Algorithm 804: Subroutines for the Computation of Mathieu Functions of Integer Orders

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Computer subroutines in C++ for computing Mathieu functions of integer orders are described. The routines can handle a large range of the order n and the parameter h. Sample test results and graphs are given.

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General Terms: Algorithms

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1. INTRODUCTION

This article presents the outline of a set of routines that enable the computation of all Mathieu functions of integer orders for large range of the order n and the parameter h. There are many algorithms and routines available in the literature [Alhargan 1996; Arscott and Shymansky 1978; Baker 1992; Blanch 1966; Leeb 1979; McLachlan 1947; Morse and Feshbach 1953; NBS 1967; Rengarajan and Lewis 1980; Shirts 1993a; 1993b; Toyama and Shogan 1984; Wimp 1984]. However, many of these algorithms and routines are limited in the range of values for the order n and the parameter h; in many cases the limit for n is 20 or less, and in a few cases the limit for n barely reaches 30. Such limitations are a real obstacle in many research areas where larger values of the order n are required. This

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article implements the algorithms that were outlined in the companion article.

The full computation of a Mathieu function is obtained in three steps:

(1) estimate and Compute Mathieu characteristic number

(2) compute Mathieu coefficients

(3) compute the function using the infinite series

The accuracy of the results can be checked using the Wronskians for Mathieu functions, as the algorithms for computing Mathieu functions are not based on the Wronskian of the functions. Tables of values that demonstrate the accuracy of the computations are given based on this method. The routines provided can handle orders up to $n = 200$ and beyond.

2. COMPUTING THE MCNS

Mathieu characteristic numbers (MCNs) a_n and b_n are computed using an iterative procedure that is outlined in the companion article (Section 3). First, an initial value is estimated for the MCN using the function `Estimatmcn`. The steps for estimating initial values of MCNs for a given n and h involve evaluations of the approximate (Eqs. (30)–(37) in the companion article) and asymptotic values (Eq. (38)), then choosing between the values as appropriate for the case of $n < 70$. For larger orders chaining is required (see companion article Eqs. (39)–(40)), which can give fairly good estimates for orders as large as $n = 1000$. The routines are expect to work even beyond this value, but for many practical problems $n = 1000$ is sufficient.

Once the estimate for MCN is obtained the function `MCNRoot` takes the estimate as input and then improves it by calling the Newton Raphson function `NewtonRaphsons` repeatedly until it reaches the required accuracy or the upper limit of the number of iterations allowed.

3. COMPUTATION OF MATHIEU COEFFICIENTS

3.1 First-Kind Mathieu Coefficients, Be_m and Bo_m

The equations for the computation of the coefficients were described in Section 4.1 of the companion article and these are implemented by the function `Coefficients`. For MCNs very close to the singularity r^2 (where r is the order of the MCN), the computation becomes unstable, and in fact for this case all coefficients are very close to zero accept $B_r \simeq 1$. This condition is taken care of at the beginning of the procedure. Also it is important to note that the number M of required coefficients should exceed the order by at least 15 (i.e., as the orders increase the number of coefficients for accurate computation also increases).

3.2 Modified First-Kind Mathieu Coefficients, Ae_m and Ao_m

The computation of the modified first-kind Mathieu Coefficients, using `MCoefficients`, requires the computation of the Mathieu coefficients Be, Bo. Then the rest is a matter of scaling according to the equations given in Section 4.2 of the companion article.

3.3 Second-Kind Coefficients, De_m and Do_m

The steps for computing the second-kind coefficients involves first the computation of the first-kind coefficients B_m. Then the coefficients are computed using the the set of equations given in Section 4.3 in the companion article. This is implemented in the function `CoefficientSec`.

3.4 Modified Second-Kind Coefficients, Ce_m and Co_m

In the similar manner the function `MCoefficientSec` implements the computation of the modified second-kind coefficients by first computing the modified first-kind coefficients A_m. Then the coefficients are computed using the the set of equations given in Section 4.4 in the companion article.

4. COMPUTING MATHIEU FUNCTIONS

Once Mathieu coefficients of the appropriate kind, type, and order have been computed, the computations of the Mathieu functions are straightforward. In the case of the circumferential functions the only requirement is the computation of the trigonometric functions cos and sin of the appropriate argument, whereas for the radial function the appropriate Bessel functions of various orders are required.

4.1 The Circumferential Functions

The function `MathuSn` is used to compute Mathieu circumferential functions. Figure 1 shows the second-kind circumferential functions for $n = 4$.

The function `MathuQn` is used to compute modified Mathieu circumferential functions. Figures 2–3 show the modified first- and second-kind circumferential functions for $n = 4$.

4.2 The Radial Functions

In computing the radial functions, arrays of Bessel functions of various kinds and orders are required; some recent work suggests the use of power series [Schneider and Marquardt 1999], though, such expressions may be sufficiently accurate for some ranges of the parameters; the ranges covered are limited to relatively small values, and therefore to be able to cover a very large range, the Bessel product series are required. To simplify the process, routines for computing arrays of Bessel functions of any required size have been developed and are included in the library.

The functions `MathuZn` and `MathuMZn` are used to compute the radial and Modified radial Mathieu functions respectively.

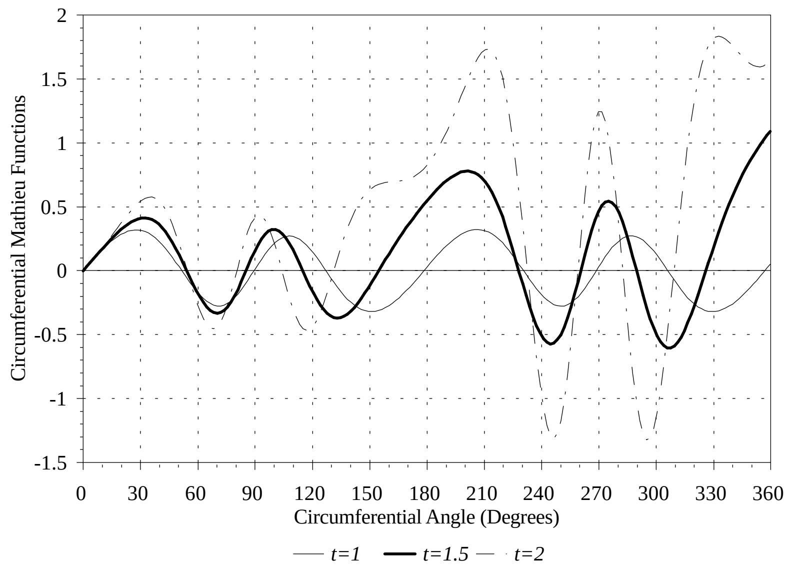

Fig. 1. Second-kind even circumferential Mathieu function Fe_4, $t = 1, 1.5, 2$ and $v = 0 - 360$.

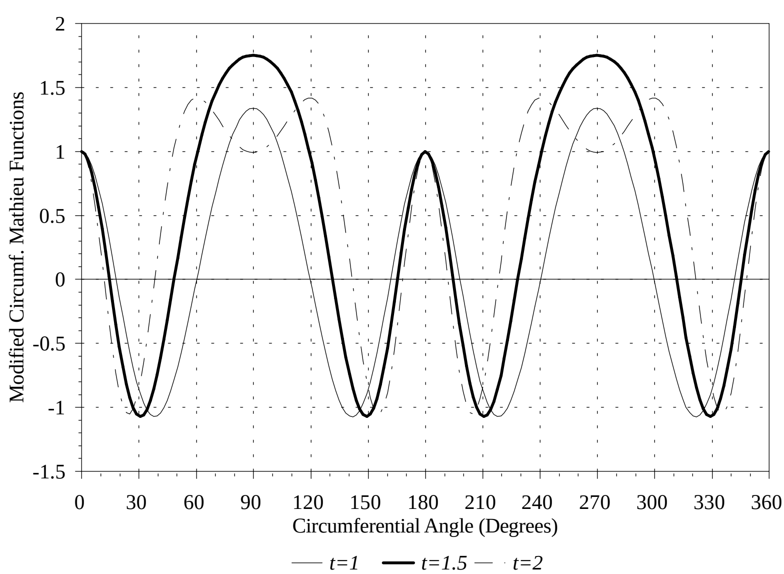

Fig. 2. Modified 1st-kind even circumferential Mathieu function Qe_4, $t = 1, 1.5, 2$ and $v = 0 - 360$.

Figures 4–5 show the modified first- and second-kind radial Mathieu functions for $n = 4$, and Tables I and II show the accuracy for the modified even and odd radial Mathieu functions using the Wranskian. The tables show the results for $n = 5, 10, 15, 20$ and $t = 0.4, 0.8, 1.2, 1.6, 2.0, 2.4$, where $h = nt$.

Fig. 3. Modified 2nd-kind even circumferential Mathieu function Ee_4, $t = 1, 1.5, 2$ and $v = 0 - 360$.

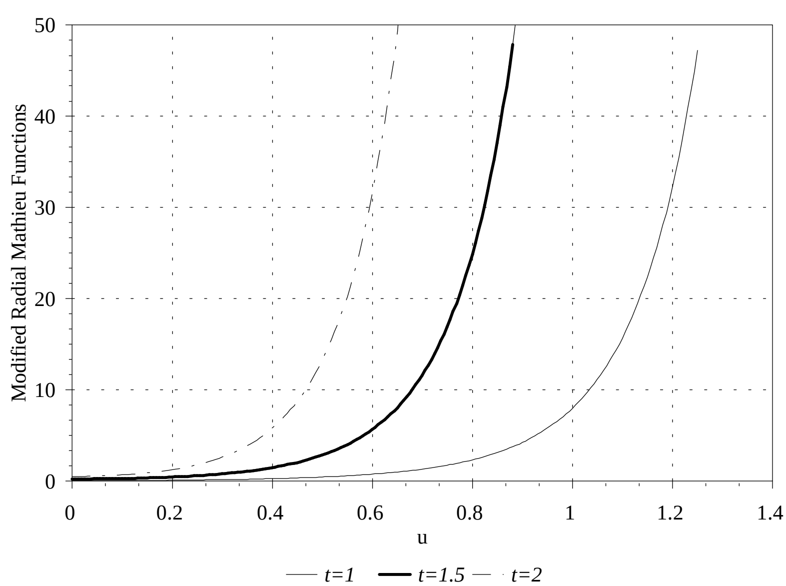

Fig. 4. Modified 1st-kind even radial Mathieu function Ie_4 with $t = 1, 1.5, 2$ and $u = 0 - 1.4$.

5. CONCLUSION

There are four main Mathieu functions: two circumferential and two radial. Each of these functions subdivides into two functions; one is even, and the other is odd. This results in eight standard functions each having three

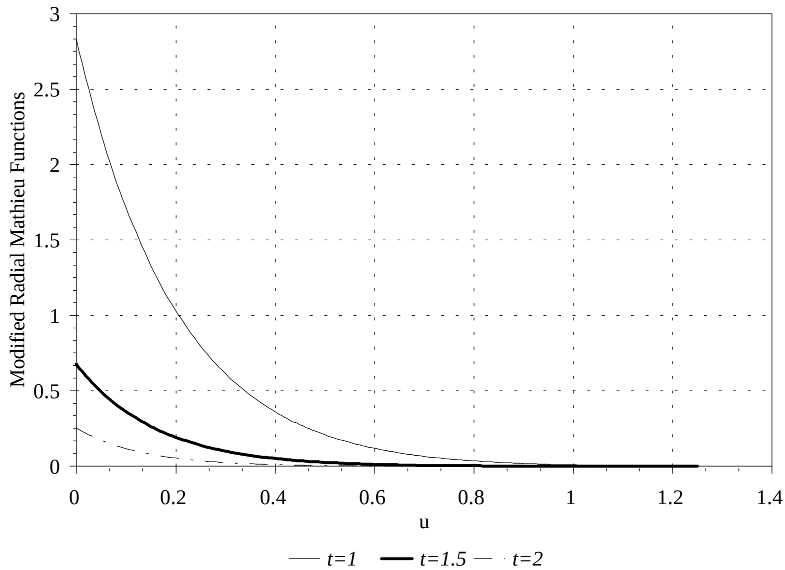

Fig. 5. Modified 2nd-kind even radial Mathieu function Ke_4 with $t = 1, 1.5, 2$ and $u = 0 - 1.4$.

Table I. Accuracy Test Using $1 - |W(\mathrm{Ie}_n\mathrm{Ke}_n)|$

$t \backslash n$	5	10	15	20
0.40	+1.110223e-16	+1.443290e-15	+3.108624e-15	+1.554312e-15
0.80	+1.554312e-15	-1.110223e-15	-1.332268e-15	-1.012523e-13
1.20	+1.443290e-15	-2.442491e-15	+3.851142e-12	-2.069922e-11
1.60	+9.992007e-16	+4.891643e-13	+3.468159e-11	+5.302665e-09
2.00	+3.552714e-15	+3.096556e-11	+1.602296e-10	+3.720313e-08
2.40	+1.942890e-14	-5.622169e-13	+1.430650e-07	+3.328065e-06

Table II. Accuracy Test Using $1 - |W(\mathrm{Io}_n\mathrm{Ko}_n)|$

$t \backslash n$	5	10	15	20
0.40	+0.000000e+00	+1.998401e-15	+1.998401e-15	-1.998401e-15
0.80	+1.665335e-15	-6.661338e-16	+1.343370e-14	-1.358913e-13
1.20	+1.998401e-15	-1.620926e-14	+8.925083e-13	-2.604050e-11
1.60	-1.398881e-14	+2.128298e-13	+2.769721e-10	+3.067323e-09
2.00	+8.326673e-15	+1.076472e-12	+9.768703e-10	+1.978108e-08
2.40	-2.220446e-16	-3.504130e-11	+5.114261e-09	+8.449368e-07

variables n, h, and v or u. Taking the modified functions into account results in a total of 16 functions. This library contains the routines for computing all mentioned functions plus routines that compute their derivative with respect to u or v. Also the library contains routines for the computation of Bessel and modified Bessel functions and their derivatives which are required for the computation of the radial Mathieu functions.

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