



On Bernardi's integral operator and the Briot–Bouquet differential subordination [☆]

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Received 30 August 2005

Available online 18 January 2006

Submitted by S. Ruscheweyh

Abstract

Let $A, B, D, E \in [-1, 1]$. Conditions on A, B, D and E are determined so that

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez} \quad \text{implies} \quad p(z) \prec \frac{1 + Az}{1 + Bz}.$$

The result is applied to Bernardi's integral operator of two classes of analytic functions.

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Keywords: Analytic functions; Univalent functions; Starlike functions; Bernardi's integral operator

1. Introduction

Let \mathcal{A} be the class of all analytic functions $f(z)$ defined in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $S^*[A, B]$ denote the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

[☆] The authors gratefully acknowledged support from the research grant IRPA 09-02-05-00020 EAR.

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or the equivalent inequality

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \left| A - B \frac{zf'(z)}{f(z)} \right| \quad (z \in \Delta, -1 \leq B < A \leq 1).$$

Functions in $S^*[A, B]$ are called the *Janowski starlike functions* [3,6].

For $0 \leq \alpha < 1$, the class $S^*[1 - 2\alpha, -1]$ is the familiar class S^*_α of starlike functions of order α , while $S^*[1 - \alpha, 0]$ is the class $S^*(\alpha)$ of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in \Delta, 0 \leq \alpha < 1).$$

For $0 < \alpha \leq 1$, $S^*[\alpha, -\alpha] =: S^*[\alpha]$ is the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \left| \frac{zf'(z)}{f(z)} + 1 \right| \quad (z \in \Delta, 0 < \alpha \leq 1).$$

For this latter class $S^*[\alpha]$, Parvatham proved the following:

Theorem 1.1. [4, Theorem 1, p. 438] *Let $c \geq 0, 0 < \alpha \leq 1$ and δ be given by*

$$\delta := \alpha \left[\frac{2 + \alpha + c(1 - \alpha)}{1 + 2\alpha + c(1 - \alpha)} \right].$$

If $f \in S^[\delta]$, then the function $F(z)$ given by Bernardi's integral*

$$F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt \tag{1.1}$$

is in $S^[\alpha]$.*

It is well known [2] that the classes of starlike, convex and close-to-convex functions are closed under Bernardi's integral operator. Since $\delta \geq \alpha$, Theorem 1.1 extends the result of Bernardi [2].

Parvatham also considered a similar problem for the class $R[\alpha]$ of functions $f \in \mathcal{A}$ satisfying

$$|f'(z) - 1| < \alpha |f'(z) + 1| \quad (z \in \Delta, 0 < \alpha \leq 1),$$

and proved the following:

Theorem 1.2. [4, Theorem 2, p. 440] *Let $c \geq 0, 0 < \alpha \leq 1$ and δ be given by*

$$\delta := \alpha \left[\frac{2 - \alpha + c(1 - \alpha)}{1 + c(1 - \alpha)} \right].$$

If $f \in R[\delta]$, then the function $F(z)$ given by Bernardi's integral (1.1) is in $R[\alpha]$.

The class $R[\alpha]$ can be extended to the bigger class $R[A, B]$ consisting of all analytic functions $f(z) \in \mathcal{A}$ satisfying

$$f'(z) < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1),$$

or in other words,

$$|f'(z) - 1| < |A - Bf'(z)| \quad (z \in \Delta, -1 \leq B < A \leq 1).$$

For $0 \leq \alpha < 1$, the class $R[1 - 2\alpha, -1]$ consists of functions $f \in \mathcal{A}$ for which

$$\Re f'(z) > \alpha \quad (z \in \Delta, 0 < \alpha \leq 1),$$

and $R[1 - \alpha, 0] =: R_\alpha$ is the class of functions $f \in \mathcal{A}$ satisfying the condition

$$|f'(z) - 1| < 1 - \alpha \quad (z \in \Delta, 0 \leq \alpha < 1).$$

When $0 < \alpha \leq 1$, the class $R[\alpha, -\alpha]$ is the class $R[\alpha]$ considered by Parvatham [4].

In this paper, we extend Theorems 1.1 and 1.2 to hold true for the more general classes $S^*[A, B]$ and $R[A, B]$, respectively. We shall in fact obtain a more general result relating to the Briot–Bouquet differential subordination, and then apply it to Bernardi’s integral operator of the classes $S^*[D, E]$ and $R[D, E]$. The proofs are, however, very computationally involved.

2. A Briot–Bouquet differential subordination

Theorem 2.1. *Let $-1 \leq B < A \leq 1$ and $-1 \leq E \leq 0 < D \leq 1$. For $\beta \geq 0$ and $\beta + \gamma > 0$, let $G := A\beta + B\gamma$, $H := (\beta + \gamma)(D - E)$, $I := (A\beta + B\gamma)(D - E) + (BD - AE)(\beta + \gamma) - kE(A - B)$, $J := (A\beta + B\gamma)(BD - AE)$, and $L := \beta + \gamma + k$. In addition, for all $k \geq 1$, let*

$$(L^2 + G^2)[(H + J)I - 4H|J|] + 4LG H J \geq LG[(H - J)^2 + I^2]. \tag{2.1}$$

Further assume that

$$\frac{[\beta(1 + A) + \gamma(1 + B) + 1](A - B)}{[\beta(1 + A) + \gamma(1 + B)][D(1 + B) - E(1 + A)] - E(A - B)} \geq 1. \tag{2.2}$$

Let $p(z)$ be analytic in Δ with $p(0) = 1$. If

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez},$$

then

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Proof. Define $P(z)$ by

$$P(z) := p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$$

and $w(z)$ by

$$w(z) := \frac{p(z) - 1}{A - Bp(z)},$$

or equivalently by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.3}$$

Then $w(z)$ is meromorphic in Δ and $w(0) = 0$. We need to show that $|w(z)| < 1$ in Δ . By a computation from (2.3), we get

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{(A - B)zw'(z)}{(1 + Bw(z))[\beta(1 + Aw(z)) + \gamma(1 + Bw(z))]}.$$

Therefore

$$\frac{P(z) - 1}{D - EP(z)} = \frac{(A - B)[(\beta + \gamma)w(z) + (A\beta + B\gamma)w^2(z) + zw'(z)]}{[(D - E) + (BD - AE)w(z)][\beta + \gamma + (A\beta + B\gamma)w(z)] - E(A - B)zw'(z)}.$$

Assume that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by [5, Lemma 1.3, p. 28], there exists $k \geq 1$ such that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this z_0 , we have

$$\left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| = \left| \frac{(A - B)[L + Gw(z_0)]}{H + Iw(z_0) + Jw(z_0)^2} \right| = (A - B)[\varphi(\cos \theta)]^{1/2},$$

where

$$\varphi(\cos \theta) := \frac{|L + Ge^{i\theta}|^2}{|He^{-i\theta} + Je^{i\theta} + I|^2} = \frac{L^2 + G^2 + 2LG \cos \theta}{H^2 + J^2 + I^2 + 2HJ \cos 2\theta + 2I(H + J) \cos \theta}.$$

In view of the fact that

$$\min\{at^2 + bt + c : -1 \leq t \leq 1\} = \begin{cases} \frac{4ac - b^2}{4a}, & \text{if } a > 0 \text{ and } |b| < 2a, \\ a - |b| + c, & \text{otherwise,} \end{cases}$$

the function

$$\varphi(t) := \frac{L^2 + G^2 + 2L G t}{4H J t^2 + 2I(H + J)t + (H - J)^2 + I^2}$$

is easily seen to be a decreasing function of $t = \cos \theta$ provided (2.1) holds. Thus we have $\varphi(t) \geq \varphi(1) = [(L + G)/(I + J + H)]^2$. Yet another calculation shows that the function $\psi(k) := (L + G)/(I + J + H)$ is an increasing function of k . Since $k \geq 1$, we have $\psi(k) \geq \psi(1)$ and therefore

$$\left| \frac{P(z_0) - 1}{D - EP(z_0)} \right| \geq \frac{[\beta(1 + A) + \gamma(1 + B) + 1](A - B)}{[\beta(1 + A) + \gamma(1 + B)][D(1 + B) - E(1 + A)] - E(A - B)},$$

which by (2.2) is greater than or equal to 1. This contradicts that $P(z) < (1 + Dz)/(1 + Ez)$ and completes the proof. \square

3. Bernardi’s integral operator on $S^*[D, E]$ and $R[D, E]$

Upon differentiating Bernardi’s integral (1.1), we obtain

$$(c + 1)f(z) = zF'(z) + cF(z).$$

Logarithmic differentiation now yields

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z) + c},$$

with $p(z) = zF'(z)/F(z)$.

Theorem 3.1. *Let the conditions of Theorem 2.1 hold with $\beta = 1$ and $\gamma = c > -1$. If $f \in S^*[D, E]$, then the function $F(z)$ given by Bernardi’s integral (1.1) is in $S^*[A, B]$.*

Observe that when $J = 0$, condition (2.1) reduces to the equivalent form

$$(LI - GH)(LH - GI) \geq 0. \tag{3.1}$$

Remark 3.1. If $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ ($0 < \alpha, \delta \leq 1$), then $G = \alpha(1 - c)$, $H = 2\delta(1 + c)$, $I = 2\alpha\delta(1 + k - c)$, $J = 0$ and $L = 1 + c + k$. Since $J = 0$, we need to verify condition (3.1). In this case, $LI - GH = 2\alpha\delta k(2 + k) > 0$. In addition, $LH - GI \geq 0$ becomes $(1 + c)(1 + c + k) \geq \alpha^2(1 - c)(1 - c + k)$. Clearly this condition holds when $c \geq 0$. In the case $-1 < c < 0$, since

$$\frac{(1 + c)(2 + c)}{(1 - c)(2 - c)} \leq \frac{(1 + c)(1 + c + k)}{(1 - c)(1 - c + k)},$$

condition (3.1) holds provided $\alpha^2 \leq (1 + c)(2 + c)/((1 - c)(2 - c))$. Thus Theorem 3.1 not only reduces to Theorem 1.1 for $c \geq 0$, but also extends it for the case $-1 < c < 0$.

Corollary 3.1. Let $-1 < c < 0$, $0 < \alpha \leq \sqrt{(1 + c)(2 + c)/((1 - c)(2 - c))}$, and δ be as in Theorem 1.1. If $f \in S^*[\delta]$, then the function $F(z)$ given by Bernardi’s integral (1.1) belongs to $S^*[\alpha]$.

Remark 3.2. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ ($0 \leq \alpha, \delta < 1$), we have $G = 1 - \alpha$, $H = (1 - \delta)(1 + c)$, $I = (1 - \alpha)(1 - \delta)$, $J = 0$ and $L = 1 + c + k$. Since $J = 0$, condition (3.1) reduces to

$$(1 + c)(1 + c + k) - (1 - \alpha)^2 \geq 0. \tag{3.2}$$

Since $(1 + c)(1 + c + k) - (1 - \alpha)^2 \geq (1 + c)(2 + c) - (1 - \alpha)^2$, inequality (3.2) holds provided $\alpha \geq 1 - \sqrt{(1 + c)(2 + c)}$. This condition holds for $c \geq (\sqrt{4(\alpha - 1)^2 + 1} - 3)/2$. This yields the following result for the class $S^*(\delta)$.

Corollary 3.2. Let $\delta := \alpha - (1 - \alpha)/(2 + c - \alpha)$, $f(z) \in S^*(\delta)$ and $F(z)$ be given by Bernardi’s integral (1.1). If $\alpha_0 \leq \alpha < 1$, then $F(z) \in S^*(\alpha)$ for all $c > -1$. Here $\alpha_0 := (3 + c - \sqrt{(3 + c)^2 - 4})/2$.

Theorem 3.2. Under the conditions stated in Theorem 2.1 with $\beta = 0$ and $\gamma = c + 1$, if $f \in R[D, E]$, then the function $F(z)$ given by Bernardi’s integral (1.1) is in $R[A, B]$.

Proof. Since

$$(c + 1)f(z) = zF'(z) + cF(z),$$

we obtain

$$f'(z) = \frac{zF''(z)}{c + 1} + F'(z). \tag{3.3}$$

The result now follows from Theorem 2.1 with $p(z) = F'(z)$, $\beta = 0$ and $\gamma = c + 1$. \square

Remark 3.3. For $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ ($0 < \alpha, \delta \leq 1$), then $G = -\alpha(1 + c)$, $H = 2\delta(1 + c)$, $I = 2\alpha\delta(k - 1 - c)$, $J = 0$ and $L = 1 + c + k$. Condition (3.1) becomes

$$4\alpha\delta^2k^2(1 + c)[(1 + c)(1 - \alpha^2) + k(1 + \alpha^2)] \geq 0,$$

which holds for any $c > -1$. This shows that Theorem 3.2 reduces to Theorem 1.2 and that the assertion even holds in the case $-1 < c < 0$.

Remark 3.4. For $A = \delta$, $B = 0$, $D = \alpha$ and $E = 0$ ($0 < \alpha, \delta \leq 1$), we have $G = I = J = 0$, $H = \alpha(1 + c)$, and $L = 1 + c + k$. In this case condition (3.1) holds for any $c > -1$. Thus Theorem 3.2 extends the earlier result of Anbudurai [1, Theorem 2.1, p. 20] even in the case $-1 < c < 0$.

Remark 3.5. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ ($0 \leq \alpha, \delta < 1$), then $G = 0$, $H = (1 - \delta)(1 + c)$, $I = 0$, $J = 0$ and $L = 1 + c + k$. Theorem 3.2 yields the following:

Corollary 3.3. *Let $c > -1$, $1/(2 + c) \leq \alpha < 1$ and $\delta := \alpha - (1 - \alpha)/(1 + c)$. If $f(z) \in R_\delta$, then $F(z) \in R_\alpha$.*

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