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Starlikeness of integral transforms and duality ☆

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ABSTRACT

For λ satisfying a certain admissibility criteria, sufficient conditions are obtained that ensure the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt$$

maps normalized analytic functions f satisfying

$$\operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma z f''(z)-\beta\right)>0$$

into the class of starlike functions. Several interesting examples of λ are considered. Connections with various earlier works are made, and the results obtained not only reduce to those earlier works, but indeed improved certain known results. As a consequence, the smallest value $\beta<1$ is obtained that ensures a function f satisfying $\operatorname{Re}(f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z))>\beta$ is starlike.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} := \{z\in\mathbb{C}\colon\ |z|<1\}$ with the normalization $f(0)=0=f'(0)-1$, and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$. A function f in $\mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin. Analytically this geometric property is equivalent to the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\quad z\in\mathbb{D}.$$

The subclass of $\mathcal{S}$ consisting of starlike functions is denoted by $\mathcal{S}^*$. For any two functions $f(z)=z+a_2z^2+\cdots$ and $g(z)=z+b_2z^2+\cdots$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f*g$ defined by

$$(f*g)(z) := z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

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For $f \in \mathcal{A}$, Fournier and Ruscheweyh [6] introduced the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt, \tag{1.1}$$

where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)\,dt = 1$. They used the Duality Principle [14,15] to prove starlikeness of the linear integral transform $V_\lambda(f)$ over functions f in the class

$$\mathcal{P}(\beta) := \bigl\{f \in \mathcal{A}\colon\ \exists\phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\bigl(f'(z) - \beta\bigr) > 0,\ z \in \mathbb{D}\bigr\}.$$

Such problems were previously handled using the theory of subordination (see for example [10]). The duality methodology seems to work best in the sense that it gives sharp estimates of the parameter β, in situations where it can be applied.

This duality technique is now popularly used by several authors to discuss similar problems. In 2001, Kim and Rønning [8] investigated starlikeness properties of the integral transform (1.1) for functions f in the class

$$\mathcal{P}_\alpha(\beta) := \left\{f \in \mathcal{A}\colon\ \exists\phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta\right) > 0,\ z \in \mathbb{D}\right\}.$$

In a recent paper Ponnusamy and Rønning [12] discussed this problem for functions f in the class

$$\mathcal{R}_\gamma(\beta) := \bigl\{f \in \mathcal{A}\colon\ \exists\phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\bigl(f'(z) + \gamma z f''(z) - \beta\bigr) > 0,\ z \in \mathbb{D}\bigr\}.$$

For $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $\beta < 1$, define the class

$$\mathcal{W}_\beta(\alpha,\gamma) := \left\{f \in \mathcal{A}\colon\ \exists\phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\right) > 0,\ z \in \mathbb{D}\right\}. \tag{1.2}$$

It is evident that $\mathcal{P}(\beta) \equiv \mathcal{W}_\beta(1,0)$, $\mathcal{P}_\alpha(\beta) \equiv \mathcal{W}_\beta(\alpha,0)$, and $\mathcal{R}_\gamma(\beta) \equiv \mathcal{W}_\beta(1+2\gamma,\gamma)$.

The class $\mathcal{W}_\beta(\alpha,\gamma)$ is closely related to the class $R(\alpha,\gamma,h)$ consisting of all functions $f \in \mathcal{A}$ satisfying

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec h(z), \quad z \in \mathbb{D},$$

with $h(z) := h_\beta(z) = (1 + (1-2\beta)z)/(1-z)$. Here $q(z) \prec h(z)$ indicates that the function q is subordinate to h, or in other words, there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $q(z) = h(w(z))$, $z \in \mathbb{D}$. In the special case $\phi = 0$ in (1.2), it is evident that $f \in R(\alpha,\gamma,h_\beta)$ if and only if zf' is in a subclass of $\mathcal{W}_\beta(\alpha,\gamma)$. Functions $f \in R(\alpha,\gamma,h)$ for a suitably normalized convex function h have a double integral representation, which was recently investigated by Ali et al. [1].

Interestingly, the general integral transform $V_\lambda(f)$ in (1.1) reduces to various well-known integral operators for specific choices of λ. For example,

$$\lambda(t) := (1+c)t^c, \quad c > -1,$$

gives the Bernardi integral operator, while the choice

$$\lambda(t) := \frac{(a+1)^p}{\Gamma(p)} t^a \left(\log\frac{1}{t}\right)^{p-1}, \quad a > -1,\ p \geqslant 0,$$

gives the Komatu operator [9]. Clearly for $p = 1$ the Komatu operator is in fact the Bernardi operator.

For a certain choice of λ, the integral operator V_λ is the convolution between a function f and the Gaussian hypergeometric function $F(a,b;c;z) := {}_2F_1(a,b;c;z)$, which is related to the general Hohlov operator [7] given by

$$H_{a,b,c}(f) := zF(a,b;c;z) * f(z).$$

In the special case $a = 1$, the operator reduces to the Carlson–Shaffer operator [5]. Here ${}_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function given by the series

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \quad z \in \mathbb{D},$$

where the Pochhammer symbol is used to indicate $(a)_n = a(a+1)_{n-1}$, $(a)_0 = 1$, and where a, b, c are real parameters with $c \neq 0, -1, -2, \ldots.$

In the present manuscript, the Duality Principle is used to investigate the starlikeness of the integral transform $V_\lambda(f)$ in (1.1) over the class $\mathcal{W}_\beta(\alpha,\gamma)$. In Section 3, the best value of $\beta < 1$ is determined ensuring that $V_\lambda(f)$ maps $\mathcal{W}_\beta(\alpha,\gamma)$ into the class of normalized univalent functions $\mathcal{S}$. Additionally, necessary and sufficient conditions are determined that ensure $V_\lambda(f)$ is starlike univalent over the class $\mathcal{W}_\beta(\alpha,\gamma)$. In Section 4, we find easier sufficient conditions for $V_\lambda(f)$ to be starlike, and Section 5 is devoted to several applications of results obtained for specific choices of the admissible function λ. In particular, the smallest value $\beta < 1$ is obtained that ensures a function f satisfying $\operatorname{Re}(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)) > \beta$ in the unit disk is starlike.

2. Preliminaries

First we introduce two constants $\mu \geqslant 0$ and $\nu \geqslant 0$ satisfying

$$\mu+\nu=\alpha-\gamma \quad \text{and} \quad \mu\nu=\gamma. \tag{2.1}$$

When $\gamma=0$, then μ is chosen to be 0, in which case, $\nu=\alpha \geqslant 0$. When $\alpha=1+2\gamma$, (2.1) yields $\mu+\nu=1+\gamma=1+\mu\nu$, or $(\mu-1)(1-\nu)=0$.

(i) For $\gamma>0$, then choosing $\mu=1$ gives $\nu=\gamma$.
(ii) For $\gamma=0$, then $\mu=0$ and $\nu=\alpha=1$.

In the sequel, whenever the particular case $\alpha=1+2\gamma$ is considered, the values of μ and ν for $\gamma>0$ will be taken as $\mu=1$ and $\nu=\gamma$ respectively, while $\mu=0$ and $\nu=1=\alpha$ in the case $\gamma=0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z)=1+\sum_{n=1}^{\infty}\frac{(n\nu+1)(n\mu+1)}{n+1}z^n, \tag{2.2}$$

and

$$\begin{aligned}\psi_{\mu,\nu}(z)=\phi_{\mu,\nu}^{-1}(z)&=1+\sum_{n=1}^{\infty}\frac{n+1}{(n\nu+1)(n\mu+1)}z^n\\ &=\int_0^1\int_0^1\frac{ds\,dt}{(1-t^\nu s^\mu z)^2}.\end{aligned} \tag{2.3}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu}*\phi_{\mu,\nu}^{-1}=z/(1-z)$. If $\gamma=0$, then $\mu=0$, $\nu=\alpha$, and it is clear that

$$\psi_{0,\alpha}(z)=1+\sum_{n=1}^{\infty}\frac{n+1}{n\alpha+1}z^n=\int_0^1\frac{dt}{(1-t^\alpha z)^2}.$$

If $\gamma>0$, then $\nu>0$, $\mu>0$, and making the change of variables $u=t^\nu$, $v=s^\mu$ results in

$$\psi_{\mu,\nu}(z)=\frac{1}{\mu\nu}\int_0^1\int_0^1\frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvz)^2}\,du\,dv.$$

Thus the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z)=\begin{cases}\frac{1}{\mu\nu}\int_0^1\int_0^1\frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvz)^2}\,du\,dv, & \gamma>0,\\ \int_0^1\frac{dt}{(1-t^\alpha z)^2}, & \gamma=0,\ \alpha\geqslant 0.\end{cases} \tag{2.4}$$

Now let g be the solution of the initial-value problem

$$\frac{d}{dt}t^{1/\nu}\big(1+g(t)\big)=\begin{cases}\frac{2}{\mu\nu}t^{1/\nu-1}\int_0^1\frac{s^{1/\mu-1}}{(1+st)^2}\,ds, & \gamma>0,\\ \frac{2}{\alpha}\frac{t^{1/\alpha-1}}{(1+t)^2}, & \gamma=0,\ \alpha>0,\end{cases} \tag{2.5}$$

satisfying $g(0)=1$. It is easily seen that the solution is given by

$$g(t)=\frac{2}{\mu\nu}\int_0^1\int_0^1\frac{s^{1/\mu-1}w^{1/\nu-1}}{(1+swt)^2}\,ds\,dw-1=2\sum_{n=0}^{\infty}\frac{(n+1)(-1)^nt^n}{(1+\mu n)(1+\nu n)}-1. \tag{2.6}$$

In particular,

$$\begin{aligned}g_\gamma(t)&=\frac{1}{\gamma}\int_0^1 s^{1/\gamma-1}\frac{1-st}{1+st}\,ds, \quad \gamma>0,\ \alpha=1+2\gamma,\\ g_\alpha(t)&=\frac{2}{\alpha}t^{-1/\alpha}\int_0^t\frac{\tau^{1/\alpha-1}}{(1+\tau)^2}\,d\tau-1, \quad \gamma=0,\ \alpha>0.\end{aligned} \tag{2.7}$$

3. Main results

Functions in the class $\mathcal{W}_\beta(\alpha,\gamma)$ generally are not starlike; indeed, they may not even be univalent. Our central result below provides conditions for univalence and starlikeness.

Theorem 3.1. *Let $\mu \geqslant 0$, $\nu \geqslant 0$ satisfy* (2.1)*, and let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\, dt, \tag{3.1}$$

where g is the solution of the initial-value problem (2.5)*. If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, then $F = V_\lambda(f) \in \mathcal{W}_0(1,0) \subset \mathcal{S}$.*

Further let

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}\, dx, \quad \nu > 0, \tag{3.2}$$

$$\Pi_{\mu,\nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu}\, dx, & \gamma > 0\ (\mu > 0,\ \nu > 0), \\ \Lambda_\alpha(t), & \gamma = 0\ (\mu = 0,\ \nu = \alpha > 0), \end{cases} \tag{3.3}$$

and assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Let

$$h(z) = \frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon| = 1.$$

Then

$$\begin{cases} \operatorname{Re} \displaystyle\int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt \geqslant 0, & \gamma > 0, \\ \operatorname{Re} \displaystyle\int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt \geqslant 0, & \gamma = 0, \end{cases} \tag{3.4}$$

if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{S}^*$. This conclusion does not hold for smaller values of β.

Proof. Since the case $\gamma = 0$ ($\mu = 0$ and $\nu = \alpha$) corresponds to [8, Theorem 2.1], it is sufficient to consider only the case $\gamma > 0$.

Let

$$H(z) = (1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma) f'(z) + \gamma z f''(z).$$

Since $\nu + \mu = \alpha - \gamma$ and $\mu\nu = \gamma$, then

$$\begin{aligned} H(z) &= \bigl(1+\gamma-(\alpha-\gamma)\bigr)\frac{f(z)}{z} + (\alpha-\gamma-\gamma) f'(z) + \gamma z f''(z) \\ &= (1+\mu\nu-\nu-\mu)\frac{f(z)}{z} + (\nu+\mu-\mu\nu) f'(z) + \mu\nu z f''(z) \\ &= \mu\nu\left(\frac{1}{\nu}-1\right)\left(\frac{1}{\mu}-1\right) z^{-1} f(z) + \mu\nu\left(\frac{1}{\nu}-1\right) f'(z) + \nu f'(z) + \mu\nu z f''(z) \\ &= \mu\nu z^{1-1/\mu}\frac{d}{dz}\left[z^{1/\mu-1/\nu+1}\left(\left(\frac{1}{\nu}-1\right) z^{1/\nu-2} f(z) + z^{1/\nu-1} f'(z)\right)\right] \\ &= \mu\nu z^{1-1/\mu}\frac{d}{dz}\left[z^{1/\mu-1/\nu+1}\frac{d}{dz}\bigl(z^{1/\nu-1} f(z)\bigr)\right]. \end{aligned}$$

With $f(z) = z + \sum_{n=2}^\infty a_n z^n$, it follows from (2.2) that

$$H(z) = 1 + \sum_{n=1}^\infty a_{n+1}(n\nu+1)(n\mu+1) z^n = f'(z) * \phi_{\mu,\nu}, \tag{3.5}$$

and (2.3) yields

$$f'(z) = H(z) * \psi_{\mu,\nu}(z). \tag{3.6}$$

Let g be given by

$$g(z) = \frac{H(z) - \beta}{1 - \beta}.$$

Since $\operatorname{Re} e^{i\phi} g(z) > 0$, without loss of generality, we may assume that

$$g(z) = \frac{1 + xz}{1 + yz}, \quad |x| = 1,\ |y| = 1. \tag{3.7}$$

Now (3.6) implies that $f'(z) = [(1-\beta)g(z) + \beta] * \psi_{\mu,\nu}$, and (3.7) readily gives

$$\frac{f(z)}{z} = \frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw} + \beta\right) dw * \psi(z), \tag{3.8}$$

where for convenience, we write $\psi := \psi_{\mu,\nu}$.

To show that $F \in \mathcal{S}$, the Noshiro–Warschawski Theorem asserts it is sufficient to prove that $F'(\mathbb{D})$ is contained in a half-plane not containing the origin. Now

$$\begin{aligned} F'(z) &= \int_0^1 \frac{\lambda(t)}{1-tz}\,dt * f'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}\,dt * \left((1-\beta)\frac{1+xz}{1+yz} + \beta\right) * \psi(z) \\ &= \int_0^1 \lambda(t)\psi(tz)\,dt * \left((1-\beta)\frac{1+xz}{1+yz} + \beta\right) = \left(\int_0^1 \lambda(t)\big[(1-\beta)\psi(tz) + \beta\big]\,dt\right) * \frac{1+xz}{1+yz}. \end{aligned}$$

It is known [15, p. 23] that the dual set of functions g given by (3.7) consists of analytic functions q satisfying $q(0) = 1$ and $\operatorname{Re} q(z) > 1/2$ in $\mathbb{D}$. Thus

$$\begin{aligned} F' \neq 0 \quad &\Longleftrightarrow \quad \operatorname{Re}\int_0^1 \lambda(t)\big[(1-\beta)\psi(tz) + \beta\big]\,dt > \frac{1}{2} \\ &\Longleftrightarrow \quad \operatorname{Re}(1-\beta)\left[\int_0^1 \lambda(t)\psi(tz)\,dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right] > 0. \end{aligned}$$

It follows from (3.1) and (2.4) that the latter condition is equivalent to

$$\operatorname{Re}\int_0^1 \lambda(t)\left[\left(\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvtz)^2}\,du\,dv\right) - \left(\frac{1+g(t)}{2}\right)\right] dt > 0. \tag{3.9}$$

Now

$$\begin{aligned} &\operatorname{Re}\int_0^1 \lambda(t)\left[\left(\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvtz)^2}\,du\,dv\right) - \left(\frac{1+g(t)}{2}\right)\right] dt \\ &\quad \geqslant \operatorname{Re}\int_0^1 \lambda(t)\left[\left(\frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1+uvt)^2}\,du\,dv\right) - \left(\frac{1+g(t)}{2}\right)\right] dt. \end{aligned} \tag{3.10}$$

The condition (2.6) implies that

$$\frac{1+g(t)}{2} = \frac{1}{\mu\nu}\int_0^1\int_0^1 \frac{w^{1/\nu-1}s^{1/\mu-1}}{(1+swt)^2}\,ds\,dw.$$

Substituting this value into (3.10) makes the integrand vanish, and so condition (3.9) holds. Consequently $F'(\mathbb{D}) \subset \operatorname{co} g(\mathbb{D})$ with g given by (3.7) ([15, p. 23], [13, Lemma 4, p. 146]), which gives $\operatorname{Re} e^{i\theta} F'(z) > 0$ for $z \in \mathbb{D}$. Hence F is close-to-convex, and thus univalent.

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, a well-known result in [15, p. 94] states that

$$F \in S^* \iff \frac{1}{z}(F * h)(z) \neq 0, \quad z \in \mathbb{D},$$

where

$$h(z) = \frac{z(1 + \frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon| = 1.$$

Hence $F \in S^*$ if and only if

$$0 \neq \frac{1}{z}\big(V_\lambda(f)(z) * h(z)\big) = \frac{1}{z}\left[\int_0^1 \lambda(t)\frac{f(tz)}{t}\,dt * h(z)\right]$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}\,dt * \frac{f(z)}{z} * \frac{h(z)}{z}.$$

From (3.8), it follows that

$$0 \neq \int_0^1 \frac{\lambda(t)}{1-tz}\,dt * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw} + \beta\right)dw * \psi(z)\right] * \frac{h(z)}{z}$$

$$= \int_0^1 \frac{\lambda(t)}{1-tz}\,dt * \frac{h(z)}{z} * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw} + \beta\right)dw\right] * \psi(z)$$

$$= \int_0^1 \lambda(t)\frac{h(tz)}{tz}\,dt * (1-\beta)\left[\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}\,dw + \frac{\beta}{1-\beta}\right] * \psi(z)$$

$$= (1-\beta)\left[\int_0^1 \lambda(t)\frac{h(tz)}{tz}\,dt + \frac{\beta}{1-\beta}\right] * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}\,dw * \psi(z).$$

Hence

$$0 \neq (1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}\,dw\right)dt + \frac{\beta}{1-\beta}\right] * \frac{1+xz}{1+yz} * \psi(z)$$

$$\iff \quad \mathrm{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}\,dw\right)dt + \frac{\beta}{1-\beta}\right] * \psi(z) > \frac{1}{2}$$

$$\iff \quad \mathrm{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}\,dw\right)dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right] * \psi(z) > 0$$

$$\iff \quad \mathrm{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}\,dw\right)dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right] * \psi(z) > 0.$$

Using (3.1), the latter condition is equivalent to

$$\mathrm{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}\,dw - \frac{1+g(t)}{2}\right)dt\right] * \psi(z) > 0.$$

From (2.3), the above inequality is equivalent to

$$\begin{aligned}
0 &< \operatorname{Re}\int_0^1 \lambda(t)\left(\sum_{n=0}^{\infty}\frac{z^n}{(n\nu+1)(n\mu+1)} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2}\right)dt \\
&= \operatorname{Re}\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{d\eta\, d\zeta}{1-z\eta^{\nu}\zeta^{\mu}} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2}\right)dt \\
&= \operatorname{Re}\int_0^1 \lambda(t)\left(\int_0^1\int_0^1 \frac{h(tz\eta^{\nu}\zeta^{\mu})}{tz\eta^{\nu}\zeta^{\mu}}\, d\eta\, d\zeta - \frac{1+g(t)}{2}\right)dt,
\end{aligned}$$

which reduces to

$$\operatorname{Re}\int_0^1 \lambda(t)\left[\int_0^1\int_0^1 \frac{1}{\mu\nu}\frac{h(tzuv)}{tzuv}u^{1/\nu-1}v^{1/\mu-1}\, dv\, du - \frac{1+g(t)}{2}\right]dt > 0.$$

A change of variable $w = tu$ leads to

$$\operatorname{Re}\int_0^1 \frac{\lambda(t)}{t^{1/\nu}}\left[\int_0^t\int_0^1 \frac{h(wzv)}{wzv}w^{1/\nu-1}v^{1/\mu-1}\, dv\, dw - \mu\nu t^{1/\nu}\frac{1+g(t)}{2}\right]dt > 0.$$

Integrating by parts with respect to t and using (2.5) gives the equivalent form

$$\operatorname{Re}\int_0^1 \Lambda_{\nu}(t)\left[\int_0^1 \frac{h(tzv)}{tzv}t^{1/\nu-1}v^{1/\mu-1}\, dv - t^{1/\nu-1}\int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2}\, ds\right]dt \geqslant 0.$$

Making the variable change $w = vt$ and $\eta = st$ reduces the above inequality to

$$\operatorname{Re}\int_0^1 \Lambda_{\nu}(t)t^{1/\nu-1/\mu-1}\left[\int_0^t \frac{h(wz)}{wz}w^{1/\mu-1}\, dw - \int_0^t \frac{\eta^{1/\mu-1}}{(1+\eta)^2}\, d\eta\right]dt \geqslant 0,$$

which after integrating by parts with respect to t yields

$$\operatorname{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt \geqslant 0.$$

Thus $F \in \mathcal{S}^*$ if and only if condition (3.4) holds.

To verify sharpness, let β_0 satisfy

$$\frac{\beta_0}{1-\beta_0} = -\int_0^1 \lambda(t)g(t)\, dt.$$

Assume that $\beta < \beta_0$ and let $f \in \mathcal{W}_{\beta}(\alpha,\gamma)$ be the solution of the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

From (3.5), it follows that

$$f(z) = z + \sum_{n=1}^{\infty}\frac{2(1-\beta)}{(n\nu+1)(n\mu+1)}z^{n+1}.$$

Thus

$$G(z) = V_{\lambda}(f)(z) = z + \sum_{n=1}^{\infty}\frac{2(1-\beta)\tau_n}{(n\nu+1)(n\mu+1)}z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t)t^n\,dt$. Now (2.6) implies that

$$\frac{\beta_0}{1-\beta_0} = -\int_0^1 \lambda(t)g(t)\,dt = -1-2\sum_{n=1}^{\infty}\frac{(n+1)(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}.$$

This means that

$$G'(-1) = 1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(n+1)(-1)^n\tau_n}{(1+\mu n)(1+\nu n)} = 1-\frac{1-\beta}{1-\beta_0} < 0.$$

Hence $G'(z)=0$ for some $z\in\mathbb{D}$, and so G is not even locally univalent in $\mathbb{D}$. Therefore the value of β in (3.1) is sharp. $\square$

Remark 3.1. Theorem 3.1 yields several known results.

(1) When $\gamma=0$, then $\mu=0$, $\nu=\alpha$, and in this particular instance, Theorem 3.1 gives Theorem 2.1 in Kim and Rønning [8].
(2) The special case $\alpha=1$ above yields a result of Fournier and Ruscheweyh [6, Theorem 2].
(3) If $\alpha=1+2\gamma$, then $\mu=1$ and $\nu=\gamma$ in the case $\gamma>0$, while $\mu=0$ and $\nu=\alpha=1$ when $\gamma=0$. In this instance, Theorem 3.1 gives Theorem 2.2 in Ponnusamy and Rønning [12].

4. Starlikeness criteria of integral transforms

An easier sufficient condition for starlikeness of the integral operator (1.1) is given in the following theorem.

Theorem 4.1. *Let $\Pi_{\mu,\nu}$ and Λ_ν be as given in Theorem* 3.1. *Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on* $[0,1]$ *and positive on* $(0,1)$. *Assume further that $\mu\geqslant 1$ and*

$$\frac{\Pi_{\mu,\nu}(t)}{1-t^2}\quad \text{is decreasing on } (0,1). \tag{4.1}$$

If β satisfies (3.1), *and $f\in\mathcal{W}_\beta(\alpha,\gamma)$, then $V_\lambda(f)\in\mathcal{S}^*$.*

Proof. The function $t^{1/\mu-1}$ is decreasing on $(0,1)$ when $\mu\geqslant 1$. Thus the condition (4.1) along with [6, Theorem 1] yield

$$\operatorname{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz}-\frac{1}{(1+t^2)}\right)dt \geqslant 0.$$

The desired conclusion now follows from Theorem 3.1. $\square$

Let us scrutinize Theorem 4.1 for helpful conditions to ensure starlikeness of $V_\lambda(f)$. Recall that for $\gamma>0$,

$$\Pi_{\mu,\nu}(t) = \int_t^1 \Lambda_\nu(y)y^{1/\nu-1-1/\mu}\,dy \quad\text{and}\quad \Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}}\,dx.$$

To apply Theorem 4.1, it is sufficient to show that the function

$$p(t) = \frac{\Pi_{\mu,\nu}(t)}{1-t^2} \tag{4.2}$$

is decreasing in the interval $(0,1)$. Note that $p(t)>0$ and

$$\frac{p'(t)}{p(t)} = -\frac{\Lambda_\nu(t)}{t^{1+1/\mu-1/\nu}\Pi_{\mu,\nu}(t)}+\frac{2t}{1-t^2}.$$

So it remains to show that $q'(t)\geqslant 0$ over $(0,1)$, where

$$q(t) := \Pi_{\mu,\nu}(t)-\frac{1-t^2}{2}\Lambda_\nu(t)t^{1/\nu-2-1/\mu}.$$

Since $q(1)=0$, this will imply that $p'(t)\leqslant 0$, and p is decreasing on $(0,1)$. Now

$$\begin{aligned} q'(t) &= \Pi'_{\mu,\nu}(t)-\frac{1}{2}\left[\left(1-t^2\right)\Lambda'_\nu(t)t^{1/\nu-2-1/\mu}+\Lambda_\nu(t)(-2t)t^{1/\nu-2-1/\mu}+\Lambda_\nu(t)\left(1-t^2\right)\left(\frac{1}{\nu}-2-\frac{1}{\mu}\right)t^{1/\nu-3-1/\mu}\right] \\ &= \frac{1-t^2}{2}t^{1/\nu-3-1/\mu}\left[\lambda(t)t^{1-1/\nu}-\left(\frac{1}{\nu}-2-\frac{1}{\mu}\right)\Lambda_\nu(t)\right]. \end{aligned}$$

So $q'(t) \geqslant 0$ is equivalent to the condition

$$\Delta(t) := -\lambda(t)t^{1-1/\nu} + \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\Lambda_\nu(t) \leqslant 0. \tag{4.3}$$

Since $\lambda(t) \geqslant 0$ gives $\Lambda_\nu(t) \geqslant 0$ for $t \in (0,1)$, condition (4.3) holds whenever $1/\nu - 2 - 1/\mu \leqslant 0$, or $\nu \geqslant \mu/(2\mu+1)$.

These observations will be used to prove the following theorem.

Theorem 4.2. *Let λ be a non-negative real-valued integrable function on $[0,1]$. Assume that Λ_ν and $\Pi_{\mu,\nu}$ given respectively by (3.2) and (3.3) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 3.1, if λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leqslant \begin{cases} 1 + \frac{1}{\mu}, & \mu \geqslant 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty), \end{cases} \tag{4.4}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{S}^*$. The conclusion does not hold for smaller values of β.

Proof. Suppose $\mu \geqslant 1$. In view of (4.3) and Theorem 4.1, the integral transform $V_\lambda(f)(z) \in \mathcal{S}^*$ for $\nu \geqslant \mu/(2\mu+1)$. It remains to find conditions on μ and ν in the range $0 \leqslant \nu < \mu/(2\mu+1)$ such that for each choice of λ, condition (4.3) is satisfied.

Now $\Delta(t)$ at $t = 1$ in (4.3) reduces to

$$\Delta(1) = -\lambda(1) + \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\Lambda_\nu(1) = -\lambda(1) \leqslant 0.$$

Hence to prove condition (4.3), it is enough to show that Δ is an increasing function in $(0,1)$. Now

$$\begin{aligned}\Delta'(t) &= -\lambda'(t)t^{1-1/\nu} - \left(1 - \frac{1}{\nu}\right)\lambda(t)t^{-1/\nu} - \left(\frac{1}{\nu} - 2 - \frac{1}{\mu}\right)\frac{\lambda(t)}{t^{1/\nu}} \\ &= -\lambda(t)t^{-1/\nu}\left[\frac{t\lambda'(t)}{\lambda(t)} - \left(1 + \frac{1}{\mu}\right)\right],\end{aligned}$$

and this is non-negative when $t\lambda'(t)/\lambda(t) \leqslant 1 + 1/\mu$.

In the case $\gamma = 0$, then $\mu = 0$, $\nu = \alpha > 0$. Let

$$k(t) := \Lambda_\alpha(t)t^{1/\alpha - 1}, \quad \text{where } \Lambda_\alpha(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\alpha}}\,dx.$$

To apply Theorem 1 in [6] along with Theorem 3.1, the function $p(t) = k(t)/(1-t^2)$ must be shown to be decreasing on the interval $(0,1)$. This will hold provided

$$q(t) := k(t) + \frac{1-t^2}{2}t^{-1}k'(t) \leqslant 0.$$

Since $q(1) = 0$, this will certainly hold if q is increasing on $(0,1)$. Now

$$q'(t) = \frac{(1-t^2)}{2}t^{-2}\big[tk''(t) - k'(t)\big],$$

and

$$\begin{aligned}tk''(t) - k'(t) &= \Lambda''_\alpha(t)t^{1/\alpha} + 2\left(\frac{1}{\alpha} - 1\right)\Lambda'_\alpha(t)t^{1/\alpha - 1} + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right)\Lambda_\alpha(t)t^{1/\alpha - 2} \\ &\quad - \Lambda'_\alpha(t)t^{1/\alpha - 1} - \left(\frac{1}{\alpha} - 1\right)\Lambda_\alpha(t)t^{1/\alpha - 2} \\ &= t^{1/\alpha - 2}\left[\Lambda''_\alpha(t)t^2 + \Lambda'_\alpha(t)t\left(\frac{2}{\alpha} - 3\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right)\Lambda_\alpha(t)\right].\end{aligned}$$

Thus $tk''(t) - k'(t)$ is non-negative if

$$\Lambda''_\alpha(t)t^2 + \Lambda'_\alpha(t)t\left(\frac{2}{\alpha} - 3\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right)\Lambda_\alpha(t) \geqslant 0.$$

The latter condition is equivalent to

$$-\lambda'(t)t^{2-1/\alpha} + \lambda(t)t^{1-1/\alpha}\left(3 - \frac{1}{\alpha}\right) + \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 3\right)\Lambda_\alpha(t) \geqslant 0. \tag{4.5}$$

Since $\Lambda_\alpha(t) \geqslant 0$ and $(1/\alpha - 1)(1/\alpha - 3) \geqslant 0$ for $\alpha \in (0, 1/3] \cup [1, \infty)$, then $q'(t) \geqslant 0$ is equivalent to

$$-\lambda'(t)t^{2-1/\alpha} + \lambda(t)t^{1-1/\alpha}\left(3 - \frac{1}{\alpha}\right) \geqslant 0 \quad \Longleftrightarrow \quad \frac{t\lambda'(t)}{\lambda(t)} \leqslant 3 - \frac{1}{\alpha}.$$

Thus (4.3) is satisfied and the proof is complete. □

Remark 4.1.

(1) For $\mu < 1$, the conditions obtained will generally be complicated, and for $\mu \geqslant 1$, the conditions coincide with those given in [12].
(2) Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 4.2 yields Corollary 3.1 in [4] and Theorem 3.1 in [12].
(3) The condition $\mu \geqslant 1$ is equivalent to $0 < \gamma \leqslant \alpha \leqslant 2\gamma + 1$.

5. Applications to certain integral transforms

In this section, various well-known integral operators are considered, and conditions for starlikeness for $f \in \mathcal{W}_\beta(\alpha, \gamma)$ under these integral operators are obtained. First let λ be defined by

$$\lambda(t) = (1 + c)t^c, \quad c > -1.$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1 + c)\int_0^1 t^{c-1} f(tz)\, dt, \quad c > -1, \tag{5.1}$$

is the Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (5.1) with $c = 0$ and $c = 1$ respectively. For this special case of λ, the following result holds.

Theorem 5.1. *Let $c > -1$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1 - \beta} = -(c + 1)\int_0^1 t^c g(t)\, dt,$$

where g is given by (2.6)*. If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = (1 + c)\int_0^1 t^{c-1} f(tz)\, dt$$

belongs to $\mathcal{S}^*$ if

$$c \leqslant \begin{cases} 1 + \frac{1}{\mu}, & \mu \geqslant 1\ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty). \end{cases}$$

The value of β is sharp.

Proof. With $\lambda(t) = (1 + c)t^c$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = t\frac{c(1 + c)t^{c-1}}{(1 + c)t^c} = c,$$

and the result now follows from Theorem 4.2. □

Taking $\gamma = 0$, $\alpha > 0$ in Theorem 5.1 leads to the following corollary:

Corollary 5.1. *Let $-1 < c \leqslant 3 - 1/\alpha$, $\alpha \in (0, 1/3] \cup [1, \infty)$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1 - \beta} = -(c + 1)\int_0^1 t^c g_\alpha(t)\, dt,$$

where g_α is given by (2.7)*. If $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\alpha(\beta)$, then the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)\,dt$$

belongs to $\mathcal{S}^*$. The value of β is sharp.

Remark 5.1. When $\alpha = 1+2\gamma$, $\gamma > 0$, and $\mu = 1$, Theorem 5.1 yields Corollary 3.2 obtained by Ponnusamy and Rønning [12], while in the case $\alpha = 1$ and $\gamma = 0$, Theorem 5.1 yields Corollary 1 in Fournier and Ruscheweyh [6].

The case $c = 0$ in Theorem 5.1 yields the following interesting result, which we state as a theorem.

Theorem 5.2. *Let $\alpha \geqslant \gamma > 0$, or $\gamma = 0$, $\alpha \geqslant 1/3$. If $F \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\big(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)\big) > \beta$$

in $\mathbb{D}$, and $\beta < 1$ satisfies

$$\frac{\beta}{1-\beta} = -\int_0^1 g(t)\,dt,$$

where g is given by (2.6)*, then F is starlike. The value of β is sharp.*

Proof. It is evident that the function $f = zF'$ belongs to the class

$$\mathcal{W}_{\beta,0}(\alpha,\gamma) = \left\{ f \in \mathcal{A}\colon \operatorname{Re}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z)\right) > \beta,\ z \in \mathbb{D}\right\}.$$

Thus

$$F(z) = \int_0^1 \frac{f(tz)}{t}\,dt,$$

and the result follows from Theorem 5.1 with $c = 0$ for the ranges $\alpha \geqslant \gamma > 0$, or $\gamma = 0$, $\alpha \geqslant 1$. Simple computations show that in fact (4.5) is satisfied in the larger range $\gamma = 0$, $\alpha \geqslant 1/3$. It is also evident from the proof of sharpness in Theorem 3.1 that indeed the extremal function in $\mathcal{W}_\beta(\alpha,\gamma)$ also belongs to the class $\mathcal{W}_{\beta,0}(\alpha,\gamma)$. □

Remark 5.2. We list two interesting special cases.

(1) If $\gamma = 0$, $\alpha \geqslant 1/3$, and $\beta = \kappa/(1+\kappa)$, where (2.6) yields

$$\kappa = -\int_0^1 g(t)\,dt = -1 - 2\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\alpha} = -\frac{1}{\alpha}\int_0^1 t^{1/\alpha - 1}\frac{1-t}{1+t}\,dt,$$

then

$$\operatorname{Re}\big(f'(z) + \alpha z f''(z)\big) > \beta \quad \Rightarrow \quad f \in \mathcal{S}^*.$$

This reduces to a result of Fournier and Ruscheweyh [6]. In particular, if $\beta = (1 - 2\ln 2)/(2(1-\ln 2)) = -0.629445$, then

$$\operatorname{Re}\big(f'(z) + z f''(z)\big) > \beta \quad \Rightarrow \quad f \in \mathcal{S}^*.$$

(2) If $\gamma = 1$, $\alpha = 3$, then $\mu = 1 = \nu$. In this case, (2.6) yields $\beta = (6 - \pi^2)/(12 - \pi^2) = -1.816378$. Thus

$$\operatorname{Re}\big(f'(z) + 3z f''(z) + z^2 f'''(z)\big) > \beta \quad \Rightarrow \quad f \in \mathcal{S}^*.$$

This sharp estimate of β improves a result of Ali et al. [1].

Theorem 5.3. *Let $b > -1$, $a > -1$, and $\alpha > 0$. Let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\,dt,$$

where g is given by (2.6) *and*

$$\lambda(t)=\begin{cases}(a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b\neq a,\\ (a+1)^2t^a\log(1/t), & b=a.\end{cases}$$

If $f\in\mathcal{W}_\beta(\alpha,\gamma)$, *then*

$$\mathcal{G}_f(a,b;z)=\begin{cases}\frac{(a+1)(b+1)}{b-a}\int_0^1 t^{a-1}(1-t^{b-a})f(tz)\,dt, & b\neq a,\\ (a+1)^2\int_0^1 t^{a-1}\log(1/t)f(tz)\,dt, & b=a,\end{cases}$$

belongs to $\mathcal{S}^*$ *if*

$$a\leqslant\begin{cases}1+\frac{1}{\mu}, & \gamma>0\ (\mu\geqslant 1),\\ 3-\frac{1}{\alpha}, & \gamma=0,\ \alpha\in(0,1/3]\cup[1,\infty).\end{cases}\tag{5.2}$$

The value of β *is sharp.*

Proof. It is easily seen that $\int_0^1\lambda(t)\,dt=1$. There are two cases to consider. When $b\neq a$, then

$$\frac{t\lambda'(t)}{\lambda(t)}=a-\frac{(b-a)t^{b-a}}{1-t^{b-a}}.$$

The function λ satisfies (4.4) if

$$a-\frac{(b-a)t^{b-a}}{1-t^{b-a}}\leqslant\begin{cases}1+\frac{1}{\mu}, & \gamma>0,\\ 3-\frac{1}{\alpha}, & \gamma=0,\ \alpha\in(0,1/3]\cup[1,\infty).\end{cases}\tag{5.3}$$

Since $t\in(0,1)$, the condition $b>a$ implies $(b-a)t^{b-a}/(1-t^{b-a})>0$, and so inequality (5.3) holds true whenever a satisfies (5.2). When $b<a$, then $(a-b)/(t^{a-b}-1)<b-a$, and hence $a-(b-a)t^{b-a}/(1-t^{b-a})<b<a$, and thus inequality (5.3) holds true whenever a satisfies (5.2).

For the case $b=a$, it is seen that

$$\frac{t\lambda'(t)}{\lambda(t)}=a-\frac{1}{\log(1/t)}.$$

Since $t<1$ implies $1/\log(1/t)\geqslant 0$, condition (4.4) is satisfied whenever a satisfies (5.2). This completes the proof. □

Remark 5.3. The conditions $b>-1$ and $a>-1$ in Theorem 5.3 yield several improvements of known results.

(1) Taking $\gamma=0$ and $\alpha>0$ in Theorem 5.3 leads to a result similar to Theorem 2.4(i) and (ii) obtained in [3] for the case $\alpha\in[1/2,1]$. The condition $b>a$ there resulted in $a\in(-1,1/\alpha-1]$. When $\alpha=1$, the range of a obtained in [3] lies in the interval $(-1,0]$, whereas the range of a obtained in Theorem 5.3 for this particular case lies in $(-1,2]$, and with the condition $b>a$ removed.
(2) Choosing $\alpha=1$ in the case above leads to improvements of Corollary 3.13(i) obtained in [2] and Corollary 3.1 in [11]. Indeed, there the conditions on a and b were $b>a>-1$, whereas in the present situation, it is only required that $b>-1$, $a>-1$.
(3) Applying Theorem 5.3 to the particular case $\alpha=1+2\gamma$, $\gamma>0$, and $\mu=1$ improves Theorem 4.1 in [4] in the sense that the condition $b>a>-1$ is now replaced by $b>-1$, $a>-1$.

For another choice of λ, let it now be given by

$$\lambda(t)=\frac{(1+a)^p}{\Gamma(p)}t^a\big(\log(1/t)\big)^{p-1},\quad a>-1,\ p\geqslant 0.$$

The integral transform V_λ in this case takes the form

$$V_\lambda(f)(z)=\frac{(1+a)^p}{\Gamma(p)}\int_0^1\left(\log\left(\frac{1}{t}\right)\right)^{p-1}t^{a-1}f(tz)\,dt,\quad a>-1,\ p\geqslant 0.$$

This is the Komatu operator, which reduces to the Bernardi integral operator if $p=1$. For this λ, the following result holds.

Theorem 5.4. *Let $-1 < a$, $\alpha > 0$, $p \geqslant 1$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \big(\log(1/t)\big)^{p-1} g(t)\, dt,$$

where g is given by (2.6)*. If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \big(\log(1/t)\big)^{p-1} t^{a-1} f(tz)\, dt$$

belongs to $\mathcal{S}^*$ if

$$a \leqslant \begin{cases} 1 + \frac{1}{\mu}, & \gamma > 0\ (\mu \geqslant 1), \\ 3 - \frac{1}{\alpha}, & \gamma = 0,\ \alpha \in (0, 1/3] \cup [1, \infty). \end{cases} \tag{5.4}$$

The value of β is sharp.

Proof. It is evident that

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(p-1)}{\log(1/t)}.$$

Since $\log(1/t) > 0$ for $t \in (0, 1)$, and $p \geqslant 1$, condition (4.4) is satisfied whenever a satisfies (5.4). □

Remark 5.4.

(1) Taking $\gamma = 0$ and $\alpha > 0$ in Theorem 5.4 gives a result similar to Theorem 2.1 in [3] and Theorem 2.3 in [8].
(2) When $\alpha = 1 + 2\gamma$, $\gamma > 0$, and $\mu = 1$, Theorem 5.4 yields Theorem 4.2 obtained by Balasubramanian et al. [4], while when $\alpha = 1$ and $\gamma = 0$, Theorem 5.4 yields Corollary 3.12(i) obtained by Balasubramanian et al. [2].

Let Φ be defined by $\Phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n (1-t)^n$, $b_n \geqslant 0$ for $n \geqslant 1$, and

$$\lambda(t) = K t^{b-1} (1-t)^{c-a-b} \Phi(1-t), \tag{5.5}$$

where K is a constant chosen such that $\int_0^1 \lambda(t)\, dt = 1$. The following result holds in this instance.

Theorem 5.5. *Let $a, b, c, \alpha > 0$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) g(t)\, dt,$$

where g is given by (2.6) *and K is a constant such that $K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1} (1-t)^{c-a-b} \Phi(1-t) \frac{f(tz)}{t}\, dt$$

belongs to $\mathcal{S}^*$ provided one of the following conditions holds:

$$\begin{aligned} &\text{(i)} \quad c < a + b \quad \textit{and} \quad 0 < b \leqslant 1, \\ &\text{(ii)} \quad c \geqslant a + b \quad \textit{and} \quad b \leqslant \begin{cases} 2 + \frac{1}{\mu}, & \gamma > 0\ (\mu \geqslant 1), \\ 4 - \frac{1}{\alpha}, & \gamma = 0,\ \alpha \in (1/4, 1/3] \cup [1, \infty). \end{cases} \end{aligned} \tag{5.6}$$

The value of β is sharp.

Proof. For λ given by (5.5),

$$\frac{t\lambda'(t)}{\lambda(t)} = (b-1) - \frac{(c-a-b)t}{1-t} - \frac{t\Phi'(1-t)}{\Phi(1-t)}.$$

For the case $c < a+b$, computing $(b-1) - ((c-a-b)t)/(1-t)$ and using the fact that $t\Phi'(1-t)/\Phi(1-t) > 0$ implies condition (4.4) is satisfied whenever $0 < b \leqslant 1$. For $c \geqslant a+b$, a similar computation shows that the condition (4.4) is satisfied whenever b satisfies (5.6). Now the result follows by applying Theorem 4.2 for this special λ. □

Taking $\gamma = 0$, $\alpha > 0$ in Theorem 5.5 leads to the following corollary:

Corollary 5.2. *Let $a, b, c, \alpha > 0$, and $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\alpha(t)\,dt,$$

where g_α is given by (2.7), and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\alpha(\beta)$, then the function

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}\,dt$$

belongs to $\mathcal{S}^*$ whenever a, b, c are related by either (i) *$c \leqslant a+b$ and $0 < b \leqslant 1$, or* (ii) *$c \geqslant a+b$ and $b \leqslant 4 - 1/\alpha$, $\alpha \in (1/4, 1/3] \cup [1, \infty)$, for all $t \in (0, 1)$. The value of β is sharp.*

Remark 5.5. For $\alpha = 1$, Corollary 5.2 improves Theorem 3.8(i) in [2] in the sense that the result now holds not only for $c \geqslant a+b$ and $0 < b \leqslant 3$, but also to the range $c \leqslant a+b$, $0 < b \leqslant 1$.

Taking $\alpha = 1+2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 5.5 reduces to the following corollary:

Corollary 5.3. *Let $a, b, c > 0$, and let $\beta < 1$ satisfy*

$$\frac{\beta}{1-\beta} = -K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)g_\gamma(t)\,dt,$$

where g_γ is given by (2.7), and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(1+2\gamma, \gamma)$, then the function

$$V_\lambda(f)(z) = K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)\frac{f(tz)}{t}\,dt$$

belongs to $\mathcal{S}^*$ whenever a, b, c are related by either (i) *$c \leqslant a+b$ and $0 < b \leqslant 1$, or* (ii) *$c \geqslant a+b$ and $0 < b \leqslant 3$, for all $t \in (0, 1)$ and $\gamma > 0$. The value of β is sharp.*

Remark 5.6. Choosing $\Phi(1-t) = F(c-a, 1-a, c-a-b+1; 1-t)$ in Theorem 5.5(ii) gives

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$$

whenever $c-a-b+1 > 0$. In this case, the function $V_\lambda(f)(z)$ reduces to the Hohlov operator given by

$$\begin{aligned} V_\lambda(f)(z) &= H_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) \\ &= K \int_0^1 t^{b-1}(1-t)^{c-a-b}F(c-a, 1-a, c-a-b+1; 1-t)\frac{f(tz)}{t}\,dt, \end{aligned}$$

where $a > 0$, $b > 0$, $c-a-b+1 > 0$. This case of Corollary 5.2 was treated in [3, Theorem 2.2(i), ($\mu = 0$)] and [8, Theorem 2.4], but the range of b provided by Corollary 5.2(ii) yields $0 < b \leqslant 3$, which is larger than the range given in [3] and [8] of $0 < b \leqslant 1$.

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