

## Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination

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**Abstract.** Several radii problems are considered for functions  $f(z) = z + a_2z^2 + \dots$  with fixed second coefficient  $a_2$ . For  $0 \leq \beta < 1$ , sharp radius of starlikeness of order  $\beta$  for several subclasses of functions are obtained. These include the class of parabolic starlike functions, the class of Janowski starlike functions, and the class of strongly starlike functions. Sharp radius of convexity of order  $\beta$  for uniformly convex functions, and sharp radius of strong-starlikeness of order  $\gamma$  for starlike functions associated with the lemniscate of Bernoulli are also obtained as special cases.

### 1. Motivation and a survey

Let  $\mathcal{A}$  denote the class of analytic functions  $f$  defined in the open unit disc  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  and normalized by the conditions  $f(0) = 0$  and  $f'(0) = 1$ . Let  $\mathcal{S}$  be its subclass consisting of univalent analytic functions. Thus functions in  $\mathcal{S}$  has the form  $f(z) = z + a_2z^2 + \dots$ . Gronwall [13] obtained lower and upper bounds for the quantities  $|f(z)|$  and  $|f'(z)|$  for univalent functions with preassigned second coefficient  $a_2$ . Corresponding results for convex functions were also obtained. Unaware of these results, Finkelstein [8] investigated the problem again and obtained similar results, except for an inaccurate lower bound for  $|f(z)|$ . Corresponding estimates for starlike functions of positive order with fixed second coefficient were obtained by Tepper [41], while for convex functions of positive order, such estimates were derived by Padmanabhan [24]. The problem for general classes of functions defined by subordination was investigated by Padmanabhan [27] in 2001. For close-to-star and close-to-convex functions, such estimates were investigated by Al-Amiri [4] and Silverman [33], respectively.

In addition to distortion and growth estimates, Tepper [41] obtained the radius of convexity for starlike functions with fixed second coefficient. This radius result was also obtained independently by Goel [11], whom additionally obtained the radius of starlikeness for functions  $f$  with fixed second coefficient satisfying  $\operatorname{Re}(f(z)/z) > 0$  for  $z \in \mathbb{D}$ . Following these works, several authors have investigated radii problems for functions with fixed second coefficient; we provide here a brief history of these works.

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2010 *Mathematics Subject Classification.* Primary 30C80; Secondary 30C45

*Keywords.* Subordination, radius of starlikeness, radius of convexity, radius of strong starlikeness, lemniscate of Bernoulli

Received: 13 January 2011; Accepted: 09 September 2011

Communicated by Miodrag Mateljević

The work presented here was supported in parts by grants from Universiti Sains Malaysia, and the National Research Foundation of Korea (No. 2010-0017111). The authors are thankful to the referee for his useful comments.

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Let  $f$  and  $g$  be analytic in  $\mathbb{D}$ . Then  $f$  is subordinate to  $g$ , written  $f(z) < g(z)$  ( $z \in \mathbb{D}$ ), if there is an analytic function  $w$ , with  $w(0) = 0$  and  $|w(z)| < 1$ , such that  $f(z) = g(w(z))$ . In particular, if  $g$  is univalent in  $\mathbb{D}$ , then  $f$  is subordinate to  $g$  provided  $f(0) = g(0)$  and  $f(\mathbb{D}) \subseteq g(\mathbb{D})$ . Noticing that several subclasses of univalent functions are characterized by the quantities  $zf'(z)/f(z)$  or  $1 + zf''(z)/f'(z)$  lying in a region in the right-half plane, Ma and Minda [18] gave a unified presentation of various subclasses of convex and starlike functions which are defined below. The region considered is the image set of an analytic function  $\varphi$  with positive real part in  $\mathbb{D}$  and normalized by the conditions  $\varphi(0) = 1$  and  $\varphi'(0) > 0$ , and maps the unit disc  $\mathbb{D}$  onto a region starlike with respect to 1 that is symmetric with respect to the real axis. For a fixed function  $\varphi$ , let

$$\mathcal{P}(\varphi) := \{p(z) = 1 + cz + \dots : p(z) < \varphi(z)\}.$$

Ma and Minda [18] considered the classes

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}(\varphi) \right\} \quad \text{and} \quad \mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}(\varphi) \right\}.$$

For  $-1 \leq B < A \leq 1$ , the function  $\varphi(z) = (1 + Az)/(1 + Bz)$  is a convex function whose image is symmetric with respect to the real axis. For this  $\varphi$ , the class  $\mathcal{ST}(\varphi)$  reduces to the familiar class consisting of Janowski starlike functions denoted by  $\mathcal{ST}(A, B)$ . The corresponding class of convex functions is denoted by  $\mathcal{CV}(A, B)$ . We shall consider these classes by relaxing the conditions on  $A, B$  to be  $|B| \leq 1$  with  $A \neq B$ . The special case  $A = 1$  and  $B = -1$  leads to the usual classes  $\mathcal{ST}$  and  $\mathcal{CV}$  of starlike and convex functions respectively, while  $A = 1 - 2\alpha, B = -1$  with  $0 \leq \alpha < 1$  yield the classes  $\mathcal{ST}(\alpha)$  and  $\mathcal{CV}(\alpha)$  of starlike and convex functions of order  $\alpha$  respectively. For  $0 < \gamma \leq 1$ , let  $\mathcal{SST}(\gamma)$  and  $\mathcal{SCV}(\gamma)$  be the classes consisting of strongly starlike and strongly convex functions of order  $\gamma$ :  $\mathcal{SST}(\gamma) = \mathcal{ST}(((1+z)/(1-z))^\gamma)$  and  $\mathcal{SCV}(\gamma) = \mathcal{CV}(((1+z)/(1-z))^\gamma)$ .

A function  $f$  is  $k$ -fold symmetric if  $f(e^{2\pi i/k}z) = e^{2\pi i/k}f(z)$ . It is clear that a function  $f \in \mathcal{A}$  is  $k$ -fold symmetric if and only if  $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots$ . Let us denote the class of functions  $p(z) = 1 + p_kz^k + p_{2k}z^{2k} + \dots$  subordinated to a function  $\varphi$  by  $\mathcal{P}_k(\varphi)$ . For  $-1 \leq B < A \leq 1$ , and  $\varphi(z) = \frac{1+Az^k}{1+Bz^k}$ , denote the class  $\mathcal{P}_k(\varphi)$  by  $\mathcal{P}_k(A, B)$ . For functions in the class  $\mathcal{P}_k(A, B)$ , it is easy to see that  $|p_{nk}| \leq (A - B)$ . For  $0 \leq b \leq 1$ , denote the class of functions  $p$  with coefficient  $p_k = b(A - B)$  by  $\mathcal{P}_{k,b}(A, B)$ . Also let  $\mathcal{ST}_{k,b}(A, B)$  denote the class of  $k$ -fold symmetric functions  $f$  with fixed coefficient  $a_{k+1} = b(A - B)/k$  satisfying  $\frac{zf'(z)}{f(z)} \in \mathcal{P}_{k,b}(A, B)$ .

The radius of convexity for the class  $\mathcal{ST}_b := \mathcal{ST}_{1,b}(1, -1)$  of starlike functions with fixed second coefficient was extended by Tepper [41] and Goel [11], while Anh [5] determined the radius of convexity for the class  $\mathcal{ST}_{k,b}(A, B)$  with certain restrictions on the parameters. The case  $k = 1$  was earlier considered by Tuan and Anh [43]. Earlier works in this problem include those by McCarty [21] who obtained the radius of convexity for the class  $\mathcal{ST}_{1,b}(1 - 2\alpha, -1)$  and for functions  $f$  with  $f' \in \mathcal{P}_{1,b}(1 - 2\alpha, -1)$ . The latter class was earlier investigated by McCarty [20], and the result therein is sharp only for  $\alpha = 0$ . For  $0 \leq \alpha < 1, 0 < \beta \leq 1$ , Juneja and Mogra [15] obtained the radius of convexity for the class of functions  $f$  with fixed second coefficient satisfying  $f' \in \mathcal{P}_{1,b}(1 - 2\alpha\beta, 1 - 2\beta)$  as well as for the class  $\mathcal{ST}(1 - 2\alpha\beta, 1 - 2\beta)$ .

Improving the results of Mogra and Juneja [22], Padmanabhan and Ganesan [26] obtained the radius of convexity for functions  $f(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \dots$  with missing initial terms and fixed  $a_{k+1}$  belonging to the class  $\mathcal{ST}(A, B)$  when  $A + B \geq 0$ . Related results were derived by Mogra and Juneja [23] that are generalization of the results obtained by McCarty [21], Goel [10], Shaffer [32], Caplineger and Causey [7], Mogra and Juneja [14], Singh [35], [36], Padmanabhan [25], and Juneja and Mogra [15].

Silverman [33] obtained distortion and covering estimates, as well as the radius of convexity for the class of close-to-convex functions  $f(z) = z + a_2z^2 + \dots$  satisfying

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} \geq \beta, \quad \beta \geq 0,$$

for some convex function  $g$  of order  $\alpha$ . Tuan and Anh [42] obtained results on certain related classes, while Silverman and Telage [34], Umarani [44], Ahuja [1] and Aouf [6] obtained results on spirallike functions. Kumar [16] investigated radius problems for functions

$$f(z) = \{(\gamma + \alpha)^{-1}z^{1-\gamma}(z^\gamma F(z)^\alpha)'\}^{\frac{1}{\alpha}},$$

where  $\alpha$  is a positive real number,  $\gamma$  is a complex number such that  $\gamma + a \neq 0$  and the function  $F$  varies over various subclasses of univalent functions with fixed second coefficient. Ali et al. [2] recently investigated differential subordination for functions with fixed initial coefficient.

In this paper, the radius of starlikeness of order  $\beta$ ,  $0 \leq \beta < 1$ , for functions in  $\mathcal{ST}(\varphi)$  with fixed second coefficient is derived. As special cases, sharp radius of starlikeness of order  $\beta$  for several subclasses of functions are obtained. These include the class of parabolic starlike functions, the class of Janowski starlike functions, and the class of strongly starlike functions. Sharp radius of convexity of order  $\beta$  for uniformly convex functions, and sharp radius of strong-starlikeness of order  $\gamma$  for starlike functions associated with the lemniscate of Bernoulli are also obtained. We shall also make connections with several earlier works.

The following extension of Schwarz lemma will be required.

**Lemma 1.1.** ([8]) *Let  $w(z) = pz + \dots$  be an analytic map of the unit disc  $\mathbb{D}$  into itself. Then  $|p| \leq 1$  and*

$$|w(z)| \leq \frac{r(r + |p|)}{1 + |p|r}, \quad |z| = r < 1.$$

Equality holds at some  $z \neq 0$  if and only if

$$w(z) = \frac{e^{-it}z(z + pe^{it})}{1 + \bar{p}e^{-it}z}, \quad t \geq 0.$$

## 2. Radii of convexity and starlikeness

Ma and Minda [18, Theorem 3, p. 164] showed that for a function  $f \in \mathcal{CV}(\varphi)$ , the absolute value of the second coefficient of  $f$  is bounded by  $\varphi'(0)/2$ . As a consequence of Alexander's relation between the classes  $\mathcal{ST}(\varphi)$  and  $\mathcal{CV}(\varphi)$ , it follows that the second coefficient of  $f \in \mathcal{ST}(\varphi)$  is bounded by  $\varphi'(0)$ . Therefore, for  $f \in \mathcal{ST}(\varphi)$ ,  $|a_2| = \varphi'(0)b$  for some  $0 \leq b \leq 1$ . For  $b$  in this range, the class  $\mathcal{ST}_b(\varphi)$  is defined to be the subclass of  $\mathcal{ST}(\varphi)$  consisting of functions  $f(z) = z + a_2z^2 + \dots$  where the second coefficient is given by  $a_2 = \varphi'(0)b$ . Let  $\mathcal{CV}_b(\varphi)$  be the subclass of  $\mathcal{CV}(\varphi)$  consisting of functions  $f(z) = z + a_2z^2 + \dots$  with  $a_2 = \varphi'(0)b/2$ .

**Theorem 2.1.** *Let  $\varphi$  be an analytic function with positive real part in  $\mathbb{D}$ ,  $\varphi(0) = 1$ ,  $\varphi'(0) > 0$ , and  $\varphi$  maps the unit disc  $\mathbb{D}$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. Further suppose that*

$$\min_{|z|=r} \operatorname{Re} \varphi(z) = \varphi(-r).$$

Then the radius of starlikeness of order  $\beta$ ,  $0 \leq \beta < 1$ , for the class  $\mathcal{ST}_b(\varphi)$  is  $R_\beta$ , where

$$R_\beta = \begin{cases} 1 & \text{if } \varphi(-1) \geq \beta \\ \frac{-2\varphi^{-1}(\beta)}{b(1+\varphi^{-1}(\beta)) + \sqrt{b^2(1+\varphi^{-1}(\beta))^2 - 4\varphi^{-1}(\beta)}} & \text{if } \varphi(-1) \leq \beta. \end{cases}$$

The result is sharp. (Here and elsewhere  $\varphi(-1) = \lim_{r \rightarrow 1^-} \varphi(r)$ .)

*Proof.* Let  $f \in \mathcal{ST}_b(\varphi)$  so that  $zf'(z)/f(z) < \varphi(z)$ . From the definition of subordination, it follows that there is an analytic function  $w$  with  $w(0) = 0$  and  $|w(z)| < 1$  in  $\mathbb{D}$  satisfying

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)) \quad (z \in \mathbb{D}).$$

Since  $f(z) = z + \varphi'(0)bz^2 + \dots$ , a calculation shows that

$$\varphi(w(z)) = \frac{zf'(z)}{f(z)} = 1 + \varphi'(0)bz + \dots,$$

and hence  $\varphi'(w(z))w'(z) = \varphi'(0)b + \dots$ . Since  $w(0) = 0$ , the above equation yields  $w'(0) = b$  and hence it takes the form  $w(z) = bz + \dots$ .

For a given  $0 \leq r < 1$ , let  $r_1$  be defined by

$$r_1 := \frac{r(r+b)}{1+br}. \tag{1}$$

Since  $w$  maps  $\mathbb{D}$  onto itself, Lemma 1.1 yields  $|w(z)| \leq r_1, |z| = r$ . It follows that  $|w(z)| \leq r_1$  if  $|z| \leq r$ , and therefore  $\{\varphi(w(z)) : |z| \leq r\} \subseteq \{\varphi(\xi) : |\xi| \leq r_1\}$ . Consequently, for  $|z| \leq r < 1$ ,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \operatorname{Re}(\varphi(w(z))) \geq \min_{|z| \leq r} \operatorname{Re} \varphi(w(z)) \geq \min_{|z| \leq r_1} \operatorname{Re} \varphi(z) = \min_{|z|=r_1} \operatorname{Re} \varphi(z) = \varphi(-r_1).$$

The last equality follows from the hypothesis that  $\min_{|z|=r} \operatorname{Re} \varphi(z) = \varphi(-r)$ .

Since the function  $\varphi$  is starlike,  $\varphi$  is univalent and hence  $\varphi'(z) \neq 0$  for all  $z \in \mathbb{D}$ ; in particular, the restriction of  $\varphi$  to the interval  $(-1, 1)$  has non-vanishing derivative  $\varphi'(r)$  for  $r \in (-1, 1)$ . Since  $\varphi'(0) > 0$ , it follows that  $\varphi'(r) > 0$  and hence  $\varphi$  is strictly increasing on  $(-1, 1)$ . If  $\varphi(-1) \geq \beta$ , then

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \varphi(-r_1) \geq \varphi(-1) \geq \beta$$

for  $|z| < 1$ , and thus in this case,  $R_\beta = 1$ .

Let us now assume that  $\varphi(-1) < \beta$ . Since  $\varphi(\mathbb{D})$  is symmetric, it maps  $(-1, 1)$  into the real axis. Since  $\varphi$  is strictly increasing, the inverse function  $\varphi^{-1} : \varphi(\mathbb{D}) \cap \mathbb{R} \rightarrow \mathbb{R}$  exists and is also increasing. Thus  $\varphi(-r_1) \geq \beta$  holds if and only if  $-r_1 \geq \varphi^{-1}(\beta)$ . The desired expression for  $R_\beta$  now readily follows by solving  $-r_1 = \varphi^{-1}(\beta)$  for  $r$ .

To verify sharpness, consider the function  $f_0 : \mathbb{D} \rightarrow \mathbb{C}$  defined by

$$f_0(z) = z \exp \left( \int_0^z \left( \frac{1}{s} \varphi \left( -\frac{s(s+b)}{1+bs} \right) - \frac{1}{s} \right) ds \right), \quad 0 \leq b \leq 1.$$

The complex number  $-z(z+b)/(1+bz) \in \mathbb{D}$  and hence the function  $f_0$  is well-defined. Clearly  $f_0(0) = 0$  and  $f_0'(0) = 1$ . A calculation shows that

$$\frac{zf_0'(z)}{f_0(z)} = \varphi \left( -\frac{z(z+b)}{1+bz} \right) \in \varphi(\mathbb{D}),$$

that is,  $f_0 \in \mathcal{ST}_b(\varphi)$ . At  $z = R_\beta$ , clearly

$$\operatorname{Re} \frac{zf_0'(z)}{f_0(z)} = \varphi \left( -\frac{R_\beta(R_\beta+b)}{1+bR_\beta} \right) = \beta,$$

and this shows that the result is sharp.  $\square$

**Remark 2.2.** Under the conditions of Theorem 2.1, it is clear from the proof that the order of starlikeness for functions in the class  $\mathcal{ST}_b(\varphi)$  is  $\varphi(-1)$ .

**Remark 2.3.** When  $b = 1$ , Theorem 2.1 reduces to a result in [9, Theorem 2.2, p. 304] for functions (with varying second coefficient) belonging to the class  $\mathcal{ST}(\varphi)$ .

**Remark 2.4.** In view of Alexander’s relation between the classes  $\mathcal{ST}_b(\varphi)$  and  $\mathcal{CV}_b(\varphi)$ , it follows that Theorem 2.1 holds for the class  $\mathcal{CV}_b(\varphi)$  if we replace the phrase “radius of starlikeness” by “radius of convexity”.

Every convex function  $f$  in  $\mathbb{D}$  maps the circle  $|z| = r < 1$  onto a convex arc. However, it need not map every circular arc about a center in  $\mathbb{D}$  onto a convex arc. This motivated the investigation of uniformly convex functions. A function  $f \in \mathcal{S}$  is uniformly convex [12] if  $f$  maps every circular arc  $\gamma$  contained in  $\mathbb{D}$  with center  $\zeta \in \mathbb{D}$ , onto a convex arc. Denote by  $\mathcal{UCV}$  the class of all uniformly convex functions. Ma and Minda [19] and Ronning [30], independently showed that a function  $f$  is uniformly convex if and only if

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{D}).$$

Thus,  $f \in \mathcal{UCV}$  if  $1 + zf''(z)/f'(z)$  lies in the parabolic region  $\Omega := \{u + iv : v^2 < 2u - 1\}$ . A corresponding class  $\mathcal{PS}$  consisting of parabolic starlike functions  $f$ , where  $f(z) = zg'(z)$  for  $g$  in  $\mathcal{UCV}$ , was introduced in [30]. Clearly a function  $f$  is in  $\mathcal{PS}$  if and only if

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}).$$

A survey of these functions may be found in [3] (see also [31]), while some radius problems associated with the classes  $\mathcal{UCV}$  and  $\mathcal{PS}$  can be found in [9, 29].

Ma and Minda [19, Theorem 4, p. 171] (see also Rønning [30, Theorem 5, p. 194]) proved that the second coefficient of functions in  $\mathcal{UCV}$  is bounded by  $4/\pi^2$ . Hence we shall consider the class  $\mathcal{UCV}_b$  of functions  $f \in \mathcal{UCV}$  of the form  $f(z) = z + a_2z^2 + \dots$  with fixed second coefficient  $a_2 = a$  given by  $a = 4b/\pi^2, 0 \leq b \leq 1$ . Similarly, the class  $\mathcal{PS}_b$  consists of functions  $f \in \mathcal{SP}$  of the form  $f(z) = z + a_2z^2 + \dots$  with fixed second coefficient  $a_2 = a$  given by  $a = 8b/\pi^2, 0 \leq b \leq 1$ .

**Corollary 2.5.** *The radius of starlikeness of order  $\beta, 0 \leq \beta < 1$ , for the class  $\mathcal{PS}_b$  is  $R_\beta$ , where*

$$R_\beta = \begin{cases} 1 & \text{if } 0 \leq \beta \leq \frac{1}{2} \\ \frac{2\rho}{(1-\rho)^b + \sqrt{(1-\rho)^2 b^2 + 4\rho}} & \text{if } \frac{1}{2} \leq \beta < 1, \end{cases}$$

with

$$\rho = \tan^2 \left( \frac{\sqrt{1-\beta}\pi}{2\sqrt{2}} \right).$$

The radius of convexity of order  $\beta, 0 \leq \beta < 1$ , for the class  $\mathcal{UCV}_b$  is also  $R_\beta$ . These results are sharp.

*Proof.* Since  $f$  is uniformly convex, it follows (see Ma and Minda [19], Rønning [30]) that

$$1 + \frac{zf''(z)}{f'(z)} < \varphi_{PAR}(z),$$

where

$$\varphi_{PAR}(z) = 1 + \frac{2}{\pi^2} \left[ \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2 = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2k+1} \right) z^n.$$

Since  $\varphi_{PAR}(\mathbb{D})$  is the parabolic region  $\Omega$ , it is clear that  $\varphi_{PAR}$  is a convex function (and therefore starlike with respect to 1). Further  $\varphi_{PAR}(\mathbb{D})$  is symmetric with respect to the real axis and  $\varphi'_{PAR}(0) > 0$ . A calculation shows that

$$\min_{|z|=r} \operatorname{Re} \varphi_{PAR}(z) = \varphi_{PAR}(-r) = 1 - \frac{8}{\pi^2} (\arctan \sqrt{r})^2.$$

The proof now follows from Theorem 2.1 by noting that the inverse of the function  $\varphi_{PAR}$  is

$$\varphi_{PAR}^{-1}(w) = -\tan^2\left(\frac{\sqrt{1-w\pi}}{2\sqrt{2}}\right),$$

and hence

$$\varphi_{PAR}^{-1}(\beta) = -\tan^2\left(\frac{\sqrt{1-\beta\pi}}{2\sqrt{2}}\right) = -\rho.$$

The result now follows from Theorem 2.1.  $\square$

**Remark 2.6.** When  $b = 1$ , Corollary 2.5 reduces to a result in [9, Theorem 2.1, p. 303] for functions (with varying second coefficient) belonging to the class  $\mathcal{UCV}$ .

For real numbers  $A, B$  with  $|B| \leq 1, B < A$ , the function  $\varphi_{A,B}(z) = (1 + Az)/(1 + Bz)$  is a convex function whose range is symmetric with respect to the real axis and  $\varphi'_{A,B}(0) = A - B > 0$ . Further  $\varphi_{A,B}^{-1}(w) = (w - 1)/(A - Bw)$ , and so  $-\varphi_{A,B}^{-1}(\beta) = (1 - \beta)/(A - B\beta)$ . The following corollary now readily follows from Theorem 2.1.

**Corollary 2.7.** The radius of starlikeness of order  $\beta, 0 \leq \beta < 1$ , for the class  $\mathcal{ST}_b(A, B)$  is  $R_\beta$ , where  $R_\beta$  is given by

$$R_\beta = \begin{cases} 1 & \text{if } 0 \leq \beta \leq \frac{1-A}{1-B} \\ \frac{2\rho}{(1-\rho)b + \sqrt{(1-\rho)^2b^2 + 4\rho}} & \text{if } \frac{1-A}{1-B} \leq \beta < 1, \end{cases}$$

with  $\rho = (1 - \beta)/(A - B\beta)$ . This result is sharp.

For  $0 \leq b \leq 1$ , let  $\mathcal{SST}_b(\gamma) = \mathcal{ST}_b(((1 + z)/(1 - z))^\gamma)$ . An application of Theorem 2.1 yields the following corollary.

**Corollary 2.8.** The radius of starlikeness of order  $\beta, 0 \leq \beta < 1$ , for the class  $\mathcal{SST}_b(\gamma), 0 < \gamma \leq 1$ , is  $R_\beta$ , where

$$R_\beta = \frac{2\rho}{(1-\rho)b + \sqrt{(1-\rho)^2b^2 + 4\rho}} \tag{2}$$

with  $\rho = (1 - \beta^{1/\gamma})/(1 + \beta^{1/\gamma})$ . This result is sharp.

**Remark 2.9.** Stankiewicz [40, Theorem 6, p. 105] obtained the radius of starlikeness of order  $\beta$  for strongly starlike functions of order  $\alpha$  for functions with varying second coefficient. This is precisely Corollary 2.8 with  $b = 1$ . (Note that there is a typographical error in his result.)

Let  $\mathcal{SL}$  be the class of functions defined by

$$\mathcal{SL} := \left\{ f \in \mathcal{A} : \left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (z \in \mathbb{D}) \right\}$$

Thus a function  $f \in \mathcal{SL}$  if  $zf'(z)/f(z)$  lies in the region bounded by the right-half of the lemniscate of Bernoulli given by  $|w^2 - 1| < 1$ . This class  $\mathcal{SL}$  was introduced by Sokół and Stankiewicz [37]. Paprocki and Sokół [28] discussed a more general class  $\mathcal{S}^*(a, b)$  consisting of normalized analytic functions  $f$  satisfying  $|[zf'(z)/f(z)]^a - b| < b, b \geq \frac{1}{2}, a \geq 1$ . Some results for functions belonging to this class can be found in [38] and [39]. In particular, we like to mention that Sokół [39, Theorem 2 (2), p. 571] has shown that the radius of starlikeness of order  $\beta$  for functions in  $\mathcal{SL}$  is  $1 - \beta^2$ . Letting  $\mathcal{SL}_b = \mathcal{ST}_b(\sqrt{1+z})$ , we extend this result of Sokół for functions with fixed second coefficient in  $\mathcal{SL}$  in the following corollary.

**Corollary 2.10.** *The radius of starlikeness of order  $\beta$ ,  $0 \leq \beta < 1$ , for the class  $\mathcal{SL}_b$  is  $R_\beta$  given by equation (2) with  $\rho = 1 - \beta^2$ . This result is sharp.*

*Proof.* Let  $f \in \mathcal{SL}$  so that  $zf'(z)/f(z) < \sqrt{1+z} =: \varphi_{SL}(z)$ . It is easy to see that

$$\operatorname{Re} \sqrt{1+re^{it}} = \frac{1}{\sqrt{2}}(\sqrt{1+r^2+2r\cos t} + 1 + r\cos t)^{1/2},$$

and it attains its minimum at  $t = \pi$ . Thus

$$\min_{|z|=r} \operatorname{Re} \varphi_{SL}(z) = \sqrt{1-r} = \varphi_{SL}(-r).$$

Also  $\varphi_{SL}(\mathbb{D})$  is the right-half of the interior of the lemniscate of Bernoulli  $|w^2 - 1| = 1$ ; therefore it is convex (and hence starlike with respect to 1), and symmetric with respect to the real line. Notice that  $\varphi_{SL}^{-1}(w) = w^2 - 1$  and hence  $\rho := -\varphi_{SL}^{-1}(\beta) = 1 - \beta^2$ . The result now follows from Theorem 2.1.  $\square$

**Remark 2.11.** When  $b = 1$ , the above corollary reduces to a result of Sokół [39, Theorem 2 (2), p. 571].

### 3. Radius of strong starlikeness

In [9], the radius of strong starlikeness for Janowski starlike functions is computed by first finding the disk in which  $w = zf'(z)/f(z)$  lies and next subjecting the disk to be contained within the sector  $|\arg w| \leq \gamma\pi/2$ . It is shown in [9] that the disk  $|w - a| \leq R_a$  is contained in the sector  $|\arg w| \leq \pi\gamma/2$  provided  $R_a \leq (\operatorname{Re} a) \sin(\pi\gamma/2) - (\operatorname{Im} a) \cos(\pi\gamma/2)$ ,  $\operatorname{Im} a \geq 0$ . This section is devoted to finding the radius of strong starlikeness for certain classes of functions with fixed second coefficient.

**Theorem 3.1.** *Let  $\varphi$  be an analytic function with positive real part in  $\mathbb{D}$  with  $\varphi(0) = 1$  and  $\varphi'(0) > 0$ . Further let  $\varphi$  map the unit disc  $\mathbb{D}$  onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the radius of strong starlikeness of order  $\gamma$ ,  $0 < \gamma \leq 1$ , for functions  $f \in \mathcal{ST}_b(\varphi)$  is  $R_\gamma$ , where  $R_\gamma$  is the solution of the following equation in  $r$ :*

$$\max\{\arg \varphi(r_1z) : |z| = 1\} = \frac{\gamma\pi}{2},$$

with  $r_1$  given by (1). The result is sharp.

*Proof.* The proof follows along similar lines as in the proof of Theorem 2.1, and hence many of the details are omitted here. Let  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ . We claim that  $\varphi$  maps  $\mathbb{D}_r$  onto a region symmetric with respect to the real line. To see this, first note that since  $\varphi$  maps  $\mathbb{D}$  onto a region symmetric with respect to the real line, the Taylor's coefficients of  $\varphi$  are real and hence  $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ ,  $B_i \in \mathbb{R}$ . Let  $w \in \varphi(\mathbb{D}_r)$  so that  $w = \varphi(rz)$ ,  $z \in \mathbb{D}$ . Then  $\bar{w} = \overline{\varphi(rz)} = \varphi(r\bar{z}) \in \varphi(\mathbb{D}_r)$ . Hence  $\max\{|\arg \varphi(r_1z)| : |z| = 1\} = \max\{\arg \varphi(r_1z) : |z| = 1\}$ .  $\square$

Theorem 3.1 is next applied to certain special cases. Our first result is for the case of starlike functions.

**Corollary 3.2.** *The radius of strong starlikeness of order  $\gamma$ ,  $0 < \gamma \leq 1$ , for starlike functions in  $\mathcal{ST}_b$  is the number  $R_\beta$  given in equation (2) with  $\rho = \tan(\pi\gamma/4)$ .*

*Proof.* Let  $f$  be a normalized starlike function so that

$$\frac{zf'(z)}{f(z)} < \frac{1+z}{1-z} =: \varphi(z).$$

Since

$$\varphi(re^{it}) = \frac{1-r^2+2ri\sin t}{1+r^2-2r\cos t},$$

it follows that

$$\max\{\arg \varphi(r_1 z) : |z| = 1\} = \arctan\left(\frac{2r_1}{1-r_1^2}\right).$$

Then the equation  $\max\{\arg \varphi(r_1 z) : |z| = 1\} = \gamma\pi/2$  becomes

$$\frac{2r_1}{1-r_1^2} = \tan\left(\frac{\gamma\pi}{2}\right), \quad 0 < \gamma < 1,$$

while  $R_\beta = 1$  if  $\gamma = 1$ . The former case leads to

$$r_1 = \rho = \csc(\pi\gamma/2) - \cot(\pi\gamma/2) = \tan\left(\frac{\pi\gamma}{4}\right).$$

Solving  $r_1 = \rho$  for  $r$  yields the desired result.  $\square$

The next result is for the class  $\mathcal{SL}$ . Sokół [39, Theorem 2 (4), p. 571] obtained the radius of strong starlikeness of order  $\gamma$  for functions in  $\mathcal{SL}$  to be  $\sin(\pi\gamma)$ . We extend this result for starlike functions with fixed second coefficient in the following corollary.

**Corollary 3.3.** *The radius of strong starlikeness of order  $\gamma$ ,  $0 < \gamma \leq 1$ , for functions in the class  $\mathcal{SL}_b$  is the number  $R_\beta$  given by equation (2) with  $\rho = \sin(\pi\gamma)$ .*

*Proof.* Let  $f \in \mathcal{SL}_b$  so that

$$\frac{zf'(z)}{f(z)} < \sqrt{1+z} =: \varphi(z).$$

Then

$$\arg \varphi(re^{it}) = \frac{1}{2} \arctan\left(\frac{r \sin t}{1+r \cos t}\right) \leq \frac{1}{2} \arctan \frac{r}{\sqrt{1-r^2}}.$$

The equation  $\max\{\arg \varphi(r_1 z) : |z| = 1\} = \gamma\pi/2$  becomes

$$\frac{r_1}{\sqrt{1-r_1^2}} = \tan(\gamma\pi), \quad \gamma \neq \frac{1}{2},$$

and thus  $r_1 = \rho = \sin(\pi\gamma)$ . Solving  $r_1 = \rho$  for  $r$  yields the desired result. The case  $\gamma = 1/2$  yields  $R_\beta = 1$ .  $\square$

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