorem 2.1. *Let* $\phi \in \Phi_H[\Omega, q]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\{\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) : z \in U\} \subset \Omega, \tag{2.1}$$

then

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z), \quad (z \in U).$$

Proof. Define the analytic function p in U by

$$p(z) := H_n^{l,m}[\alpha_1]f(z). \tag{2.2}$$

In view of the relation

$$\alpha_1 H_n^{l,m}[\alpha_1+1]f(z) = z[H_n^{l,m}[\alpha_1]f(z)]' + (\alpha_1 - n)H_n^{l,m}[\alpha_1]f(z), \tag{2.3}$$

from Eq. (2.2), we get

$$H_n^{l,m}[\alpha_1+1]f(z) = \frac{zp'(z) + (\alpha_1-n)p(z)}{\alpha_1}. \tag{2.4}$$

Further computations show that

$$H_n^{l,m}[\alpha_1+2]f(z) = \frac{z^2p''(z) + 2(\alpha_1-n+1)zp'(z) + (\alpha_1-n)(\alpha_1-n+1)p(z)}{\alpha_1(\alpha_1+1)}. \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s + (\alpha_1-n)r}{\alpha_1}, \quad w = \frac{t + 2(\alpha_1-n+1)s + (\alpha_1-p)(\alpha_1-n+1)r}{\alpha_1(\alpha_1+1)}. \tag{2.6}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \phi(u,v,w;z)\\ &= \phi\left(r, \frac{s+(\alpha_1-n)r}{\alpha_1}, \frac{t + 2(\alpha_1-n+1)s + (\alpha_1-n)(\alpha_1-n+1)r}{\alpha_1(\alpha_1+1)}; z\right).\end{aligned} \tag{2.7}$$

The proof shall make use of Theorem 1.1. Using Eqs. (2.2), (2.4) and (2.5), from Eq. (2.7), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z). \tag{2.8}$$

Hence Eq. (2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Note that

$$\frac{t}{s}+1=\frac{\alpha_1(\alpha_1+1)w+(n-\alpha_1)(\alpha_1-n+1)u}{\alpha_1 v+(n-\alpha_1)u}-(2(\alpha_1-n)+1),$$

and hence $\psi \in \Psi_n[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z). \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let* $\phi \in \Phi_H[h, q]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) \prec h(z), \tag{2.9}$$

then

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z).$$

Our next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.1. *Let* $\Omega \subset \mathbb{C}$, q *be univalent in* U *and* $q(0)=0$. *Let* $\phi \in \Phi_H[\Omega, q_\rho]$ *for some* $\rho \in (0,1)$ *where* $q_\rho(z) = q(\rho z)$. *If* $f \in \mathcal{A}_n$ *and*

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) \in \Omega,$$

then

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $H_n^{l,m}[\alpha_1]f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. $\square$

Theorem 2.3. *Let* h *and* q *be univalent in* U, *with* $q(0)=0$ *and set* $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$. *Let* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ *satisfy one of the following conditions*:

1. $\phi \in \Phi_H[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*
2. *there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}_n$ *satisfies Eq.* (2.9), *then*

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z).$$

Proof. The result is similar to the proof of [17, Theorem 2.3d, p. 30] and is therefore omitted. $\square$

The next theorem yields the best dominant of the differential subordination (2.9).

Theorem 2.4. *Let h be univalent in U. Let* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and* ψ *be given by Eq.* (2.7). *Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{2.10}$$

has a solution q with $q(0)=0$ *and satisfy one of the following conditions*:

1. $q \in \mathcal{Q}_0$ *and* $\phi \in \Phi_H[h, q]$,
2. *q is univalent in U and* $\phi \in \Phi_H[h, q_\rho]$ *for some* $\rho \in (0,1)$, *or*
3. *q is univalent in U and there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}_n$ *satisfies Eq.* (2.9), *then*

$$H_n^{l,m}[\alpha_1]f(z) \prec q(z),$$

and q is the best dominant.

Proof. Following the same arguments in [17, Theorem 2.3e, p. 31], we deduce that q is a dominant from Theorems 2.2 and 2.3. Since q satisfies Eq. (2.10) it is also a solution of Eq. (2.9) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

In the particular case $q(z)=Mz$, $M>0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, is described below.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\alpha_1-n}{\alpha_1}Me^{i\theta}, \frac{L+(\alpha_1-n+1)(2k+\alpha_1-n)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right) \notin \Omega, \tag{2.11}$$

whenever $z \in U, \theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ, $\alpha_1 \in \mathbb{C}(\alpha_1 \neq 0, -1)$ and $k \geq n$.

Corollary 2.2. *Let* $\phi \in \Phi_H[\Omega, M]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) \in \Omega,$$

then

$$|H_n^{l,m}[\alpha_1]f(z)| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$.

Corollary 2.3. *Let* $\phi \in \Phi_H[M]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$|\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)| < M,$$

then

$$|H_n^{l,m}[\alpha_1]f(z)| < M.$$

Corollary 2.4. *If* $\operatorname{Re}\alpha_1 \geq 0$ *and* $f \in \mathcal{A}_n$ *satisfies*

$$|H_n^{l,m}[\alpha_1+1]f(z)| < M,$$

then

$$|H_n^{l,m}[\alpha_1]f(z)| < M.$$

Proof. This follows from Corollary 2.3 by taking $\phi(u,v,w;z) = v$. □

Remark 2.1. When $\Omega = U$ and $M=1$, Corollary 2.2 reduces to [1, Theorem 1, p. 269]. When $\Omega = U$, $n=1$, $l=2$, $m=1$, $\alpha_1 = \alpha+1, \alpha_2 = 1(\alpha > -1)$, $\beta_1 = 1$ and $M=1$, Corollary 2.2 reduces to [14, Theorem 1, p. 230]. When $M=1$, Corollary 2.4 is the same as [1, Corollary 3, p. 271]

Corollary 2.5. *Let* $M > 0$ *and* $0 \neq \alpha_1 \in \mathbb{C}$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$|H_n^{l,m}[\alpha_1+1]f(z) + \left(\frac{n}{\alpha_1}-1\right)H_n^{l,m}[\alpha_1]f(z)| < \frac{Mn}{|\alpha_1|}, \quad \textit{then } |H_n^{l,m}[\alpha_1]f(z)| < M. \tag{2.12}$$

Proof. Let $\phi(u,v,w;z) = v + (n/\alpha_1 - 1)u$ and $\Omega = h(U)$ where $h(z) = M/\alpha_1 z$, $M > 0$. To use Corollary 2.2, we need to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (2.11) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta}, \frac{k+\alpha_1-n}{\alpha_1}Me^{i\theta}, \frac{L+(\alpha_1-n+1)(2k+\alpha_1-n)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right)\right| = \frac{kM}{|\alpha_1|} \geq \frac{Mn}{|\alpha_1|},$$

$z \in U$, $\theta \in \mathbb{R}$, $\alpha_1 \in \mathbb{C}(\alpha_1 \neq 0, -1)$ and $k \geq n$. Hence by Corollary 2.2, we deduce the required result. □

We can use Theorem 2.4 to present a different proof of Corollary 2.5, and to also show that the result is sharp. The differential equation

$$zq'(z) + \left(\frac{n}{\alpha_1}-1\right)q(z) = \frac{Mn}{\alpha_1}z$$

has a univalent solution $q(z) = Mz$. By using Theorem 2.4, we see that $q(z) = Mz$ is the best dominant of Eq. (2.12).

Note that

$$H_n^{(2,1)}(1,1;1)f(z) = f(z),$$

$$H_n^{(2,1)}(2,1;1)f(z) = zf'(z) + (1-n)f(z),$$

$$H_n^{(2,1)}(3,1;1)f(z) = \tfrac{1}{2}[z^2 f''(z) + 2(2-n)zf'(z) + (1-n)(2-n)f(z)].$$

By taking $l=2$, $m=1$, $\alpha_1=\alpha_2=\beta_1=1$, Eq. (2.12) shows that for $f\in\mathcal{A}_n$, whenever

$$zf'(z)+n\left(\frac{1-\alpha_1}{\alpha_1}\right)f(z)\prec\frac{Mz}{|\alpha_1|},\quad \text{then } f(z)\prec Mz.$$

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{Q}_0\cap\mathcal{H}_0$. The class of admissible functions $\Phi_{H,1}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ satisfying the admissibility condition

$$\phi(u,v,w;z)\notin\Omega,$$

whenever

$$u=q(\zeta),\quad v=[k\zeta q'(\zeta)+(\alpha_1-1)q(\zeta)]/\alpha_1\quad(\alpha_1\in\mathbb{C},\alpha_1\neq 0,-1),$$

$$\operatorname{Re}\left\{\frac{\alpha_1[(\alpha_1+1)w+(1-\alpha_1)u]}{v\alpha_1+(1-\alpha_1)u}+1-2\alpha_1\right\}\geq k\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z\in U$, $\zeta\in\partial U\backslash E(q)$ and $k\geq 1$.

Theorem 2.5. *Let $\phi\in\Phi_{H,1}[\Omega,q]$. If $f\in\mathcal{A}_n$ satisfies*

$$\left\{\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}};z\right):z\in U\right\}\subset\Omega,\tag{2.13}$$

then

$$\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}\prec q(z).$$

Proof. Define the analytic function p in U by

$$p(z):=\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}.\tag{2.14}$$

By making use of Eqs. (2.3) and (2.14), we get

$$\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}=\frac{1}{\alpha_1}(zp'(z)+(\alpha_1-1)p(z)).\tag{2.15}$$

Further computations show that

$$\frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}=\frac{1}{\alpha_1(\alpha_1+1)}(z^2p''(z)+2\alpha_1 zp'(z)+\alpha_1(\alpha_1-1)p(z)).\tag{2.16}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r, v=\frac{s+(\alpha_1-1)r}{\alpha_1},\quad w=\frac{t+2\alpha_1 s+\alpha_1(\alpha_1-1)r}{\alpha_1(\alpha_1+1)}.\tag{2.17}$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)=\phi\left(r,\frac{s+(\alpha_1-1)r}{\alpha_1},\frac{t+2\alpha_1 s+\alpha_1(\alpha_1-1)r}{\alpha_1(\alpha_1+1)};z\right).\tag{2.18}$$

The proof shall make use of Theorem 1.1. Using Eqs. (2.14)–(2.16), from Eq. (2.18), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right). \tag{2.19}$$

Hence Eq. (2.13) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Note that

$$\frac{t}{s} + 1 = \frac{\alpha_1[(\alpha_1+1)w + (1-\alpha_1)u]}{v\alpha_1 + (1-\alpha_1)u} + 1 - 2\alpha_1,$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or

$$\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}} \prec q(z). \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$. In the particular case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$ is denoted by $\Phi_{H,1}[\Omega, M]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let $\phi \in \Phi_{H,1}[h, q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right) \prec h(z), \tag{2.20}$$

then

$$\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}} \prec q(z).$$

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}Me^{i\theta}, \frac{L + \alpha_1(2k+\alpha_1-1)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right) \notin \Omega, \tag{2.21}$$

whenever $z \in U, \theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ, $\alpha_1 \in \mathbb{C}$ $(\alpha_1 \neq 0, -1)$ and $k \geq 1$.

Corollary 2.6. *Let $\phi \in \Phi_{H,1}[\Omega, M]$. If $f \in \mathcal{A}_n$ satisfies*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right) \in \Omega,$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}\right| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$.

Corollary 2.7. *Let* $\phi \in \Phi_{H,1}[M]$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right)\right| < M,$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}\right| < M.$$

Corollary 2.8. *If* $\operatorname{Re}\alpha_1 \geq 0$ *and* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}\right| < M,$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}\right| < M.$$

Proof. This follows from Corollary 2.7 by taking $\phi(u, v, w; z) = v$. □

Remark 2.2. When $\Omega = U$, $n=1$, $l=2$, $m=1$, $\alpha_1 = \alpha+1, \alpha_2 = 1 (\alpha > -1)$, $\beta_1 = 1$ and $M=1$, Corollary 2.6 reduces to [14, Theorem 1, p. 230]. When $n=1$, $l=2$, $m=1$, $\alpha_1 = \alpha+1, \alpha_2 = 1 (\alpha > -1)$, $\beta_1 = 1$ and $M=1$, Corollary 2.8 is the same as [14, Corollary 1, p. 231].

Example 2.1. If $\delta \geq 0$ and $f \in \mathcal{A}$ satisfies

$$\left|\delta\left(\frac{zf''(z)}{f'(z)}+1\right) + (1-\delta)\frac{zf'(z)}{f(z)}\right| < 1, \quad \text{then } |f(z)| < 1. \tag{2.22}$$

Proof. Let $\phi(u, v, w; z) = \delta(2w/v-1) + (1-\delta)v/u$ for all real $\delta \geq 0$, $n=1$, $l=2$, $m=1$, $\alpha_1 = 1, \alpha_2 = 1$, $\beta_1 = 1$, $M=1$ and $\Omega = h(U)$ where $h(z)=z$. To use Corollary 2.6, we need to show that $\phi \in \Phi_{H,1}[\Omega, M] \equiv \Phi_{H,1}[U]$, that is, the admissibility condition (2.21) is satisfied. This follows since

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}Me^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right)\right| \\
&\quad = \left|\delta\left(\frac{Le^{-i\theta}}{k}+1\right) + (1-\delta)k\right| \geq \delta + (1-\delta)k + \frac{\delta}{k}\operatorname{Re}(Le^{-i\theta}) \\
&\quad \geq \delta + (1-\delta)k + \frac{\delta}{k}k(k-1) = k \geq 1,
\end{aligned}$$

$z \in U, \theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq k(k-1)$ and $k \geq 1$. Hence by Corollary 2.6, we deduce the required result. □

Definition 2.5. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying the admissibility condition

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = \frac{1}{\alpha_1 + 1}\left(1 + \alpha_1 q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right) \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1, -2, q(\zeta) \neq 0),$$

$$\mathrm{Re}\left\{\frac{[(\alpha_1 + 1)(w - v) + w - 1](1 + \alpha_1)v}{(1 + \alpha_1)v - (1 + \alpha_1 u)} + (1 + \alpha_1)v - (2\alpha_1 u + 1)\right\} \geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 2.7. *Let $\phi \in \Phi_{H,2}[\Omega, q]$. If $f \in \mathcal{A}_n$ satisfies*

$$\left\{\phi\left(\frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 2]f(z)}{H_n^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 3]f(z)}{H_n^{l,m}[\alpha_1 + 2]f(z)}; z\right) : z \in U\right\} \subset \Omega, \tag{2.23}$$

then

$$\frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)} \prec q(z).$$

Proof. Define the analytic function p in U by

$$p(z) := \frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}. \tag{2.24}$$

Using Eq. (2.24), we get

$$\frac{zp'(z)}{p(z)} := \frac{z[H_n^{l,m}[\alpha_1 + 1]f(z)]'}{H_n^{l,m}[\alpha_1 + 1]f(z)} - \frac{z[H_n^{l,m}[\alpha_1]f(z)]'}{H_n^{l,m}[\alpha_1]f(z)}. \tag{2.25}$$

By making use of Eq. (2.3) in Eq. (2.25), we get

$$\frac{H_n^{l,m}[\alpha_1 + 2]f(z)}{H_n^{l,m}[\alpha_1 + 1]f(z)} = \frac{1}{\alpha_1 + 1}\left(\alpha_1 p(z) + 1 + \frac{zp'(z)}{p(z)}\right). \tag{2.26}$$

Further computations show that

$$\frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)}=\frac{1}{\alpha_1+2}\left(2+\alpha_1 p(z)+\frac{zp'(z)}{p(z)}+\frac{\alpha_1 zp'(z)+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^2p''(z)}{p(z)}}{1+\alpha_1 p(z)+\frac{zp'(z)}{p(z)}}\right). \tag{2.27}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r, v=\frac{1}{\alpha_1+1}\left(1+\alpha_1 r+\frac{s}{r}\right),$$

$$w=\frac{1}{\alpha_1+2}\left(2+\alpha_1 r+\frac{s}{r}+\frac{\alpha_1 s+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}}{1+\alpha_1 r+\frac{s}{r}}\right). \tag{2.28}$$

Let

$$\begin{aligned}\psi(r,s,t;z)&:=\phi(u,v,w;z)\\ &=\phi\left(r,\frac{1}{\alpha_1+1}\left[\alpha_1 r+1+\frac{s}{r}\right],\frac{1}{\alpha_1+2}\left(2+\alpha_1 r+\frac{s}{r}+\frac{\alpha_1 s+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}}{1+\alpha_1 r+\frac{s}{r}}\right);z\right).\end{aligned} \tag{2.29}$$

The proof shall make use of Theorem 1.1. Using Eqs. (2.24), (2.26) and (2.27), from Eqs. (2.29), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)},\frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)};z\right). \tag{2.30}$$

Hence Eq. (2.23) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{H,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Note that

$$\frac{t}{s}+1=\frac{[(\alpha_1+1)(w-v)+w-1](1+\alpha_1)v}{(1+\alpha_1)v-(1+\alpha_1 u)}+(1+\alpha_1)v-(2\alpha_1 u+1),$$

and hence $\psi\in\Psi[\Omega,q]$. By Theorem 1.1, $p(z)\prec q(z)$ or

$$\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}\prec q(z). \qquad \square$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{H,2}[h(U),q]$ is written as $\Phi_{H,2}[h,q]$. In the particular case $q(z)=1+Mz$, $M>0$, the class of admissible functions $\Phi_{H,2}[\Omega,q]$ is denoted by $\Phi_{H,2}[\Omega,M]$. The following result is an immediate consequence of Theorem 2.7.

Theorem 2.8. *Let* $\phi\in\Phi_{H,2}[h,q]$. *If* $f\in\mathcal{A}_n$ *satisfies*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)},\frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)};z\right)\prec h(z), \tag{2.31}$$

then

$$\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}\prec q(z).$$

Definition 2.6. Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_{H,2}[\Omega,M]$ consists of those functions $\phi:\mathbb{C}^3\times U\to\mathbb{C}$ such that

$$\begin{aligned}\phi\Bigg(1+Me^{i\theta},1+\frac{k+\alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta})}Me^{i\theta},1+\frac{\alpha_1(1+Me^{i\theta})+k}{(\alpha_1+2)(1+Me^{i\theta})}Me^{i\theta}\\+\frac{(M+e^{-i\theta})[Le^{-i\theta}+kM(\alpha_1+1)+\alpha_1kM^2e^{i\theta}]-k^2M^2}{(\alpha_1+2)(M+e^{-i\theta})[\alpha_1M^2e^{i\theta}+(1+\alpha_1)e^{-i\theta}+M(1+2\alpha_1+k)]};z\Bigg)\notin\Omega,\end{aligned} \tag{2.32}$$

whenever $z\in U,\theta\in\mathbb{R}$, $\mathrm{Re}(Le^{-i\theta})\geq(k-1)kM$ for all real θ, $\alpha_1\in\mathbb{C}(\alpha_1\neq0,-1,-2)$ and $k\geq1$.

Corollary 2.9. *Let* $\phi\in\Phi_{H,2}[\Omega,M]$. *If* $f\in\mathcal{A}_n$ *satisfies*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)},\frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)};z\right)\in\Omega,$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}-1\right|<M.$$

In the special case $\Omega=q(U)=\{\omega:|\omega-1|<M\}$, the class $\Phi_{H,2}[\Omega,M]$ is simply denoted by $\Phi_{H,2}[M]$.

Corollary 2.10. *Let* $\phi\in\Phi_{H,2}[M]$. *If* $f\in\mathcal{A}_n$ *satisfies*

$$\left|\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)},\frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)};z\right)-1\right|<M,$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}-1\right|<M.$$

Corollary 2.11. *If* $M>0$, $\alpha_1\in\mathbb{C}(\alpha_1\neq0,-1)$ *and* $f\in\mathcal{A}_n$ *satisfies*

$$\left|\frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)}-\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}\right|<\frac{M^2}{|1+\alpha_1|(1+M)}, \tag{2.33}$$

then

$$\left|\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}-1\right|<M.$$

Proof. This follows from Corollary 2.9 by taking $\phi(u,v,w;z)=v-u$ and $\Omega=h(U)$ where $h(z)=(M^2/|1+\alpha_1|(1+M))z$. To use Corollary 2.9 we need to show that $\phi\in\Phi_{H,2}[\Omega,M]$, that is, the admissibility condition (2.32) is satisfied. This follows since

$$|\phi(u,v,w;z)|=\left|-1-Me^{i\theta}+1+\frac{k+\alpha_1(1+Me^{i\theta})}{(1+\alpha_1)(1+Me^{i\theta})}Me^{i\theta}\right|=\frac{M}{|1+\alpha_1|}\left|\frac{k-1-Me^{i\theta}}{1+Me^{i\theta}}\right|$$
$$\geq\frac{M}{|1+\alpha_1|}\left|\frac{k-1-M}{1+M}\right|\geq\frac{M}{|1+\alpha_1|}\left|\frac{1}{1+M}-1\right|=\frac{M^2}{|1+\alpha_1|(1+M)},$$

$z\in U$, $\theta\in\mathbb{R}$, $\alpha_1\in\mathbb{C}$ $(\alpha_1\neq 0,-1)$, $k\neq 1+M$ and $k\geq 1$. Hence by Corollary 2.9, we deduce the required result. $\square$

By taking $l=2$, $m=1$, $\alpha_1=\alpha_2=\beta_1=1$, Eq. (2.33) shows that for $f\in\mathcal{A}_n$, whenever

$$\frac{\dfrac{zf'(z)}{f(z)}\left[\dfrac{zf''(z)}{f'(z)}-2\dfrac{zf'(z)}{f(z)}+n\right]}{\dfrac{zf'(z)}{f(z)}-n+1}\prec -n+\frac{M^2}{1+M}z \quad \text{then} \quad \frac{zf'(z)}{f(z)}\prec Mz+n.$$

Example 2.2. If $f\in\mathcal{A}$, then

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right)\right|<1\Rightarrow\left|\frac{zf'(z)}{f(z)}-1\right|<1. \tag{2.34}$$

Proof. This follows from Corollary 2.9 by taking $\phi(u,v,w;z)=u(2v-1-u)$, $n=1$, $l=2$, $m=1$, $\alpha_1=1,\alpha_2=1$, $\beta_1=1$, $M=1$ and $\Omega=h(U)$ where $h(z)=z$. $\square$

Example 2.3. If $M>0$ and $f\in\mathcal{A}$ satisfies

$$\left|\frac{\dfrac{zf''(z)}{f'(z)}+1}{\dfrac{zf'(z)}{f(z)}}-1\right|<M, \tag{2.35}$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<M. \tag{2.36}$$

Proof. This follows from Corollary 2.10 by taking $\phi(u,v,w;z)=(2v-1)/u$, $n=1$, $l=2$, $m=1$, $\alpha_1=1,\alpha_2=1$ and $\beta_1=1$. $\square$

3. Superordination of the Dziok–Srivastava linear operator

The dual problem of differential subordination, that is, differential superordination of the Dziok–Srivastava linear operator is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;\zeta) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\alpha_1 - p)q(z)}{m\alpha_1} \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1),$$

$$\mathrm{Re}\left\{\frac{\alpha_1(\alpha_1+1)w + (n-\alpha_1)(\alpha_1-n+1)u}{\alpha_1 v + (n-\alpha_1)u} - [2(n-\alpha_1)+1]\right\} \leq \frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n$.

Theorem 3.1. *Let $\phi \in \Phi'_H[\Omega,q]$. If $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f \in \mathcal{Q}_0$ and*

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)$$

is univalent in U, then

$$\Omega \subset \{\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) : z \in U\} \tag{3.1}$$

implies

$$q(z) \prec H_n^{l,m}[\alpha_1]f(z).$$

Proof. From (2.8) and (3.1), we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}.$$

From (2.6), we see that the admissibility condition for $\phi \in \Phi'_H[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'_n[\Omega,q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec H_n^{l,m}[\alpha_1]f(z). \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_H[h(U),q]$ is written as $\Phi'_H[h,q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let h be analytic in U and $\phi \in \Phi'_H[h,q]$. If $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_0$ and $\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)$ is univalent in U, then*

$$h(z) \prec \phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) \tag{3.2}$$

implies

$$q(z) \prec H_n^{l,m}[\alpha_1]f(z).$$

Theorems 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ.

Theorem 3.3. *Let h be analytic in U and $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and ψ be given by Eq.* (2.7). *Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_H[h, q]$, $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_0$ and

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)$$

is univalent in U, then

$$h(z) \prec \phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)$$

implies

$$q(z) \prec H_n^{l,m}[\alpha_1]f(z)$$

and $q(z)$ is the best subordinant.

Proof. The result is similar to the proof of Theorem 2.4 and is therefore omitted. □

Combining Theorems 2.2 and 3.2, we obtain the following sandwich-type theorem.

Corollary 3.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent function in U, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z) \in \mathcal{H}[0, p] \cap \mathcal{Q}_0$ and*

$$\phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z)$$

is univalent in U, then

$$h_1(z) \prec \phi(H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1+1]f(z), H_n^{l,m}[\alpha_1+2]f(z); z) \prec h_2(z),$$

implies

$$q_1(z) \prec H_n^{l,m}[\alpha_1]f(z) \prec q_2(z).$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega, \tag{3.3}$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(\alpha_1 - 1)q(z)}{m\alpha_1} \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1),$$

$$\mathrm{Re}\left\{\frac{\alpha_1[(\alpha_1+1)w+(1-\alpha_1)u]}{\alpha_1 v+(1-\alpha_1)u}+(1-2\alpha_1)\right\}\le\frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z\in U$, $\zeta\in\partial U$ and $m\ge 1$.

Next we will give the dual result of Theorem 2.5 for differential superordination.

Theorem 3.4. *Let* $\phi\in\Phi'_{H,1}[\Omega,q]$. *If* $f\in\mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z)/z^{n-1}\in\mathcal{Q}_0$ *and*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}};z\right)$$

is univalent in U, *then*

$$\Omega\subset\left\{\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}};z\right):z\in U\right\}\tag{3.4}$$

implies

$$q(z)\prec\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}.$$

Proof. From (2.19) and (3.4), we have

$$\Omega\subset\{\phi(p(z),zp'(z),z^2p''(z);z):z\in U\}.$$

From (2.17), we see that the admissibility condition for $\phi\in\Phi'_{H,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi\in\Psi'[\Omega,q]$, and by Theorem 1.2, $q(z)\prec p(z)$ or

$$q(z)\prec H_n^{l,m}[\alpha_1]f(z).\qquad\square$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{H,1}[h(U),q]$ is written as $\Phi'_{H,1}[h,q]$. The following result is an immediate consequence of Theorem 3.4.

Theorem 3.5. *Let* $q\in\mathcal{H}_0$, h *is analytic on* U *and* $\phi\in\Phi'_{H,1}[h,q]$. *If* $f\in\mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z)/z^{n-1}\in\mathcal{Q}_0$ *and* $\phi(H_n^{l,m}[\alpha_1]f(z)/z^{n-1},H_n^{l,m}[\alpha_1+1]f(z)/z^{n-1},H_n^{l,m}[\alpha_1+2]f(z)/z^{n-1};z)$ *is univalent in* U, *then*

$$h(z)\prec\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}},\frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}};z\right)\tag{3.5}$$

implies

$$q(z)\prec\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}.$$

Combining Theorems 2.6 and 3.5, we obtain the following sandwich-type theorem.

Corollary 3.2. *Let* h_1 *and* q_1 *be analytic functions in* U, h_2 *be univalent function in* U, $q_2\in\mathcal{Q}_0$ *with* $q_1(0)=q_2(0)=0$ *and* $\phi\in\Phi_{H,1}[h_2,q_2]\cap\Phi'_{H,1}[h_1,q_1]$. *If* $f\in\mathcal{A}_n$, $H_n^{l,m}[\alpha_1]f(z)/z^{n-1}\in\mathcal{H}_0\cap$

$\mathcal{Q}_0$ *and*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right)$$

is univalent in U, *then*

$$h_1(z) \prec \phi\left(\frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+1]f(z)}{z^{n-1}}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{z^{n-1}}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{H_n^{l,m}[\alpha_1]f(z)}{z^{n-1}} \prec q_2(z).$$

Definition 3.3. Let Ω be a set in $\mathbb{C}$, $q(z)\neq 0$, $zq'(z)\neq 0$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{1}{\alpha_1+1}\left(1+\alpha_1 q(z) + \frac{zq'(z)}{mq(z)}\right) \quad (\alpha_1 \in \mathbb{C}, \alpha_1 \neq 0, -1, -2),$$

$$\mathrm{Re}\left\{\frac{[(\alpha_1+1)(w-v)+w-1](1+\alpha_1)v}{(1+\alpha_1)v-(1+\alpha_1 u)} + (1+\alpha_1)v-(2\alpha_1 u+1)\right\} \leq \frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Now we will give the dual result of Theorem 2.7 for differential superordination.

Theorem 3.6. *Let* $\phi \in \Phi'_{H,2}[\Omega, q]$. *If* $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1+1]f(z)/H_n^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)}, \frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)}; z\right)$$

is univalent in U, *then*

$$\Omega \subset \left\{\phi\left(\frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1+2]f(z)}{H_n^{l,m}[\alpha_1+1]f(z)}, \frac{H_n^{l,m}[\alpha_1+3]f(z)}{H_n^{l,m}[\alpha_1+2]f(z)}; z\right) : z \in U\right\} \tag{3.6}$$

implies

$$q(z) \prec \frac{H_n^{l,m}[\alpha_1+1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}.$$

Proof. From (2.30) and (3.6), we have

$$\Omega \subset \{\phi(p(z), zp'(z), z^2p''(z); z) : z \in U\}.$$

In view of (2.28), the admissibility condition for $\phi \in \Phi'_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}. \qquad \square$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi'_{H,2}[h(U), q]$ is written as $\Phi'_{H,2}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.6.

Theorem 3.7. *Let $q \in \mathcal{H}$, h be analytic in U and $\phi \in \Phi'_{H,2}[h, q]$. If $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1 + 1]f(z)/H_n^{l,m}[\alpha_1]f(z) \in \mathcal{Q}_1$ and $\phi(H_n^{l,m}[\alpha_1 + 1]f(z)/H_n^{l,m}[\alpha_1]f(z), H_n^{l,m}[\alpha_1 + 2]f(z)/H_n^{l,m}[\alpha_1 + 1]f(z), H_n^{l,m}[\alpha_1 + 3]f(z)/H_n^{l,m}[\alpha_1 + 2]f(z); z)$ is univalent in U, then*

$$h(z) \prec \phi\left(\frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 2]f(z)}{H_n^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 3]f(z)}{H_n^{l,m}[\alpha_1 + 2]f(z)}; z\right) \tag{3.7}$$

implies

$$q(z) \prec \frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}.$$

Combining Theorems 2.8 and 3.7 we obtain the following sandwich-type theorem.

Corollary 3.3. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent function in U, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,2}[h_2, q_2] \cap \Phi'_{H,2}[h_1, q_1]$. If $f \in \mathcal{A}_n$, $H_n^{l,m}[\alpha_1 + 1]f(z)/H_n^{l,m}[\alpha_1]f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 2]f(z)}{H_n^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 3]f(z)}{H_n^{l,m}[\alpha_1 + 2]f(z)}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 2]f(z)}{H_n^{l,m}[\alpha_1 + 1]f(z)}, \frac{H_n^{l,m}[\alpha_1 + 3]f(z)}{H_n^{l,m}[\alpha_1 + 2]f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{H_n^{l,m}[\alpha_1 + 1]f(z)}{H_n^{l,m}[\alpha_1]f(z)} \prec q_2(z).$$

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