Coefficient Inequalities for Starlikeness and Convexity

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Abstract

For an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$, the range of β is determined so that the function f is either starlike or convex of order α. Several related problems are also investigated. Applications of these results to Gaussian hypergeometric functions are also provided.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. A function $f \in \mathcal{A}$ has Taylor's series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be subclasses of $\mathcal{S}$ consisting of starlike functions of order α and convex functions of order α, respectively defined analytically by the following equalities:

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}, \quad \text{and} \quad \mathcal{C}(\alpha) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the familiar classes of starlike and convex functions respectively. Closely related are the following classes of functions:

$$\mathcal{S}^*_\alpha := \left\{ f \in \mathcal{S} : \left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \right\}, \quad \text{and} \quad \mathcal{C}_\alpha := \left\{ f \in \mathcal{S} : \left|\frac{zf''(z)}{f'(z)}\right| < 1 - \alpha \right\}.$$

Note that $\mathcal{S}^*_\alpha \subseteq \mathcal{S}^*(\alpha)$ and $\mathcal{C}_\alpha \subseteq \mathcal{C}(\alpha)$. For $\beta < 1$, $\alpha \in \mathbb{R}$, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{R}(\alpha, \beta)$ if the function f satisfies the following inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right) > \beta. \tag{2}$$

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Clearly, $\mathcal{R}(0,\beta)=\mathcal{S}^*(\beta)$. For $\beta \geq -\alpha/2$, Li and Owa [5] proved that $\mathcal{R}(\alpha,\beta)\subset\mathcal{S}^*$.

A function $f\in\mathcal{S}$ is *k-uniformly convex,* $(0\leq k<\infty)$, if f maps every circular arc γ contained in $\mathbb{D}$ with center ζ, $|\zeta|\leq k$, onto a convex arc. The class of all k-uniformly convex functions is denoted by $k-\mathcal{UCV}$. Goodman [3] introduced the class $\mathcal{UCV}:=1-\mathcal{UCV}$ while the class $k-\mathcal{UCV}$ was introduced by Kanas and Wisniowska [7]. They [7, Theorem 2.2, p. 329] (see [2] for details) have shown that $f\in k-\mathcal{UCV}$ if and only if the function f satisfies the following inequality:

$$k\left|\frac{zf''(z)}{f'(z)}\right|<\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right).$$

By making use of this result, the following sufficient condition for a function to be k-uniformly convex was proved in [7]:

Theorem 1.1 ([7, Theorem 3.3, p. 334])**.** *If the function $f(z)=z+\sum_{n=2}^\infty a_nz^n$ satisfies the inequality $\sum_{n=2}^\infty n(n-1)|a_n|\leq 1/(k+2)$, $(0\leq k<\infty)$, then $f\in k-\mathcal{UCV}$. The bound $1/(k+2)$ cannot be replaced by a larger number.*

Goodman [3, Theorem 6] proved the result in the case $k=1$ for functions to be uniformly convex. Theorem 1.1 in the special case $k=0$ shows that the corresponding constant is $1/2$ for functions f to be convex. A function $f\in\mathcal{A}$ is *parabolic starlike of order* α [1] if

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-2\alpha+\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

Theorem 1.2 (Ali [1, Theorem 3.1, p. 564])**.** *If the function $f(z)=z+\sum_{n=2}^\infty a_nz^n$ satisfies the inequality $\sum_{n=2}^\infty(n-1)|a_n|\leq(1-\alpha)/(2-\alpha)$, then the function f is parabolic starlike of order α. The bound $(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.*

Motivated by Theorems 1.1 and 1.2, the range of β is determined for the analytic function $f(z)=z+\sum_{n=2}^\infty a_nz^n$ satisfying the inequality $\sum_{n=2}^\infty n(n-1)|a_n|\leq\beta$ to be either starlike or convex of order α. Similar problems were investigated for functions satisfying certain other coefficient inequality. The reverse implications are investigated for analytic functions with negative coefficients. Finally, applications of these results to hypergeometric functions are also provided.

The following theorem that gives necessary and sufficient conditions for functions to belong to certain subclasses of starlike and convex functions will be need in the sequel.

Theorem 1.3 ([8, Theorem 2, p. 961], and [10, Theorem 1 and Corollary, p. 110])**.**

1. *If the function $f(z)=z+\sum_{n=2}^\infty a_nz^n$ satisfies the inequality*

$$\sum_{n=2}^\infty(n-\alpha)|a_n|\leq 1-\alpha, \tag{3}$$

then $f\in\mathcal{S}^*_\alpha$. If $a_n\leq 0$, then the condition (3) is a necessary condition for $f\in\mathcal{S}^*(\alpha)$.

2. *Similarly, if the function f satisfies the inequality*

$$\sum_{n=2}^\infty n(n-\alpha)|a_n|\leq 1-\alpha, \tag{4}$$

then $f\in\mathcal{C}_\alpha$. If $a_n\leq 0$, then the condition (4) is a necessary condition for $f\in\mathcal{C}(\alpha)$.

The necessary and sufficient conditions in Theorem 1.3(1) was proved by Merkes, Robertson, and Scott [8, Theorem 2, p. 961] in 1962 and proved independently by Silverman [10, Theorem 1, p. 110] later in 1975. The corresponding result in Theorem 1.3(2) follows by the Alexandar's result and it was proved in [10, Corollary, p. 110].

2. Sufficient conditions for starlikeness and convexity

The following theorem provides sufficient coefficient inequality for functions to be in the classes $\mathcal{C}_\alpha$ or $\mathcal{S}^*_\alpha$.

Theorem 2.1. *Let $\alpha \in [0,1)$. If the function $f \in \mathcal{A}$ given by (1) satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \beta < 1, \tag{5}$$

then the following holds.

(1) The function f belongs to the class $\mathcal{C}_\alpha$ for $\beta \le (1-\alpha)/(2-\alpha)$, and the bound $(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.
(2) The function f belongs to the class $\mathcal{S}^*_\alpha$ for $\beta \le 2(1-\alpha)/(2-\alpha)$, and the bound $2(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.

Proof. (1) Let the function f satisfy the inequality (5) with $\beta \le (1-\alpha)/(2-\alpha)$. Then, the Equation (5) together with (1) shows that

$$|f'(z) - 1| \le \sum_{n=2}^{\infty} n|a_n| \le \sum_{n=2}^{\infty} n(n-1)|a_n| < 1,$$

and so $\operatorname{Re} f'(z) > 0$. This shows that $f \in \mathcal{S}$. Since the inequality

$$n - \alpha \le (2-\alpha)(n-1) \tag{6}$$

holds for $n \ge 2$, the inequality (5) leads to

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le (2-\alpha)\sum_{n=2}^{\infty} n(n-1)|a_n| \le (2-\alpha)\beta \le 1-\alpha.$$

Thus, by Theorem 1.3(2), $f \in \mathcal{C}_\alpha$. The function $f_0 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = z - \frac{1}{2}\frac{1-\alpha}{2-\alpha}z^2$$

satisfies the hypothesis of Theorem 1.3 and therefore $f_0 \in \mathcal{C}_\alpha$. This function f_0 shows that the bound for β cannot be replaced by a larger number.

(2) Now, let the function f satisfy the inequality (5) with $\beta \le 2(1-\alpha)/(2-\alpha)$. When $n \ge 2$, the inequality (6) holds and this leads to

$$(n-\alpha) \le \frac{n(n-\alpha)}{2} \le \frac{(2-\alpha)n(n-1)}{2} \quad (n \ge 2)$$

and hence

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \le \frac{(2-\alpha)}{2}\sum_{n=2}^{\infty} n(n-1)|a_n| \le (1-\alpha).$$

By Theorem 1.3(1), $f \in \mathcal{S}^*_\alpha$. The function

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha}z^2 \in \mathcal{S}^*_\alpha$$

shows that the result is sharp. ∎

Corollary 2.2. [7, Theorem 3.3, p. 334] *If* $f \in \mathcal{A}$ *given by* (1) *satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2},$$

then $f \in k-\mathcal{UCV}$*. Further, the bound* $1/(k+2)$ *cannot be replaced by a larger number.*

Proof. By Theorem 2.1(1), it follows that $f \in \mathcal{C}_{k/(k+1)}$ and hence the following inequality holds:

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{k+1}. \tag{7}$$

The inequality (7) yields

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \frac{k}{k+1} = 1 - \frac{1}{k+1} < 1 - \left|\frac{zf''(z)}{f'(z)}\right| < 1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right),$$

and this proves that $f \in k-\mathcal{UCV}$. ∎

Since $f \in \mathcal{C}_\alpha$ if and only if $zf' \in \mathcal{S}^*_\alpha$, Theorem 2.1 (1) immediately yields the following result. It also follows from Theorem 1.3(1) and the inequality $n-\alpha \leq (2-\alpha)(n-1)$, $n \geq 2$.

Corollary 2.3. *Let* $\alpha \in [0,1)$*. If* $f \in \mathcal{A}$ *is given by* (1) *and*

$$\sum_{n=2}^{\infty} (n-1)|a_n| \leq \frac{1-\alpha}{2-\alpha},$$

then $f \in \mathcal{S}^*_\alpha$*. Further, the bound* $(1-\alpha)/(2-\alpha)$ *cannot be replaced by a larger number.*

Remark 2.4. Theorem 1.2 for the class of parabolic starlike functions of order ρ was obtained by Ali [1, Theorem 3.1, p. 564] by using a two variable characterization of a corresponding class of uniformly convex functions. However, Theorem 1.2 follows directly from the Corollary 2.3 and his sufficient condition [1, Theorem 2.2, p. 563] for functions to be parabolic starlike of order ρ.

Theorem 2.5. *Let* $\alpha \in [0,1)$ *and* $f \in \mathcal{A}$ *be given by* (1)*.*

(1) *If the inequality* $\sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha$ *holds, then* $f \in \mathcal{S}^*_\alpha$*.*
(2) *If the inequality* $\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha$ *holds, then* $f \in \mathcal{C}_\alpha$*.*
(3) *If the inequality* $\sum_{n=2}^{\infty} n^2|a_n| \leq 4(1-\alpha)/(2-\alpha)$ *holds, then* $f \in \mathcal{S}^*_\alpha$ *and the bound* $4(1-\alpha)/(2-\alpha)$ *is sharp.*

Proof. The first two parts follow from Theorem 1.3 and the simple inequality $n-\alpha < n$. The third part follows from Theorem 1.3(1) and the identity: $(n-\alpha) \leq n^2(2-\alpha)/4 \quad (n \geq 2)$. The result is sharp for the function f_0 given by

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha}z^2.$$

∎

3. The subclass $\mathcal{R}(\alpha, \beta)$

Recall that the class $\mathcal{R}(\alpha,\beta)$ consists of functions f satisfying the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right) > \beta, \quad (\beta<1,\ \alpha\in\mathbb{R}). \tag{8}$$

The following Lemma 3.1 provides a sufficient coefficient condition for functions f to belong to the class $\mathcal{R}(\alpha,\beta)$. In this section, conditions are determined so that the sufficient coefficient condition in Lemma 3.1 implies starlikeness and convexity of certain order. Also the sufficient coefficient inequalities are considered for functions to belong to the class $\mathcal{R}(\alpha,\beta)$.

Lemma 3.1. [6, cf. Theorem 6, p. 412] *Let $\beta<1$, and $\alpha\in\mathbb{R}$. If $f\in\mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n| \le 1-\beta, \tag{9}$$

then $f\in\mathcal{R}(\alpha,\beta)$.

It should be remarked that Lemma 3.1 reduces to Theorem 1.3(1) in the special case $\alpha=0$. The following theorem provides sufficient coefficient conditions for functions to belong to either $\mathcal{R}(\alpha,\beta)\cap\mathcal{S}^*{}_\eta$ or $\mathcal{R}(\alpha,\beta)\cap\mathcal{C}_\eta$.

Theorem 3.2. *Let $\beta<1$ and $\alpha>0$. If the function $f\in\mathcal{A}$ satisfies the inequality* (9)*, then the following holds.*

(1) The function f is in the class $\mathcal{S}^*{}_\eta$ for $\eta\le(2\alpha+\beta)/(2\alpha+1)$ and the bound $(2\alpha+\beta)/(2\alpha+1)$ is sharp.

(2) The function f is in the class $\mathcal{C}_\eta$ for $\eta\le(\alpha-1+\beta)/\alpha$, $\beta>0$.

Proof. (1) If $\eta\le\eta_0:=(2\alpha+\beta)/(2\alpha+1)$, then $\mathcal{S}^*_{\eta_0}\subset\mathcal{S}^*_{\eta}$. Hence it is enough to prove that $f\in\mathcal{S}^*_{\eta_0}$. The inequality

$$(2\alpha+1)n-2\alpha\le\alpha n^2+(1-\alpha)n \quad (n\ge2,\ \alpha\ge0)$$

together with (9) shows that

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\eta_0)|a_n| &= \sum_{n=2}^{\infty}\frac{(2\alpha+1)n-2\alpha-\beta}{2\alpha+1}|a_n|\\ &\le\sum_{n=2}^{\infty}\frac{\alpha n^2+(1-\alpha)n-\beta}{2\alpha+1}|a_n|\\ &\le\frac{1-\beta}{2\alpha+1}=1-\eta_0.\end{aligned}$$

Thus, by Theorem 1.3(1), $f\in\mathcal{S}^*{}_{\eta_0}$. The result is sharp for the function $f_0\in\mathcal{S}^*(\eta_0)$ given by

$$f_0(z)=z-\frac{1-\beta}{2\alpha+2-\beta}z^2.$$

(2) If $\eta\le\eta_0:=(\alpha-1+\beta)/\alpha$, then $\mathcal{C}_{\eta_0}\subset\mathcal{C}_\eta$. Hence it is enough to prove that $f\in\mathcal{C}_{\eta_0}$. The inequality

$$\alpha n^2+(1-\alpha)n-n\beta\le\alpha n^2+(1-\alpha)n-\beta \quad (n\ge2,\quad \beta\ge0),$$

together with (9) yields

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-\eta_0)|a_n| &= \frac{1}{\alpha}\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-n\beta\big)|a_n|\\ &\leq \frac{1}{\alpha}\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)\,|a_n|\\ &\leq \frac{1-\beta}{\alpha} = 1-\eta_0.\end{aligned}$$

Thus, by Theorem 1.3(2), $f \in \mathcal{C}_{\eta_0}$. ∎

Along the same line as Theorem 2.1, the following theorem provides a sufficient coefficient inequality for functions to belong to the class $\mathcal{R}(\alpha,\beta)$.

Theorem 3.3. *Let $\beta < 1$, $\alpha \in \mathbb{R}$ and $f \in \mathcal{A}$.*

(1) If the function f satisfies the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq 2(1-\beta)/(2\alpha+2-\beta)$, then $f \in \mathcal{R}(\alpha,\beta)$. The bound $2(1-\beta)/(2\alpha+2-\beta)$ is sharp.

(2) Let $\alpha \leq 1$ and $\eta \in \mathbb{R}$ be defined by

$$\eta = \begin{cases} 4(1-\beta)/(3\alpha+1), & \alpha+\beta>1,\\ 4(1-\beta)/(2\alpha+2-\beta), & \alpha+\beta\leq 1.\end{cases}$$

If the function f satisfies the inequality $\sum_{n=2}^{\infty} n^2|a_n| \leq \eta$, then $f \in \mathcal{R}(\alpha,\beta)$. When $\alpha+\beta \leq 1$, the result is sharp.

Proof. (1) Let the function f satisfy the inequality

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq 2(1-\beta)/(2\alpha+2-\beta).$$

Since, for $n \geq 2$,

$$2\alpha n^2+2(1-\alpha)n-2\beta \leq (2\alpha+2-\beta)n(n-1),$$

it follows that

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n| \leq \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(2\alpha+2-\beta)|a_n| \leq 1-\beta,$$

and so, by Lemma 3.1, $f \in \mathcal{R}(\alpha,\beta)$. The result is sharp for the function $f_0 \in \mathcal{R}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta}z^2.$$

(2) Let $\alpha+\beta > 1$ and the function f satisfy the inequality

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 4(1-\beta)/(3\alpha+1).$$

In this case, the use of the inequality

$$4\left(\alpha n^2+(1-\alpha)n-\beta\right) \leq (3\alpha+1)n^2 \quad (n \geq 2)$$

readily yields

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n| \le \frac{3\alpha+1}{4}\sum_{n=2}^{\infty} n^2|a_n| \le 1-\beta.$$

Lemma 3.1 then shows that $f \in \mathcal{R}(\alpha,\beta)$.

Now, let $\alpha+\beta<1$ and the function f satisfy $\sum_{n=2}^{\infty} n^2|a_n| \le 4(1-\beta)/(2\alpha+2-\beta)$. In this case, the inequality

$$4\big(\alpha n^2+(1-\alpha)n-\beta\big) \le n^2(2\alpha+2-\beta) \quad (n\ge 2)$$

shows that

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n| \le \frac{1}{4}\sum_{n=2}^{\infty} n^2(2\alpha+2-\beta)|a_n| \le 1-\beta,$$

and hence, by Lemma 3.1, $f \in \mathcal{R}(\alpha,\beta)$. The function $f_0 \in \mathcal{R}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta}z^2$$

shows that the result is sharp. ∎

4. Functions with negative coefficients

In this section, certain classes of functions with negative coefficients are investigated. The class of functions with negative coefficients, denoted by $\mathcal{T}$, consists of the functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0). \tag{10}$$

Denote by $\mathcal{TS}^*(\alpha)$, $\mathcal{TS}^*_\alpha$ and $\mathcal{TC}(\alpha)$, and $\mathcal{TC}_\alpha$ the respective subclasses of functions with negative coefficients in $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_\alpha$ and $\mathcal{C}_\alpha$ and $\mathcal{C}(\alpha)$. For starlike and convex functions functions with negative coefficients, Silverman [10] proved the following theorem.

Theorem 4.1. *Let $\alpha \in [0,1)$. If $f \in \mathcal{T}$ is given by* (10), *then*

$$f \in \mathcal{TS}^*(\alpha) \Longleftrightarrow f \in \mathcal{TS}^*_\alpha \Longleftrightarrow \sum_{n=2}^{\infty}(n-\alpha)a_n \le 1-\alpha,$$

and

$$f \in \mathcal{TC}(\alpha) \Longleftrightarrow f \in \mathcal{TC}_\alpha \Longleftrightarrow \sum_{n=2}^{\infty}n(n-\alpha)a_n \le 1-\alpha.$$

For functions with negative coefficients, the next theorem proves the equivalence of the inequalities $\sum_{n=2}^{\infty} n(n-1)a_n \le \beta$ and $|f''(z)| < \beta$.

Theorem 4.2. *Let $\beta > 0$. If the function $f \in \mathcal{T}$ is given by* (10), *then*

$$|f''(z)| \le \beta \Longleftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n \le \beta.$$

Proof. If f satisfies the coefficient inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq \beta$, then

$$|f''(z)| \leq \sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-2} \leq \sum_{n=2}^{\infty} n(n-1)a_n \leq \beta.$$

The converse follows, by allowing $z \to 1^-$, in

$$|f''(z)| = \left|\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right| \leq \beta. \qquad \blacksquare$$

Remark 4.3. It is well known that if the function $f \in \mathcal{A}$ satisfies the inequality $|f''(z)| \leq \beta$ for $0 < \beta \leq 1$, then $f \in \mathcal{S}^*$ and if $|f''(z)| \leq \beta$ for $0 < \beta \leq 1/2$, then $f \in \mathcal{C}$ [12, Theorem 1, p.1861].

Theorem 4.4. *If the function $f \in \mathcal{TC}(\alpha)$, $0 \leq \alpha < 1$, then the following holds:*

(1) The inequality $\sum_{n=2}^{\infty} na_n \leq (1-\alpha)/(2-\alpha)$ holds and the bound $(1-\alpha)/(2-\alpha)$ is sharp.
(2) The inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq 1-\alpha$ holds.
(3) The inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)/2(2-\alpha)$ holds and the bound $(1-\alpha)/2(2-\alpha)$ is sharp.
(4) The inequality $\sum_{n=2}^{\infty} n^2a_n \leq 2(1-\alpha)/(2-\alpha)$ holds and the bound $2(1-\alpha)/(2-\alpha)$ is sharp.

Proof. The results follow respectively from Theorem 4.1 and the simple inequalities $2-\alpha \leq n-\alpha$, $n-1 \leq n-\alpha$, $2(2-\alpha)(n-1) \leq n(n-\alpha)$, and $n^2(2-\alpha) \leq 2n(n-\alpha)$ satisfied for $n \geq 2$. The sharpness follows by considering the function f_0 given by

$$f_0(z) = z - \frac{1}{2}\frac{1-\alpha}{2-\alpha}z^2. \qquad \blacksquare$$

Alexander theorem between $\mathcal{TC}(\alpha)$ and $\mathcal{TS}^*(\alpha)$ immediately yields the following corollary.

Corollary 4.5. *If the function $f \in \mathcal{TS}^*(\alpha)$, $0 \leq \alpha < 1$, then the following holds.*

(1) The inequality $\sum_{n=2}^{\infty} a_n \leq (1-\alpha)/(2-\alpha)$ holds and the bound $(1-\alpha)/(2-\alpha)$ is sharp.
(2) The inequality $\sum_{n=2}^{\infty}(n-1)a_n \leq 1-\alpha$ holds.
(3) The inequality $\sum_{n=2}^{\infty} na_n \leq 2(1-\alpha)/(2-\alpha)$ holds and the bound $2(1-\alpha)/(2-\alpha)$ is sharp.

In the remaining part of this section, the properties of functions with negative coefficients belonging to the class $\mathcal{R}(\alpha,\beta)$ are investigated. The class of all functions with negative coefficients belonging to the class $\mathcal{R}(\alpha,\beta)$ is denoted in the sequel by $\mathcal{TR}(\alpha,\beta)$. To begin with, the following lemma is needed.

Lemma 4.6. *[6, Theorem 8, p.414] If $\beta < 1$, $\alpha \in \mathbb{R}$. If $f \in \mathcal{T}$, then*

$$f \in \mathcal{TR}(\alpha,\beta) \Longleftrightarrow \sum_{n=2}^{\infty}\left(\alpha n^2 + (1-\alpha)n - \beta\right)a_n \leq 1-\beta.$$

Corollary 4.7. *If $f \in \mathcal{TR}(\alpha,\beta)$ with $\beta < 1$, $\alpha > 0$, then the following holds:*

(1) The function $f \in \mathcal{TS}^*{}_\eta$ for $\eta \leq (2\alpha+\beta)/(2\alpha+1)$ and the bound $(2\alpha+\beta)/(2\alpha+1)$ is sharp.

(2) The function $f \in \mathcal{TC}_\eta$ *for* $\eta \le (\alpha - 1 + \beta)/\alpha$.

Proof. The result follows from Lemma 4.6 and Theorem 3.2. ∎

The next result shows that $\mathcal{TC}\big((2\alpha+3\beta-2)/(2\alpha+\beta)\big) \subset \mathcal{TR}(\alpha,\beta)$ for $0 \le \beta < 1$, $\alpha \in \mathbb{R}$.

Theorem 4.8. *Let* $0 \le \beta < 1$, *and* $\alpha > 0$. *If* $\eta \ge (2\alpha+3\beta-2)/(2\alpha+\beta)$, *then* $\mathcal{TC}(\eta) \subseteq \mathcal{TR}(\alpha,\beta)$.

Proof. For $\eta_0 \le \eta$, $\mathcal{TC}(\eta) \subset \mathcal{TC}(\eta_0)$ and therefore it is enough to prove $\mathcal{TC}(\eta_0) \subseteq \mathcal{TR}(\alpha,\beta)$ where $\eta_0 = (2\alpha+3\beta-2)/(2\alpha+\beta)$. For $n \ge 2$, the inequality

$$2\alpha n^2 + 2(1-\alpha)n - 2\beta \le n\big((2\alpha+\beta)n - (2\alpha+3\beta-2)\big)$$

holds and therefore

$$\begin{aligned}\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)a_n &\le \frac{1}{2}\sum_{n=2}^{\infty} n\big((2\alpha+\beta)n-(2\alpha+3\beta-2)\big)a_n\\ &= \frac{2\alpha+\beta}{2}\sum_{n=2}^{\infty} n(n-\eta_0)a_n\\ &\le \frac{2\alpha+\beta}{2}(1-\eta_0)\\ &= 1-\beta,\end{aligned}$$

and, by Lemma 4.6, $f \in \mathcal{TR}(\alpha,\beta)$. ∎

Theorem 4.9. *Let* $\beta < 1$, *and* $\alpha \in \mathbb{R}$. *If* $f \in \mathcal{TR}(\alpha,\beta)$, *then*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n \le (1-\beta)/\alpha$ *when* $\alpha > 0$.
(2) $\sum_{n=2}^{\infty}(n-1)a_n \le \eta$ *where*

$$\eta = \begin{cases}(1-\beta)/(1-\alpha), & \beta < 3\alpha+1, \quad 0 \le \alpha < 1\\ (1-\beta)/(2\alpha+2-\beta), & \beta \ge 3\alpha+1, \quad 0 \le \alpha.\end{cases}$$

The result for $\beta > 3\alpha+1$ *is sharp.*
(3) For $0 \le \alpha \le 1$, $\sum_{n=2}^{\infty} n^2 a_n \le \eta$ *where*

$$\eta = \begin{cases}(1-\beta)/\alpha, & \beta < 2(1-\alpha), \alpha > 0\\ 4(1-\beta)/(2\alpha+2-\beta), & \beta \ge 2(1-\alpha), , \beta \ge 0.\end{cases}$$

The result for $\beta > 2(1-\alpha)$ *is sharp.*
(4) $\sum_{n=2}^{\infty} n a_n \le 2(1-\beta)/(2\alpha+2-\beta)$, $\alpha, \beta \ge 0$. *The result is sharp.*

Proof. Since $f \in \mathcal{TR}(\alpha,\beta)$, by Lemma 4.6, the following inequality holds:

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)a_n \le 1-\beta.$$

This inequality is used throughout the proof of this theorem.

(1) Since

$$\alpha n(n-1) \le \alpha n^2 + (1-\alpha)n - \beta \quad n \ge 2,$$

it readily follows that

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{\alpha} a_n \leq \frac{1-\beta}{\alpha}.$$

(2) If $\beta < 3\alpha + 1$, then, for $n \geq 2$,

$$(n-1)(1-\alpha) \leq \alpha n(n-1) + n - \beta \quad (n \geq 2)$$

and an use of this inequality shows that

$$\sum_{n=2}^{\infty} (n-1)a_n \leq \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{1-\alpha} a_n \leq \frac{1-\beta}{1-\alpha}.$$

If $\beta > 3\alpha + 1$, then the inequality

$$(n-1)(2\alpha + 2 - \beta) \leq \alpha n^2 + n(1-\alpha) - \beta \quad (n \geq 2)$$

shows that

$$\sum_{n=2}^{\infty} (n-1)a_n \leq \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{2\alpha + 2 - \beta} a_n \leq \frac{1-\beta}{2\alpha + 2 - \beta}.$$

(3) If $\beta < 2(1-\alpha)$, the result follows from inequality

$$\alpha n^2 \leq \alpha n^2 + 2(1-\alpha) - \beta \leq \alpha n^2 + n(1-\alpha) - \beta.$$

Using this inequality, it follows that

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{\alpha} a_n \leq \frac{1-\beta}{\alpha}.$$

In the case $\beta \geq 2(1-\alpha)$, the inequality

$$n^2(2\alpha + 2 - \beta) \leq 4(\alpha n^2 + (1-\alpha)n - \beta) \quad (n \geq 2)$$

shows that

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty} \frac{4(\alpha n^2 + (1-\alpha)n - \beta)}{2\alpha + 2 - \beta} a_n \leq \frac{4(1-\beta)}{2\alpha + 2 - \beta}.$$

(4) For $\alpha, \beta > 0$, the inequality

$$(2\alpha + 2 - \beta)n \leq 2\big(\alpha n^2 + (1-\alpha)n - \beta\big)$$

shows that

$$\sum_{n=2}^{\infty} n a_n \leq \sum_{n=2}^{\infty} \frac{2\big(\alpha n^2 + (1-\alpha)n - \beta\big)}{2\alpha + 2 - \beta} a_n \leq \frac{2(1-\beta)}{2\alpha + 2 - \beta}.$$

The sharpness can be seen by considering the function f_0 given by

$$f(z) = z - \frac{1-\beta}{2\alpha + 2 - \beta} z^2 \in \mathcal{TR}(\alpha, \beta). \qquad \blacksquare$$

5. Applications to Gaussian hypergeometric functions

For $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* is defined by

$$F(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots,$$

where $(\lambda)_n$ is Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (n = 0, 1, 2, \dots).$$

The series converges absolutely on $\mathbb{D}$. It also converges on $|z| = 1$ when $\operatorname{Re}(c-a-b) > 0$. For $\operatorname{Re}(c-a-b) > 0$, the value of the hypergeometric function $F(a,b;c;z)$ at $z = 1$ is related to Gamma function by the following Gauss summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, \dots). \tag{11}$$

By making use of Theorem 1.3, Silverman [11] determined conditions on a, b, c so that the function $zF(a,b;c;z)$ belongs to certain subclasses of the starlike and convex functions. In the following theorem, the conditions on the parameters a, b, c are determined so that the function $zF(a,b;c;z)$ belongs to the class $\mathcal{R}(\alpha,\beta)$. For other classes investigated in this paper, similar results are true but the details are omitted here. The proof follows directly by the corresponding theorems in the previous sections, the Gauss summation formula for the Gaussian hypergeometric functions and some algebraic manipulation; the details of the proofs are omitted as they are similar to those of Silverman [11] and Kim and Ponnusamy [4].

Theorem 5.1. *Let $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$ satisfy either*

$$F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{2|ab|}{c-|a|-|b|-1}\right) \leq \frac{2(1-\beta)}{2\alpha+2-\beta},$$

for $c > |a|+|b|+2$, $\alpha \geq 0$, $\beta < 1$, or

$$F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{3|ab|}{c-|a|-|b|-1} + 1\right) \leq \frac{6-5\beta+2\alpha}{2\alpha+2-\beta},$$

for $c > |a|+|b|+2$, $1-\alpha \geq \beta$, $\alpha \in [0,1)$, then the function $zF(a,b;c;z) \in \mathcal{R}(\alpha,\beta)$. In the case $b = \overline{a}$, the range of c in either case can be improved to $c > \max\{0, 2(1+\operatorname{Re} a)\}$.

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