Appendix A

Software Supplement: Computations in Finance and Insurance

The software is offered in the form of an open source code. It can be downloaded from

www.crcpress.com/e_products/downloads/download.asp?cat_no = C429

Any C++ compiler can be used under a Linux or Windows operating system. This code can be easily read and modified.

Software berprog.cpp: Forecast of stock prices in a Bernoulli market

Suppose that price dynamics of stock S in a Bernoulli market are given by the following recurrence formula

$$S_{i+1} = S_i\,(1+\rho)\,, \qquad 0 \le i \le n\,,$$

where ρ is the profitability of S and it takes values b or a with probabilities p and $(1-p)$, respectively. Assuming that the initial price is S_0, we forecast the average price of S at time n.

The inputs are:

1. S_0, the initial price of S;
2. a and b, values of possible change (as a percentage) in price of S;
3. p, the probability for ρ to take value b;
4. n, time horizon.

The forecast is based on the properties of conditional expectations $E(Y|X)$ of independent random variables. Namely, we compute

$$S_{\text{av}} = E\left(\frac{S_1 + S_2 + \cdots + S_n}{n}\,\middle|\, S_0\right).$$

In a two-step case we have

$$S_{\rm av} = E\left(\frac{S_1 + S_2}{2}\,\middle|\, S_0\right) = E\left(\frac{S_0(1+\rho_1) + S_0(1+\rho_1)(1+\rho_2)}{2}\,\middle|\, S_0\right)$$

$$= \frac{S_0}{2}\Big[E(1+\rho_1) + E(1+\rho_1)\,E(1+\rho_2)\Big]$$

$$= \frac{S_0}{2}\Big[(1+a)p + (1+b)(1-p)$$

$$+\big((1+a)p + (1+b)(1-p)\big)\,\big((1+a)p + (1+b)(1-p)\big)\Big]$$

$$= \frac{S_0}{2}\Big[(1+a)p + (1+b)(1-p) + \big((1+a)p + (1+b)(1-p)\big)^2\Big]$$

Software binoptprice.cpp:Pricing options in a binomial market

Consider a one-step binomial (B,S)-market. Dynamics of a bank account and stock price are given by

$$B_1 = B_0\,(1+r)\,, \qquad \text{and} \qquad S_1 = S_0\,(1+\rho)\,,$$

where r is a fixed rate of interest and ρ is the profitability of S that takes values b or a with probabilities p and $(1-p)$, respectively. Quantities a, b r must satisfy the inequality $-1 < a < r < b$. Suppose that $B_0 = 1$. Consider a European call option with the contingent claim

$$f_1 = (S_1 - K)^+ = \max(0, S_1 - K)\,,$$

where K is a strike price. Let $K = S_0$. The intuitive price for this option is

$$E\left(\frac{f_1}{1+r}\right) = \frac{p\,(S_0\,b)^+ + (1-p)\,(S_0\,a)^+}{1+r}\,.$$

Alternatively, using the minimal hedging approach (see Section 1.4), one can construct a strategy $\pi_0(b_0, g_0)$ such that

$$X_1^\pi\left(X_0^\pi\right) = f_1\,.$$

In this case

$$X_0^\pi = \beta_0^*\,B_0 + \gamma_0^*\,S_0\,,$$

where

$$\gamma_0^* = \frac{f_1^{(1)} - f_1^{(2)}}{S_0(\rho_1 - \rho_2)}, \quad \text{and} \quad \beta_0^* = \frac{1}{(1+r)\,B_0}\Big(f_1^{(1)} - \gamma_0^* S_0(1+\rho_1)\Big).$$

Finally, consider a risk-neutral probability E^* such that

$$E^*\left(\frac{S_1}{B_1}\right) = S_0.$$

This implies

$$p^* = \frac{(1+r)\,B_0 - 1 - a}{b-a}.$$

If $B_0 = 1$ then

$$p^* = \frac{r-a}{b-a}.$$

Thus, risk-neutral price is given by

$$C = E^*\left(\frac{f_1}{1+r}\right) = \frac{f_1^{(1)}p^* + f_1^{(2)}(1-p^*)}{1+r}.$$

The inputs are

1. S_0, the initial price of S;
2. a and b, values of possible change (as a percentage) in price of S;
3. p, the probability for ρ to take value b;
4. r, the rate of interest.

The outputs are

1. values of the contingent claim;
2. intuitive price;
3. initial capital of the minimal hedge;
4. risk-neutral price;
5. risk-neutral probability.

Software CRRoptprice.cpp: Call-put parity in a Cox-Ross-Rubinstein market

Now we consider a N-step binomial (B, S)-market. The Cox-Ross-Rubinstein formula

$$C_N = S_0\, B(k_0, N, \tilde{p}) - K\,(1+r)^{-N} B(k_0, N, p^*)$$

gives the risk-neutral price of a European call option. Here p^* is a risk-neutral probability:

$$p^* = \frac{r-a}{b-a} \qquad \text{and} \qquad \tilde{p} = \frac{1+b}{1+a} p^* .$$

Recall (see Section 1.4) that

$$B(j, N, p) := \sum_{k=j}^{N} \binom{N}{k} p^k \,(1-p)^{N-k} ,$$

constant k_0 is defined by

$$k_0 = \min \left\{ k \leqslant N : S_0 (1+b)^k (1+a)^{N-k} \geqslant K \right\}$$

so that

$$k_0 = \left[\ln \frac{K}{S_0 (1+a)^N} \Big/ \ln \frac{1+b}{1+a} \right] + 1 .$$

Using the call-put parity, we can price a European put option with the claim

$$f_N = (K - S_N)^+ .$$

Namely, price of this option is given by

$$P_N = C_N - S_0 + K(1+r)^{-N} .$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time.

The output contains prices of the European call and put options.

Software amoptprice.cpp: Pricing an American call option

Consider an N-step binomial (B, S)-market. Recall (see Section 1.5) that price of an American call option with the sequence of claims

$$f_n = (S_n - K)^+ , \quad n \leq N ,$$

is defined by

$$C_N^{\mathrm{am}} = \sup_{\tau \in \mathcal{M}_0^N} C(f_\tau) = \sup_{\tau \in \mathcal{M}_0^N} E^* \left(\frac{f_\tau}{(1+r)^\tau} \right) .$$

Computing

$$Y_n = \max \left\{ \frac{f_n}{(1+r)^n} , E^* \left(Y_{n+1} \middle| \mathcal{F}_n \right) \right\}$$

we obtain

$$C_N^{\mathrm{am}} = Y_0 .$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time.

The output consists of a price of the American call option.

Software spread.cpp: Computing spreads in a market with constraints

Let (B_1, B_2, S) be a Cox-Ross-Rubinstein market with constraints (see Section 2.2). We find the interval $[C_*, C^*]$ of all arbitrage-free prices, and therefore compute spread $C^* - C_*$ as a measure of incompleteness of the market.

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;

3. a and b, values of possible change (as a percentage) in price of S;
4. r_1 and r_2, the rates of interest on saving and credit accounts, respectively;
5. N, the terminal time.

The output contains prices of the European call option corresponding to given rates r_1 and r_2, and the spread of the market.

Software BandSgreek.cpp: Pricing contingent claims using the Black-Scholes formula and computing Greek parameters in the continuous case

In Section 2.6 we studied pricing of contingent claims in a Black-Scholes model of a (B, S)-market. The 'fair' arbitrage-free price of a European call option is given by the Black-Scholes formula:

$$C_T = S_0\,\Phi\big(y_+\big) - K\,e^{-r\,T}\,\Phi\big(y_-\big)\,,$$

where

$$y_\pm = \frac{\ln(S_0/K) + T(r \pm \sigma^2/2)}{\sigma\sqrt{T}}.$$

The price of a European put option:

$$P_T(K, \sigma, S_0) = C_T(-K, -\sigma, -S_0)$$

is derived from the call-put parity relation.

The following 'Greeks' are used by the risk management practitioners:

Theta:

$$\theta = \frac{\partial C_T}{\partial t} = \frac{S_t\,\sigma\,\varphi\big(y_+(t)\big)}{2\,\sqrt{T-t}} - K\,r\,e^{-r\,(T-t)}\,\Phi\big(y_-(t)\big)\,,$$

Delta:

$$\Delta = \frac{\partial C_T}{\partial S} = \Phi\big(y_+(t)\big)\,,$$

Rho:

$$\rho = \frac{\partial C_T}{\partial r} = K\,(T-t)\,e^{-r\,(T-t)}\,\Phi\big(y_-(t)\big)\,,$$

Vega:

$$\Upsilon = \frac{\partial C_T}{\partial \sigma} = S_t\,\varphi\big(y_+(t)\big)\,\sqrt{T-t}\,,$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. σ, the volatility of the market;
4. r, the rate of interest;
5. T, the terminal time;
6. t, intermediate time.

The output contains prices of the European call and put options and values of 'Greek' parameters.

Software BandSdiv.cpp: Pricing contingent claims using the Black-Scholes formula in a model with dividends

In the case when a holder of asset S receives dividends, the Black-Scholes formula gives the price of a European call option in the following form

$$C_T(\delta) = S_0\, e^{-\delta T}\, \Phi\left(\frac{\ln(S_0/K) + T\,(r - \delta + \sigma^2/2)}{\sigma\sqrt{T}}\right)$$
$$-K\, e^{-rT}\, \Phi\left(\frac{\ln(S_0/K) + T\,(r - \delta - \sigma^2/2)}{\sigma\sqrt{T}}\right).$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. σ, the volatility of the market;
4. r, the rate of interest;
5. T, the terminal time;
6. δ, the dividends rate.

The output contains price of the European call option in the market with dividends.

Software Indicators.cpp: Some indicators used in technical analysis

This software supplements Section 2.9. We compute the following indicators:

1. Simple Moving Average (SMA);
2. Weighted Moving Average (WMA);
3. Exponential Moving Average (EMA);
4. Bollinger Bands;
5. Moving Average Convergence/Divergence (MACD);
6. Relative Strength Index (RSI);
7. Parabolic Time Price system (PTP);
8. Volume Price Trend (VPT).

The user has to specify which indicator must be calculated. The software contains two data files: 'data.txt' and 'datav.txt'. File 'data.txt' contains close prices from daily bar charts for shares of Microsoft for the period from April, 2001 till April, 2002. File 'datav.txt' contains the complete information of the same bar charts: daily open, close, low, high prices and trading volumes. First six indicators use file 'data.txt', and the last two use file 'datav.txt'. The resulting text file can be processed by various graphical tools, e.g. Gnuplot, MetaStock etc.

Simple Moving Average represents the average value of a quantity during a specified period of time. At time j it is computed as

$$SMA_j = \frac{1}{N} \sum_{i=j-N}^{j} v_i \,,$$

where v_i, $j - N \le i \le j$, are values of this quantity. Input N defines the time horizon. The output file 'series' contains the initial time series, and file 'ma' contains the time series for moving average.

Weighted Moving Average is a modification of the SMA:

$$WMA_j = \sum_{i=1}^{N} \varpi_i \, v_i \,,$$

where ϖ_i are weights with $\sum_{i=1}^{N} \varpi_i = 1$. As above, time horizon N corresponds to some analysis time j. Usually, time points that are closer to the analysis time j, have heavier weights. We use the following formula

$$\varpi_i = \frac{i}{\sum_{k=1}^{N} k}, \qquad i = 1, \ldots, N.$$

The output file 'series' contains the initial time series, and file 'wma' contains the time series for weighted moving average.

Exponential Moving Average is the most widely used modification of the WMA. It uses all preceding values, but times that are distant from the analysis time j, correspond to very small weights. The Exponential Moving Average is defined by the formula

$$EMA_j = (1 - \alpha)\, EMA_{j-1} + \alpha\, v_j\,,$$

where

$$\alpha = \frac{2}{N + 1}\,.$$

Clearly, this is the simplest Moving Average indicator. File 'ema' contains the time series for exponential moving average.

Bollinger Bands. First, one can use any of the Moving Average indicators to construct a Middle Band with values m_j. Then Upper and Lower Bands are defined by

$$u_j = m_j + k\,\sigma_j \qquad \text{and} \qquad l_j = m_j - k\,\sigma_j\,,$$

where

$$\sigma_j = \sqrt{\frac{\sum_{i=j-N}^{j} (v_i - m_j)^2}{N}}\,.$$

The inputs include the order of averaging N and coefficient k which reflects the sensitivity of the indicator. The output file 'bbands' contains Middle, Upper and Lower Bands.

Moving Average Convergence/Divergence is constructed in the following way:

1. compute short Moving Average;
2. compute long Moving Average;
3. compute quick line by subtracting long Moving Average from the short Moving Average;
4. compute signal line by smoothing quick line with the help of Moving Average;
5. compute MACD as the difference between signal and quick lines – this is contained in the output file 'macd'.

Relative Strength Index is computed in terms of average increase and decrease of price over some period of time.

Average increase is given by

$$U_j = \begin{cases} \frac{U_{j-1}(N-1)+(v_j-v_{j-1})}{N} & \text{if } v_j > v_{j-1} \\ \frac{U_{j-1}(N-1)}{N} & \text{if } v_j < v_{j-1}\,. \end{cases}$$

Similarly, for average decrease we have

$$D_j = \begin{cases} \frac{D_{j-1}(N-1)+(v_{j-1}-v_j)}{N} & \text{if } v_j < v_{j-1} \\ \frac{D_{j-1}(N-1)}{N} & \text{if } v_j > v_{j-1}\,. \end{cases}$$

Then

$$RSI_j = 100 - \frac{100}{1 + U_j/D_j}\,.$$

The output file 'rsi' contains the values of RSI.

Parabolic Time Price system is represented by a line that is positioned either above or below the price graph, which identifies decreasing or increasing trends, respectively. The close price C_j is determined daily by the recurrence relation

$$C_j = C_{j-1} + \mathcal{A}\left(\mathcal{E}_{j-1} - C_{j-1}\right),$$

where $\mathcal{E}$ is the critical level of daily trading: in a long position it is equal to the highest price since buying, in a short position it is the lowest price since selling. Constant $\mathcal{A}$ is the averaging factor. It determines how fast the close price should be shifted toward open position, and it depends on the number of new peaks since buying and new lows since selling. The initial value of $\mathcal{A}$ is usually set to be 0.02. Increase or decrease of the initial value respectively increases or decreases the sensitivity of PTP line. Note that in this computer version of constructing PTP lines, by open position we understand the corresponding dynamics of a trend.

This software does not have any input parameters, but the input data in file 'datav.txt' must be in the form (open, low, high, close).

Volume Price Trend reflects overbuying or overselling in the market. It is computed as

$$VPT_j = VPT_{j-1} + V_j\,\frac{P_j - P_{j-1}}{P_{j-1}}\,,$$

where V_j and P_j are values of volume and price respectively.

Software elenshure.cpp: Pure endowment assurance

This software is the computer realization of Worked Example 1.5 from Section 1.4.

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time;
6. p_x, the probability of surviving.

The output is the price of the insurance policy.

Software var.cpp: Computing the Value at Risk

This software uses historical modelling and Monte-Carlo modelling for computing values of Value at Risk. Both approaches have similar structure, the only difference is that in historical modelling one uses the real data for determining the distribution of losses and profits, whereas, in Monte-Carlo modelling, it is assumed price movements are normally distributed.

The software computes the Value at Risk for one asset. File 'vardata.txt' contains the input data. After assessing the volume of the input data, the software asks to define the time horizon. Given this information it then determines the number of possible scenarios. Depending on the number of scenarios requested by the user, the software creates as many independent scenarios as possible and computes the distribution of profits and losses. Further, it requests the value of probability for which we compute the Value at Risk, i.e., losses that correspond to this probability will not exceed the corresponding value of the Value at Risk. Finally, since the input data is discrete in time, the software requests the size of the data confidence interval in terms of probability of being in this interval and then calculates the confidence interval for the Value at Risk.

Software RiskPremInd.cpp: Computing premiums in the model of individual risk

This software is the computer realization of Example from Section 3.1. It requests the level of bankruptcy's probability and n, the number of policies. The initial data is contained in file 'RPIdata.txt' and it consists of uniformly distributed independent random variables that are used for computing the distribution of X^{ind}, the total payoff to the policy holders. The output is the price of the insurance policy.

Appendix B

Problems and Solutions

B.1 Problems for Chapter 1

Problem B.1.1 *Suppose that the effective annual rate of interest is* 10%. *Find the present value of a 3-year bond with face value* \$500 *and with annual coupon payments of* \$100.

SOLUTION We have

$$B_0 = \frac{100}{1+0.1} + \frac{100}{(1+0.1)^2} + \frac{100}{(1+0.1)^3} + \frac{500}{(1+0.1)^3} \approx 624\,(\$)\,.$$

□

Problem B.1.2 *An investor buys two European put options with strike price* \$40 *and one European call option with strike price* \$50 *on the same stock* S *with the same expiry date* N. *The total price of these options is* \$10. *Write down the gain-loss function and discuss the possible outcomes.*

SOLUTION If S_N is the price of the asset at time N, then the gain-loss function has the form

$$\mathcal{V}(S_N) = 2\left(40 - S_N\right)^+ + \left(S_N - 50\right)^+ - 10\,.$$

We have the following cases.

(1) If $S_N < 40$, then the put options are exercised and the call option is not. In this case

$$\mathcal{V}(S_N) = 2\left(40 - S_N\right)^+ - 10 = 70 - 2\,S_N\,,$$

and a profit is earned if $S_N < 35$.

(2) If $40 \le S_N \le 50$, then none of the options are exercised, and the premium is lost:

$$\mathcal{V}(S_N) = -10\,.$$

(3) If $S_N > 50$, then only the call option is exercised:

$$\mathcal{V}(S_N) = \big(S_N - 50\big)^+ - 10 = S_N - 60\,,$$

and a profit is earned if $S_N > 60$.

Thus, this investment strategy reflects the investor's expectation that the price of S will be less than \$35. Note that the maximal possible loss cannot be higher than the amount of premium. ∎

Problem B.1.3 *The joint distribution of profitabilities α and β is given in the following table.*

$\alpha \diagdown \beta$	-0.1	0	0.1
-0.2	0.1	0	0.4
0.1	0.3	0.1	0.1

Find their individual distributions, average of β and the conditional expectation $E(\beta|\alpha)$.

SOLUTION From the given table we compute:

$$\begin{aligned}
&P\big(\{\omega:\ \alpha = -0.2\}\big) = 0.1 + 0.4 = 0.5\\
&P\big(\{\omega:\ \alpha = 0.1\}\big) = 0.3 + 0.1 + 0.1 = 0.5\\
&P\big(\{\omega:\ \beta = -0.1\}\big) = 0.1 + 0.3 = 0.4\\
&P\big(\{\omega:\ \beta = 0\}\big) = 0.1\\
&P\big(\{\omega:\ \beta = 0.1\}\big) = 0.4 + 0.1 = 0.5\,,
\end{aligned}$$

which implies

$$E(\beta) = -0.1 \times 0.4 + 0.1 \times 0.5 = 0.01\,.$$

The conditional expectation $E(\beta|\alpha)$ can be written in the form (see [41], for example):

$$E(\beta|\alpha) = E\big(\beta\big|\{\omega:\ \alpha = -0.2\}\big)\, I_{\{\omega:\ \alpha=-0.2\}} + E\big(\beta\big|\{\omega:\ \alpha = 0.1\}\big)\, I_{\{\omega:\ \alpha=0.1\}}\,.$$

Computing

$$\begin{aligned}
E\big(\beta|\{\omega:\ \alpha = -0.2\}\big) &= -0.1\, P\big(\{\omega:\ \beta = -0.1\}\big|\{\omega:\ \alpha = -0.2\}\big)\\
&\qquad +0.1\, P\big(\{\omega:\ \beta = 0.1\}\big|\{\omega:\ \alpha = -0.2\}\big)\\
&= \frac{-0.1 \times 0.1 + 0.1 \times 0.4}{0.5} = 0.06\,,
\end{aligned}$$

and

$$E\big(\beta|\{\omega:\ \alpha=0.1\}\big) = -0.1\, P\big(\{\omega:\ \beta=-0.1\}\big|\{\omega:\ \alpha=0.1\}\big)$$

$$+0.1\, P\big(\{\omega:\ \beta=0.1\}\big|\{\omega:\ \alpha=0.1\}\big)$$

$$= \frac{-0.1\times 0.3 + 0.1\times 0.1}{0.5} = -0.04\,,$$

we obtain

$$E(\beta|\alpha) = 0.06\, I_{\{\omega:\ \alpha=-0.2\}} - 0.04\, I_{\{\omega:\ \alpha=0.1\}}\,.$$

Note that

$$E\big(E(\beta|\alpha)\big) = 0.06\times 0.5 - 0.04\times 0.5 = 0.01 = E(\beta)\,.$$

∎

Problem B.1.4 *Suppose that analysis of the market data suggests that the price of a certain asset S will increase by 2% in one month's time with probability p, or will decrease by 1% with probability $1-p$. Find all values of p such that an investment in this asset will be, on average, more profitable than an investment in a bank account with effective monthly interest rate of 1%.*

SOLUTION The average monthly profitability of an investment in asset S is equal to

$$0.02\,p - 0.01\,(1-p) = 0.03\,p - 0.01\,.$$

Hence p must satisfy

$$0.03\,p - 0.01 \geq 0.01\,,$$

or $p \geq 2/3$. ∎

Problem B.1.5 *As in Section 1.3 we consider a binomial (B,S)-market. Suppose we are given the following values of its parameters:*

$$a=-0.4\,,\quad b=0.6\,,\quad r=0.2\,,\quad B_0=1\,,\quad S_0=200\,.$$

Find the price and the minimal hedge of a 'look back' European call option with the contingent claim

$$f_2 = \big(S_2 - K_2\big)^+\,,\quad \textit{where}\quad K_2=\min\{S_0,\,S_1,\,S_2\}\,.$$

SOLUTION First we compute the risk-neutral probability

$$p^* = \frac{r-a}{b-r} = \frac{0.2+0.4}{0.6+0.4} = 0.6\,.$$

Now we write all possible prices of S and values of claim f_2 in the following table.

event	**probability**	ρ_1	ρ_2	S_0	S_1	S_2	K_2	f_2
ω_1	0.16	−0.4	−0.4	200	120	72	72	0
ω_2	0.24	−0.4	0.6	200	120	192	120	72
ω_3	0.24	0.6	−0.4	200	320	192	192	0
ω_4	0.36	0.6	0.6	200	320	512	200	312

We compute price of this claim:

$$C_2 = \frac{E^*(f_2)}{(1+r)^2} = \frac{0.16 \times 0 + 0.24 \times 72 + 0.24 \times 0 + 0.36 \times 312}{1.44} = 90\,.$$

Next we find the minimal hedge $\pi^* = \left(\beta_n, \gamma_n\right)_{n=1}^{2}$ that replicates claim f_2. Consider time $n = 1$. Since the value of profitability ρ_1 is known at this time, we can construct the pair $\left(\beta_2, \gamma_2\right)$. Indeed, we have that hedge π^* replicates f_2:

$$X_2^{\pi^*} = f_2\,,$$

which can be written in the form of the following system

$$\begin{cases} X_2^{\pi^*}(\omega_1) = \beta_2(\omega_1)\,(1+r)^2 + \gamma_2(\omega_1)\,S_2(\omega_1) \\ X_2^{\pi^*}(\omega_2) = \beta_2(\omega_2)\,(1+r)^2 + \gamma_2(\omega_2)\,S_2(\omega_2) \\ X_2^{\pi^*}(\omega_3) = \beta_2(\omega_3)\,(1+r)^2 + \gamma_2(\omega_3)\,S_2(\omega_3) \\ X_2^{\pi^*}(\omega_4) = \beta_2(\omega_4)\,(1+r)^2 + \gamma_2(\omega_4)\,S_2(\omega_4)\,. \end{cases}$$

Substituting all known values we obtain

$$\begin{cases} 0 = \beta_2(\omega_1)\,(1+0.2)^2 + \gamma_2(\omega_1)\,72 \\ 72 = \beta_2(\omega_2)\,(1+0.2)^2 + \gamma_2(\omega_2)\,192 \\ 0 = \beta_2(\omega_3)\,(1+0.2)^2 + \gamma_2(\omega_3)\,192 \\ 312 = \beta_2(\omega_4)\,(1+0.2)^2 + \gamma_2(\omega_4)\,512\,. \end{cases}$$

Since random variables β_2 and γ_2 do not depend on ρ_2, we also have

$$\beta_2(\omega_1) = \beta_2(\omega_2)\,, \qquad \beta_2(\omega_3) = \beta_2(\omega_4)\,,$$

and

$$\gamma_2(\omega_1) = \gamma_2(\omega_2)\,, \qquad \gamma_2(\omega_3) = \gamma_2(\omega_4)\,.$$

Hence

$$\begin{cases} \beta_2(\omega_1) = \beta_2(\omega_2) = -30 \\ \gamma_2(\omega_1) = \gamma_2(\omega_2) = 0.6 \\ \beta_2(\omega_3) = \beta_2(\omega_4) = -130 \\ \gamma_2(\omega_3) = \gamma_2(\omega_4) = 39/40\,. \end{cases}$$

The pair (β_1, γ_1) is chosen at time $n = 0$ and does not depend on prices of S. Since π^* is self-financing, we have

$$\begin{cases} \beta_1\,(1+r) + \gamma_1\, S_1(\omega_1) = \beta_2(\omega_1)\,(1+r) + \gamma_2(\omega_1)\, S_1(\omega_1) \\ \beta_1\,(1+r) + \gamma_1\, S_1(\omega_3) = \beta_2(\omega_3)\,(1+r) + \gamma_2(\omega_3)\, S_1(\omega_3)\,, \end{cases}$$

which reduces to

$$\begin{cases} \beta_1\,(1+0.2) + \gamma_1\, 120 = 36 \\ \beta_1\,(1+0.2) + \gamma_1\, 320 = 156\,. \end{cases}$$

Hence

$$\beta_1 = -30 \qquad \text{and} \qquad \gamma_1 = 0.6\,.$$

Note that the initial capital of this hedging strategy

$$X_0^{\pi^*} = -30 + 0.6 \times 200 = 90$$

coincides with the price C_2. ∎

Problem B.1.6 *Let the rate of interest be $r \geq 0$ and suppose that the price of an asset S has the following dynamics*

Ω	$n = 0$	$n = 1$	$n = 2$
ω_1	$S_0 = 10$	$S_1 = 12$	$S_2 = 15$
ω_2	$S_0 = 10$	$S_1 = 12$	$S_2 = 10$
ω_3	$S_0 = 10$	$S_1 = 6$	$S_2 = 10$
ω_4	$S_0 = 10$	$S_1 = 6$	$S_2 = 3$

1. *Find an expression for risk-neutral probability.*
2. *Find all values of $r \geq 0$ that admit the existence of a risk-neutral probability.*
3. *Consider an American call option with the sequence of claims*

$$f_0 = (S_0 - 9)^+\,, \qquad f_1 = (S_1 - 9)^+\,, \qquad f_2 = (S_2 - 10)^+\,.$$

Price this option, find the minimal hedge and the stopping times for $r = 0$.

SOLUTION

1. An expression for risk-neutral probability $P^*(r) = (p_1^*(r), p_2^*(r), p_3^*(r), p_4^*(r))$ can be found from the equalities

$$E^*\left(\frac{S_1}{1+r}\right) = S_0 \qquad \text{and} \qquad E^*\left(\frac{S_2}{(1+r)^2}\middle|\sigma(S_1)\right) = \frac{S_1}{1+r}\,,$$

which reduce to the following system

$$\begin{cases} 12\,(p_1^* + p_2^*) + 6\,(p_3^* + p_4^*) = 10\,(1+r) \\ \frac{15\,p_1^* + 10\,p_2^*}{p_1^* + p_2^*} = 12\,(1+r) \\ \frac{10\,p_3^* + 3\,p_4^*}{p_3^* + p_4^*} = 6\,(1+r) \\ p_1^* + p_2^* + p_3^* + p_4^* = 1\,. \end{cases}$$

Solving this system, we obtain

$$p_1^*(r) = \frac{2+5\,r}{3}\;\frac{2+12\,r}{5}\,, \qquad p_2^*(r) = \frac{2+5\,r}{3}\;\frac{3-12\,r}{5}\,,$$

$$p_3^*(r) = \frac{1-5\,r}{3}\;\frac{3+6\,r}{7}\,, \qquad p_4^*(r) = \frac{1-5\,r}{3}\;\frac{4-6\,r}{7}\,.$$

2. The fact that all these probabilities must be strictly positive:

$$p_1^*(r) > 0\,, \quad p_2^*(r) > 0\,, \quad p_3^*(r) > 0\,, \quad p_4^*(r) > 0\,,$$

implies that $0 \le r < 0.2$.

3. Now since $P^*(0) = \big(4/15,\, 2/5,\, 1/7,\, 4/21\big)$, we can compute all possible values of our contingent claim:

event	**probability**	S_0	f_0	S_1	f_1	S_2	f_2
ω_1	4/15	10	1	12	3	15	5
ω_2	2/5	10	1	12	3	10	0
ω_3	1/7	10	1	6	0	10	0
ω_4	4/21	10	1	6	0	3	0

For pricing this option we compute

$$E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 12\}\big) = E^*\big(f_2\big|\{\omega:\ S_1 = 12\}\big) = \frac{5\,p_1^* + 0\,p_2^*}{p_1^* + p_2^*} = 2\,,$$

$$E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 6\}\big) = E^*\big(f_2\big|\{\omega:\ S_1 = 6\}\big) = \frac{0\,p_3^* + 0\,p_4^*}{p_3^* + p_4^*} = 0\,,$$

where $Y_i^{\pi^*}$ is the capital of the minimal hedge π^* at time i. Hence

$$Y_1^{\pi^*}(\omega_1) = Y_1^{\pi^*}(\omega_2) = \max\big\{f_1,\ E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 12\}\big)\big\} = 3\,,$$

$$Y_1^{\pi^*}(\omega_3) = Y_1^{\pi^*}(\omega_4) = \max\big\{f_1,\ E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 6\}\big)\big\} = 0\,.$$

This implies

$$E^*\left(Y_1^{\pi^*} \middle| \mathcal{F}_0\right) = 3\,(p_1^* + p_2^*) + 0\,(p_3^* + p_4^*) = 3\left(\frac{4}{15} + \frac{2}{5}\right) = 2\,,$$

and

$$Y_0^{\pi^*} = \max\left\{f_0,\ E^*\left(Y_1^{\pi^*} \middle| \mathcal{F}_0\right)\right\} = 2\,,$$

therefore the price is $C_2 = 2$.

To construct the minimal hedge π^* we first compute the stopping times:

$$\tau_n^* = \min\left\{i:\ n \le i \le 2 \quad \text{and} \quad Y_i^{\pi^*} = f_i\right\},$$

so

$$\tau^* = \tau_1^* = 1, \qquad \tau_2^* = 2\,.$$

Now due to equality $Y_1^{\pi^*} = f_1$, we have

$$Y_1^{\pi^*}(\omega) = \beta_1 + \gamma_1\, S_1(\omega) = f_1(\omega)$$

or

$$\begin{cases} Y_1^{\pi^*}(\omega_1) = Y_1^{\pi^*}(\omega_2) = \beta_1 + \gamma_1\, 12 = 3 \\ \\ Y_1^{\pi^*}(\omega_3) = Y_1^{\pi^*}(\omega_4) = \beta_1 + \gamma_1\, 6 = 0\,. \end{cases}$$

Hence $\beta_1 = -3$ and $\gamma_1 = 0.5$. Also note that the initial capital $Y_0^{\pi^*} = -3 + 10 \times 0.5 = 2$ coincides with the price of the option.

∎

Problem B.1.7 *Consider a single-period binomial* (B, S)*-market with* $B_0 = 1$*,* $S_0 = 300$*,* $r = 0.1$ *and*

$$S_1 = \begin{cases} 350 \ \textit{with probability}\ 0.6 \\ 250 \ \textit{with probability}\ 0.4\,. \end{cases}$$

As in Section 1.6 use the logarithmic utility function to find an optimal strategy with the initial capital 200.

SOLUTION First we compute parameters

$$a = \frac{250 - 300}{300} = -\frac{1}{6} \quad \text{and} \quad b = \frac{350 - 300}{300} = \frac{1}{6}\,.$$

The average profitability of S with respect to the initial probability is

$$m = \frac{4}{10}\ \frac{-1}{6} + \frac{6}{10}\ \frac{1}{6} = \frac{1}{30}\,.$$

Then, according to formula (1.5), the proportion of risky capital in the required strategy must be

$$\alpha^* = \frac{(1+r)\,(m-r)}{(r-a)\,(b-r)} = -4.125\,.$$

On the other hand,

$$\alpha^* = \gamma^*\,\frac{S_0^*}{X_0^{\pi^*}}\,,$$

hence,

$$\gamma^* = -4.125\,\frac{200}{300} = -2.75\,.$$

The non-risky component β^* can be found from the condition of self-financing:

$$X_0^{\pi^*} = \beta^* + \gamma^*\,S_0\,,$$

which implies

$$\beta^* = 200 + 2.75 \times 300 = 1025\,.$$

□

Problem B.1.8 *Repeat Problem B.1.7 with* $B_0 = 1$, $S_0 = 100$, $r = 0.2$ *and*

$$S_1 = \begin{cases} 150 & \textit{with probability } 0.7 \\ 80 & \textit{with probability } 0.3\,. \end{cases}$$

SOLUTION In this case we have

$$a = \frac{80-100}{100} = -0.2 \quad \text{and} \quad b = \frac{150-100}{100} = 0.5\,.$$

Then

$$m = 0.7 \times 0.5 - 0.3 \times 0.2 = 0.29\,,$$

and

$$\alpha^* = \frac{(1+r)\,(m-r)}{(r-a)\,(b-r)} = 0.9\,.$$

Thus

$$\gamma^* = \alpha^*\,\frac{X_0^{\pi^*}}{S_0^*} = 0.9\,\frac{200}{100} = 1.8\,,$$

and

$$\beta^* = X_0^{\pi^*} - \gamma^*\,S_0 = 200 - 1.8 \times 100 = 20\,.$$

□

B.2 Problems for Chapter 2

Problem B.2.1 *Consider a single-period (B, S)-market with $B_0 = 1$, $S_0 = 10$, $r = 0.2$ and*

$$S_1(\omega_1) = 6, \quad S_1(\omega_2) = 12, \quad S_1(\omega_3) = 18\,.$$

Find risk-neutral probability P^*.

SOLUTION An expression for P^* can be found from the equality

$$E^*\left(\frac{S_1}{1+r}\right) = S_0\,,$$

which can be written in the form

$$p_1^*\,S_1(\omega_1) + p_2^*\,S_1(\omega_2) + p_3^*\,S_1(\omega_3) = S_0\,(1+r)\,.$$

Since $p_1^* + p_2^* + p_3^* = 1$, we have

$$6\,p_1^* + 12\,p_2^* + 18\,(1 - p_1^* - p_2^*) = 12\,,$$

and therefore $p_2^* = 1 - 2\,p_1^*$. Now let $p_1^* = \lambda$, then we have

$$p_2^* = 1 - 2\,\lambda\,, \qquad p_3^* = \lambda\,.$$

Since all these probabilities must be strictly positive, this implies that

$$0 < \lambda < 1/2\,.$$

Thus, we obtain a one-parameter family of risk-neutral probabilities

$$P_\lambda^* = \big(\lambda,\, 1 - 2\,\lambda,\, \lambda\big)\,, \qquad 0 < \lambda < 1/2\,.$$

∎

Problem B.2.2 *Consider a single-period (B, S)-market with a non-risky asset B and two risky assets S^1 and S^2, where*

$$\begin{aligned}
&B_0 = 1\,, \qquad r = 0.2\,,\\
&S_0^1 = 150, \quad S_1^1(\omega_1) = 200, \quad S_1^1(\omega_2) = 190, \quad S_1^1(\omega_3) = 170\,,\\
&S_0^2 = 200, \quad S_1^2(\omega_1) = 270, \quad S_1^2(\omega_2) = 250, \quad S_1^2(\omega_3) = 230\,.
\end{aligned}$$

Find risk-neutral probability P^*. If it does not exist, find an arbitrage strategy.

SOLUTION If there exists a risk-neutral probability P^*, then we must have

$$E^*\left(\frac{S_1^1}{1+r}\right) = S_0^1 \qquad \text{and} \qquad E^*\left(\frac{S_1^2}{1+r}\right) = S_0^2\,,$$

which can be written in the form of the following system

$$\begin{cases} p_1^*\, S_1^1(\omega_1) + p_2^*\, S_1^1(\omega_2) + p_3^*\, S_1^1(\omega_3) = S_0^1\,(1+r) \\ p_1^*\, S_1^2(\omega_1) + p_2^*\, S_1^2(\omega_2) + p_3^*\, S_1^2(\omega_3) = S_0^2\,(1+r)\,. \end{cases}$$

Since $p_1^* + p_2^* + p_3^* = 1$, then this system reduces to

$$\begin{cases} 200\,p_1^* + 190\,p_2^* + 170\,(1 - p_1^* - p_2^*) = 180 \\ 270\,p_1^* + 250\,p_2^* + 230\,(1 - p_1^* - p_2^*) = 240\,, \end{cases}$$

or equivalently

$$\begin{cases} 30\,p_1^* + 20\,p_2^* = 10 \\ 40\,p_1^* + 20\,p_2^* = 10\,, \end{cases}$$

which implies $p_1^* = 0$. This contradicts the assumption that all initial and risk-neutral probabilities must be strictly positive. Thus, there is no risk-neutral probability P^* that is equivalent to the initial probability P.

An arbitrage strategy $\pi = (\beta,\, \gamma_1,\, \gamma_2)$ can be found from the system

$$\begin{cases} \beta + \gamma_1\, S_0^1 + \gamma_2\, S_0^2 = 0 \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_1) + \gamma_2\, S_1^2(\omega_1) \geq 0 \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_2) + \gamma_2\, S_1^2(\omega_2) \geq 0 \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_3) + \gamma_2\, S_1^2(\omega_3) \geq 0\,, \end{cases}$$

that reflects the fact that a strategy with zero initial capital can have non-negative values at time 1. We are looking for strategies such that at least one of the inequalities above is strict. Substituting given data we obtain

$$\begin{cases} \beta = -150\,\gamma_1 - 200\,\gamma_2 \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 200\,\gamma_1 + 270\,\gamma_2 \geq 0 \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 190\,\gamma_1 + 250\,\gamma_2 \geq 0 \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 170\,\gamma_1 + 230\,\gamma_2 \geq 0\,, \end{cases}$$

or

$$\begin{cases} \beta = -150\,\gamma_1 - 200\,\gamma_2 \\ 20\,\gamma_1 + 30\,\gamma_2 \geq 0 \\ 10\,\gamma_1 + 10\,\gamma_2 \geq 0 \\ -10\,\gamma_1 - 10\,\gamma_2 \geq 0\,. \end{cases}$$

This system is satisfied by

$$\beta = -50\,a, \quad \gamma_1 = -a, \quad \gamma_2 = a,$$

with any $a \geq 0$. Hence we obtain a one-parameter family of arbitrage strategies:

$$\pi = (\beta,\, \gamma_1,\, \gamma_2) = (-50\,a,\ -a,\ a)\,, \qquad a > 0\,.$$

In other words, at time 0, an investor borrows a units of asset S^1, sells them at current price, also borrows $50\,a$ (\$) from a bank and invests all this capital in asset S^2. At time 2 the investor makes a strictly positive profit in the case of ω_1 and otherwise loses nothing. ∎

Problem B.2.3 *Consider a single-period (B, S)-market with $B_0 = 1$, $S_0 = 100$, $r = 0$ and*

$$S_1(\omega_1) = 80, \quad S_1(\omega_2) = 90, \quad S_1(\omega_3) = 180\,.$$

Is there a hedging strategy for a European call option with

$$f_1 = \big(S_1 - 100\big)^+\,?$$

SOLUTION We have the following possible values of claim f_1:

$$f_1(\omega_1) = 0, \quad f_1(\omega_2) = 0, \quad f_1(\omega_3) = 80\,.$$

Suppose that $\pi = (\beta,\, \gamma)$ is a hedging strategy, then

$$\begin{cases} \beta\,(1+r) + \gamma\, S_1(\omega_1) = f_1(\omega_1) \\ \beta\,(1+r) + \gamma\, S_1(\omega_2) = f_1(\omega_2) \\ \beta\,(1+r) + \gamma\, S_1(\omega_3) = f_1(\omega_3) \end{cases}$$

or

$$\begin{cases} \beta + 80\,\gamma = 0 \\ \beta + 90\,\gamma = 0 \\ \beta + 180\,\gamma = 80\,, \end{cases}$$

which is an inconsistent system. Hence, there is no strategy that can hedge this claim. ∎

Problem B.2.4 *Consider a single-period* (B, S)*-market with* $B_0 = 1$, $S_0 = 200$ *and*

$$S_1(\omega_1) = 150, \quad S_1(\omega_2) = 190, \quad S_1(\omega_3) = 250\,.$$

Find all values of r *that admit the existence of a risk-neutral probability* P^*.

SOLUTION We have the equality

$$E^*\left(\frac{S_1}{1+r}\right) = S_0\,,$$

which can be written in the form

$$p_1^*\, S_1(\omega_1) + p_2^*\, S_1(\omega_2) + p_3^*\, S_1(\omega_3) = S_0\,(1+r)\,.$$

Since $p_1^* + p_2^* + p_3^* = 1$, we have

$$150\, p_1^* + 190\, p_2^* + 250\,(1 - p_1^* - p_2^*) = 200\,(1+r)\,,$$

and therefore

$$r = \frac{5 - 10\, p_1^* - 6\, p_2^*}{20}\,.$$

We also have

$$p_1^* > 0\,, \quad p_2^* > 0\,, \quad p_1^* + p_2^* < 1\,,$$

which implies

$$-\frac{1}{4} < r < \frac{1}{4}\,.$$

∎

Problem B.2.5 *As in Section 2.6 consider the Black-Scholes model of a* (B, S)*-market, and compare the optimal investment strategy with the minimal hedge of an European call option with* $f_T = \big(S_T - K\big)^+$.

SOLUTION As we showed in Section 2.6, the proportion of risky capital in the optimal investment strategy is given by

$$\alpha^* = \frac{\mu - r}{\sigma^2}\,.$$

Observe that if $\mu = r$, then $\gamma_t^* = 0$ for all $t \le T$.

On the other hand, for the minimal hedge of an European call option we have

$$\gamma_t = \Phi\left(\frac{\ln(S_t/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

In particular, if $S_0 > K$, then

$$\gamma_0 = \Phi\left(\frac{\ln(S_0/K) + T\,(r + \sigma^2/2)}{\sigma\sqrt{T}}\right) > \frac{1}{2} > \gamma_0^* = 0\,,$$

which means that these two strategies do not coincide. ∎

Problem B.2.6 *Consider the Black-Scholes model of a (B, S)-market with $T = 215/365$, $S_0 = 100$, $\mu = r$. Calculate premium for a pure endowment assurance with a guaranteed minimal payment $K = 80$ in the cases when $r = 0.1$ or $r = 0.2$, and $\sigma = 0.1$ or $\sigma = 0.8$.*

SOLUTION In Section 3.4 we derived formula (3.18) for calculating premiums:

$$U_x(T) = p_x(T)\,K\,e^{-rT} + p_x(T)\,C_T\,,$$

where $p_x(T)$ is the probability that an individual of age x survives to age $x + T$, and

$$C_T = \Big[S_0\,\Phi\big(d_+(0)\big) - K\,e^{-rT}\,\Phi\big(d_-(0)\big)\Big]$$

is the price of a European call option with the strike price K. Recall that all required values of C_T were computed in Worked Example 2.4, Section 2.6.

Now, let $x = 30$, for example. Then from a life table one can find the value of $p_{30}(1)$, say $p_{30}(1) \approx 0.9987$. Thus, for given values of r and σ we obtain the following values of $U_{30}(1)$:

$r \diagdown \sigma$	0.1	0.8
0.1	99.87	110.78
0.2	99.88	109.01

∎

Problem B.2.7 *Repeat the previous problem for the discrete Gaussian model of a (B, S)-market.*

SOLUTION Using results of Sections 2.4 and 3.4 we obtain

$$U_x(T) = p_x(T)\,(1 + r)^{-T}\Big[K + S_0\,(1 + r)^T\,\Phi\big(d_+(0)\big) - K\,\Phi\big(d_-(0)\big)\Big]\,.$$

For $p_{30}(1) \approx 0.9987$ and for given values of r and σ we obtain the following values of $U_{30}(1)$:

$r \setminus \sigma$	0.1	0.8
0.1	99.87	110.92
0.2	99.87	109.32

∎

Problem B.2.8 *In the framework of the Black-Scholes model of a (B, S)-market consider an investment portfolio π with the initial capital x. Estimate the asymptotic profitability of π:*

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta, \qquad \delta \in (0,1]\,.$$

SOLUTION First we note that by Lyapunov's inequality (see, for example, [41]), we have

$$E\big(X_T^\pi(x)\big)^\delta \le \big(E X_T^\pi(x)\big)^\delta\,.$$

Suppose that the initial probability P is a martingale probability and that strategy π is self-financing. Then

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta \\
&\qquad \le \delta \limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big) = \delta \limsup_{T\to\infty} \frac{1}{T} \ln \left[E\left(\frac{X_T^\pi(x)}{B_T}\right) B_T\right] \\
&\qquad = \delta \limsup_{T\to\infty} \frac{1}{T} \Big[\ln a + \ln B_T\Big] = \delta \limsup_{T\to\infty} \frac{r\,T}{T} = \delta\, r\,,
\end{aligned}$$

where a is some constant.

On the other hand, if we invest only in non-risky asset B, we have

$$X_T^\pi(x) = x\, B_0\, e^{rT}\,,$$

so

$$\ln E\big(X_T^\pi(x)\big)^\delta = \ln\Big[x^\delta\, B_0^\delta\, e^{\delta r T}\Big] = b + \delta\, r\, T$$

for some constant b.

Thus

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta \ge \limsup_{T\to\infty} \frac{1}{T} \big[b + \delta\, r\, T\big] = \delta\, r\,,$$

so therefore

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta = \delta\, r\,,$$

and it does not depend on the initial capital x. ∎

B.3 Problems for Chapter 3

Problem B.3.1 *Consider the binomial model of a* (B, S)*-market with* $S_0 = 100,\ B_0 = 1,\ r = 0.2$ *and*

$$\rho = \begin{cases} 0.5 & \textit{with probability } 0.4 \\ -0.3 & \textit{with probability } 0.6 \,. \end{cases}$$

Calculate the premium for a pure endowment assurance with a guaranteed minimal payment $K = 100$ *in the cases when* $N = 1$ *and* $N = 2$.

SOLUTION First we compute risk-neutral probability

$$p^* = \frac{0.2 + 0.3}{0.5 + 0.3} = \frac{5}{8} \,.$$

As in Problem B.2.6, suppose that the age of life assured is $x = 30$, so that the probabilities of survival to age $30 + N$ are

$$p_{30}(1) \approx 0.9987 \qquad \text{and} \qquad p_{30}(2) \approx 0.997 \,,$$

for $N = 1$ and $N = 2$, respectively.

Computing

$$E^* \left(\frac{\max\{S_1 \,,\, K\}}{1 + r} \right) = \frac{5}{8} \; \frac{100\,(1 + 0.5)}{1.2} + \frac{3}{8} \; \frac{100}{1.2} = 109.375$$

and

$$E^* \left(\frac{\max\{S_2 \,,\, K\}}{(1 + r)^2} \right)$$
$$= \left(\frac{5}{8} \right)^2 \; \frac{100\,(1 + 0.5)^2}{1.44} + 2\, \frac{5}{8} \; \frac{3}{8} \; \frac{100\,(1 + 0.5)\,0.7}{1.44} + \left(\frac{3}{8} \right)^2 \; \frac{100}{1.44}$$
$$\approx 87.89 \,,$$

we calculate the required premiums:

$$U_{30}(1) = p_{30}(1)\, E^* \left(\frac{\max\{S_1 \,,\, K\}}{1 + r} \right) = 0.9987 \times 109.375 \approx 109,23$$

and

$$U_{30}(2) = p_{30}(2)\, E^* \left(\frac{\max\{S_2 \,,\, K\}}{(1 + r)^2} \right) = 0.997 \times 87.89 \approx 87.63 \,.$$

∎

Problem B.3.2 *Suppose that an insurance company issues* 90 *independent identical policies, and suppose that the average amount of claims is* $ 300 *with standard deviation* $ 100. *Estimate the probability of total claim amount* S *to be greater than* $ 29, 000.

SOLUTION We compute expectation and variance of S:

$$E(S) = 300 \times 90 = 27,000 \qquad \text{and} \qquad V(S) = 100 \times 100 \times 90 = 900,000\,.$$

Since S is a sum of 90 independent identically distributed random variables, then normalized random variable

$$\frac{S - E(S)}{\sqrt{V(S)}}$$

is asymptotically normal. Thus the required probability is

$$\alpha \approx 1 - \Phi\left(\frac{29,000 - 27,000}{\sqrt{900,000}}\right) \approx 0.02\,.$$

∎

Problem B.3.3 *Suppose that an insurance company issues* 100 *independent identical policies. Find probabilistic characteristics of an individual claim* X *given the following statistical data:*

	amount of claim	number of claims
1	0 – 400	2
2	400 – 800	24
3	800 – 1200	32
4	1200 – 1600	21
5	1600 – 2000	10
6	2000 – 2400	6
7	2400 – 2800	3
8	2800 – 3200	1
9	3200 – 3600	1
10	> 3600	0

SOLUTION Let us assume that both claims from the first group (0-400) were 200, all 24 claims from the second group (400-800) were 600, etc. Then we compute

$$E(X) = 200\,\frac{2}{100} + 600\,\frac{24}{100} + \ldots + 3400\,\frac{1}{100} = 1216\,,$$

and

$$V(X) = \left[200^2 \, \frac{2}{100} + 600^2 \, \frac{24}{100} + \ldots + 3400^2 \, \frac{1}{100}\right] - 1216^2 = 362944\,.$$

□

Problem B.3.4 *Suppose that an insurance company issued* 1000 *independent identical policies, and as a result,* 120 *claims were received during the last 12 months. Find the probability of not receiving a claim from an individual policy holder during the next 9 months.*

SOLUTION

1. Suppose that the number of claims received from an individual policy holder during any 3 months is modelled by a Poisson distribution with parameter q. Then, assuming that numbers of claims that correspond to non-intersecting periods of time are independent, we have that the number of claims received from an individual policy holder during any 12 months can be represented by a Poisson distribution with parameter $4\,q$. For a portfolio of 1000 policies we have a Poisson distribution with parameter $4000\,q$. Thus

$$q = \frac{120}{4000} = 0.03\,,$$

which implies that the number of claims received from an individual policy holder during any 9 months has the Poisson distribution with parameter 0.09. In particular, the probability of not receiving a claim from an individual policy holder during the next 9 months is

$$\alpha = \frac{(0.09)^0}{0!}\, e^{-0.09} \approx 0.91\,.$$

2. Alternatively, we can use the Bernoulli distribution for modelling ξ_1, the number of claims received from an individual policy holder during any 3 months:

$$\xi_1 = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p\,. \end{cases}$$

Then the number of claims received from an individual policy holder during any 12 months has binomial distribution

$$\xi_1 + \xi_2 + \xi_3 + \xi_4\,,$$

and the total number of claims has binomial distribution

$$S = \xi_1 + \xi_2 + \xi_3 + \ldots + \xi_{4000}\,,$$

where all ξ_i are independent and distributed identically to ξ_1. Thus

$$E(S) = 4000\,p = 120 \qquad \text{and hence} \qquad p = \frac{120}{4000} = 0.03\,.$$

We then have

$$P\big(\{\omega:\ \xi_1 + \xi_2 + \xi_3 = k\}\big) = \binom{3}{k}\,(0.03)^k\,(0.97)^{3-k}\,, \qquad k = 0, 1, 2, 3\,.$$

In particular, the probability of not receiving a claim from an individual policy holder during the next 9 months is

$$P\big(\{\omega:\ \xi_1 + \xi_2 + \xi_3 = 0\}\big) = (0.97)^3 \approx 0.91\,.$$

∎

Problem B.3.5 *Suppose that the following table describes the frequency of receiving claims by an insurance company during one year:*

number of claims	**number of policies**
0	3288
1	642
2	66
3	4

Find the probability of receiving only one claim from two independent policies during the next year.

SOLUTION

1. Suppose that the number of claims received during the year is modelled by a Poisson distribution with parameter q. Let N be the number of claims received from one policy. Then

$$E(N) = 642\,\frac{1}{4000} + 66\,\frac{2}{4000} + 4\,\frac{3}{4000} = 0.1965\,,$$

which implies that the number of claims received from two independent policies has Poisson distribution with parameter 0.393. Therefore, the probability of receiving only one claim from two independent policies during the next year is

$$\alpha = \frac{(0.393)^1}{1!}\,e^{-0.393} \approx 0.265\,.$$

2. Alternatively, suppose that the number of claims from one policy per year has binomial distribution:

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3=k\}\big)=\binom{3}{k}\,(0.03)^k\,(0.97)^{3-k}\,,\qquad k=0,1,2,3\,,$$

where ξ_i are independent Bernoulli random variables:

$$\xi_i=\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\,.\end{cases}$$

Then the average number of claims from two independent policies per year is 0.393.

On the other hand,

$$E\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6\}\big)=6\,p\,,$$

therefore $p=0.393/6$. Hence we obtain the following distribution

$$P\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6=k\}\big)=\binom{6}{k}\left(\frac{0.393}{6}\right)^k\left(1-\frac{0.393}{6}\right)^{6-k},$$

$k=0,1,2,\ldots,6$. In particular, the probability of receiving only one claim from two independent policies per year is

$$P\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6=1\}\big)=\binom{6}{1}\left(\frac{0.393}{6}\right)^1\left(1-\frac{0.393}{6}\right)^5\approx 0.28\,.$$

□

Problem B.3.6 *Suppose that an insurance company issued* 4000 *independent identical policies. Find the expected number of policies that will result in* 0, 1, 2 *and* 3 *claims per year if*

(1) *the number of claims from one policy per year has Poisson distribution with parameter* 0.1965*;*

(2) *the number of claims from one policy per year has a binomial distribution with the average* 0.1965.

SOLUTION

1. We compute the probabilities of receiving $0, 1, 2$ and 3 claims from one policy per year:

$$p_0 = \frac{(0.1965)^0}{0!}\; e^{-0.1965} \approx 0.82\,,$$

$$p_1 = \frac{(0.1965)^1}{1!}\; e^{-0.1965} \approx 0.16\,,$$

$$p_2 = \frac{(0.1965)^2}{2!}\; e^{-0.1965} \approx 0.016\,,$$

$$p_3 = \frac{(0.1965)^3}{3!}\; e^{-0.1965} \approx 0.001\,.$$

Multiplying these probabilities by 4000 we arrive at the following table.

number of claims	**number of policies**
0	3280
1	640
2	64
3	4

2. In the binomial case we have

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = k\}\big) = \binom{3}{k}\left(\frac{0.393}{6}\right)^k\left(1-\frac{0.393}{6}\right)^{3-k},\ k = 0, 1, 2, 3\,,$$

so that the probabilities of receiving $0, 1, 2$ and 3 claims from one policy per year are given by

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 0\}\big) = \left(1-\frac{0.393}{6}\right)^3 \approx 0.82\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 1\}\big) = 3\left(\frac{0.393}{6}\right)\left(1-\frac{0.393}{6}\right)^2 \approx 0.17\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 2\}\big) = 3\left(\frac{0.393}{6}\right)^2\left(1-\frac{0.393}{6}\right)^1 \approx 0.012\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 3\}\big) = \left(\frac{0.393}{6}\right)^3 \approx 0.0003\,.$$

Multiplying these probabilities by 4000 we obtain

number of claims	**number of policies**
0	3280
1	680
2	48
3	1

□

Problem B.3.7 *Suppose that an insurance company issued* 1000 *independent identical policies. Further, suppose that the probability of receiving a claim from one policy is* 0.5*, and that each policy allows no more than one claim to be made. Find the probability of the total number of claims to be between* 470 *and* 530*.*

SOLUTION The required probability is equal to

$$\sum_{k=470}^{530} \binom{1000}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{1000-k} .$$

We can use De Moivre-Laplace Limit Theorem to compute an approximate value of this expression. Let S be the total number of received claims, then

$$E(S) = 1000/2 = 500 , \qquad \text{and} \qquad \sqrt{V(S)} = \sqrt{1000\, \frac{1}{2}\, \frac{1}{2}} \approx 15.81 .$$

So

$$P\big(\{\omega :\ 470 \le S \le 530\}\big) \approx \Phi\left(\frac{530 - 500 + 0.5}{15.81}\right) \Phi\left(\frac{470 - 500 - 0.5}{15.81}\right) \approx 0.95 .$$

□

Problem B.3.8 *An insurance company estimated that the probability of receiving a claim from one policy during one year is* 0.01 *and the average amount of a claim is* \$980*. Suppose that the company issues* 1000 *independent identical one-year policies. Find the probability of the total amount of claims to be more than* \$14, 850*.*

SOLUTION We use the individual risk model. Let S be the total amount of claims, and suppose that its normalized value has a standard normal distribution. Then

$$E(S) = 0.01 \times 980 \times 1000 = 9800 ,$$

and

$$\sqrt{V(S)} = \sqrt{0.01 \times 0.99 \times 980 \times 980 \times 1000} \approx 3083\,.$$

Therefore

$$P\big(\{\omega:\ S > 14,850\}\big) = P\left(\left\{\omega:\ \frac{S-9800}{3083} > 1.64\right\}\right) \approx 1 - \Phi(1.64) \approx 0.05\,.$$

∎

Problem B.3.9 *Suppose that the following table describes q, the frequency of receiving claims by an insurance company during one year:*

number of claims	number of policies
0	3280
1	640
2	64
3	4

Determine a 95% *confidence interval for q.*

SOLUTION Let N be the number of claims. Suppose that N has the Poisson distribution with parameter q. Then

$$E(N) = 4000\,q \qquad \text{and} \qquad V(N) = 4000\,q\,.$$

Hence, the random variable

$$\frac{N - 4000\,q}{20\,\sqrt{q}}$$

is asymptotically normal. Since

$$N = 642 + 66 \times 2 + 4 \times 3 = 786\,,$$

we obtain

$$-1.96 < \frac{786 - 4000\,q}{20\,\sqrt{q}} < 1.96\,,$$

which implies

$$0.19 < q < 0.2\,.$$

∎

Problem B.3.10 *Consider three policies with claims X_1, X_2 and X_3, respectively. Suppose*

$$P(\{\omega:\ X_1=0\})=0.5\,,\quad P(\{\omega:\ X_1=100\})=0.5\,,$$
$$P(\{\omega:\ X_2=0\})=0.8\,,\quad P(\{\omega:\ X_2=250\})=0.2\,,$$
$$P(\{\omega:\ X_3=0\})=0.4\,,\quad P(\{\omega:\ X_3=100\})=0.4\,,\quad P(\{\omega:\ X_3=50\})=0.2\,.$$

Find the most and the least risky policy.

SOLUTION Clearly, the average value of each X_i is 50, so we will compare their variances:

$$V(X_1)=0.5\times 100\times 100-2500=2500\,,$$
$$V(X_2)=0.2\times 250\times 250-2500=10,000\,,$$
$$V(X_3)=0.4\times 100\times 100+0.2\times 50\times 50-2500=2000\,.$$

Thus, we conclude that the second policy is the most risky, and the third is the least risky policy. □

Problem B.3.11 *Consider two independent policies with the following distributions of claims*

$$P(\{\omega:\ X_1=100\})=0.6\,,\quad P(\{\omega:\ X_1=200\})=0.4\,,$$
$$P(\{\omega:\ X_2=100\})=0.7\,,\quad P(\{\omega:\ X_2=200\})=0.3\,.$$

Suppose that the probability of receiving a claim from the first policy is 0.1 *and from the second one is* 0.2*. Find the distribution of claims for the portfolio formed by these two policies.*

SOLUTION The possible amounts of claims for this portfolio are 0, 100, 200, 300 and 400. So we have

$$p_0=P(\{\omega:\ X_1+X_2=0\})=0.9\times 0.8=0.72\,,$$
$$p_{100}=P(\{\omega:\ X_1+X_2=100\})=0.9\times 0.2\times 0.7+0.1\times 0.8\times 0.6=0.174\,,$$
$$p_{200}=P(\{\omega:\ X_1+X_2=200\})$$
$$=0.1\times 0.2\times 0.6\times 0.7+0.9\times 0.2\times 0.3+0.1\times 0.8\times 0.4=0.0944\,,$$
$$p_{300}=P(\{\omega:\ X_1+X_2=300\})$$
$$=0.1\times 0.2\times 0.6\times 0.3+0.1\times 0.2\times 0.7\times 0.4=0.0092\,,$$
$$p_{400}=P(\{\omega:\ X_1+X_2=400\})=0.1\times 0.2\times 0.4\times 0.3=0.0024\,.$$

□

Problem B.3.12 *In the framework of the individual risk model consider a portfolio of* 50 *independent identical claims. Suppose that premiums are calculated according to the Expectation principle (see Section 3.1) with the security loading coefficient* 0.1. *Assuming that exactly one claim is received from each policy holder, find the probability of solvency in the following cases:*

(a) *each claim has an exponential distribution with average* 100*;*

(b) *each claim has a normal distribution with average* 100 *and variance* 400*;*

(c) *each claim has a uniform distribution in the interval* $[70,\ 130]$.

SOLUTION First, we observe that the total premium income in each of the cases is

$$\Pi = 100\,(1+0.1)\,50 = 5500\,.$$

The total claim amounts are all $100 \times 50 = 5000$ and their standard deviations are

(a) $\sigma_1 = 100\,\sqrt{50} \approx 707.1$;

(b) $\sigma_2 = 20\,\sqrt{50} \approx 141.4$;

(c) $\sigma_3 = 60\,\sqrt{50/12} \approx 122.5$.

Now, since normalized total claim amounts are asymptotically normal, then the required probabilities are

(a) $\alpha_1 \approx 1 - \Phi\left(\frac{5500-5000}{707.1}\right) \approx 1 - \Phi\big(0.707\big) \approx 0.24$;

(b) $\alpha_2 \approx 1 - \Phi\left(\frac{5500-5000}{141.4}\right) \approx 1 - \Phi\big(3.54\big) \approx 0.0002$;

(c) $\alpha_3 \approx 1 - \Phi\left(\frac{5500-5000}{122.5}\right) \approx 1 - \Phi\big(4.08\big) \approx 0.00002$.

$\square$

Problem B.3.13 *In the framework of a binomial model consider two insurance companies. Suppose that the claims of the first company are distributed according to the Poisson law with average* 2, *and that the probability of receiving a claim equal to* 0.1. *For the second company we assume the same probability of receiving a claim and the following distribution of claims:* $P\big(\{\omega : \ X = 2\}\big) = 1$. *Given that both companies receive the premium of* 1 *and have zero initial capitals, find the corresponding probabilities of solvency:* $\phi(0,1)$, $\phi(0,2)$ *and* $\phi(0)$. *(See Section 3.2 for details.)*

SOLUTION Clearly, the first company will be solvent after one time step if it receives either a claim of 1 or no claims. The second company will be solvent only if it receives no claims during this period. Hence

$$\phi(0,1) = 0.1\,\frac{2}{1}\,e^{-2} + 0.9 \approx 0.94$$

for the first company, and

$$\hat{\phi}(0,1) = 0.9$$

for the second company.

Next, the first company will stay solvent after two time steps if any of the following events will occur

A: no claims on step one, no claims on step two;

B: no claims on step one, a claim of 1 on step two;

C: no claims on step one, a claim of 2 on step two;

D: a claim of 1 on step one, a claim of 1 on step two;

E: a claim of 1 on step one, no claims on step two.

Computing the probabilities of these events:

$$\begin{aligned}
P(A) &= 0.9 \times 0.9 = 0.81\,,\\
P(B) &= 0.9 \times 0.1 \times \frac{2}{1} \times e^{-2} \approx 0.037\,,\\
P(C) &= 0.9 \times 0.1 \times \frac{2^2}{2} \times e^{-2} \approx 0.061\,,\\
P(D) &= 0.1 \times \frac{8}{1} \times e^{-8} \times 0.1 \times \frac{8}{1} \times e^{-8} \approx 0.002\,,\\
P(E) &= P(B) \approx 0.037\,,
\end{aligned}$$

we conclude that the probability of solvency after two time steps is

$$\phi(0,2) = P(A) + P(B) + P(C) + P(D) + P(E) \approx 0.91\,.$$

For the second company we have events

F: no claims on step one, no claims on step two;

G: no claims on step one, a claim of 2 on step two;

with probabilities

$$P(F) = 0.81\,, \quad \text{and} \quad P(G) = 0.9 \times 0.1 \times 1 = 0.09\,.$$

Therefore the probability of its solvency after two time steps is

$$\hat{\phi}(0,2) = P(F) + P(G) = 0.9\,.$$

Finally, we compute

$$\phi(0) = \frac{1 - 0.1 \times 2}{1 - 0.1} \approx 0.89 \quad \text{and} \quad \hat{\phi}(0) = \frac{1 - 0.1 \times 2}{1 - 0.1} \approx 0.89\,.$$

□

Problem B.3.14 *Consider the Cramér-Lundberg model (see Section 3.2) with the premium income* $\Pi(t) = t$ *and with the claims flow represented by a Poisson process with intensity* 0.5. *Suppose that the average claim amount is* 1 *with variance* 5. *Estimate the Cramér-Lundberg coefficient (see Cramér-Lundberg inequality (3.2)).*

SOLUTION We have that the Cramér-Lundberg coefficient r satisfies the equation

$$0.5 + r = 0.5 \int_0^\infty e^{r\,x}\, dF(x)\,,$$

where F satisfies the following conditions:

$$\int_0^\infty x\, dF(x) = 1 \quad \text{and} \quad \int_0^\infty x^2\, dF(x) = 5 + 1 = 6\,.$$

Hence

$$\int_0^\infty e^{r\,x}\, dF(x) \geq \int_0^\infty \left(1 + r\,x + \frac{r^2\,x^2}{2}\right) dF(x) = 1 + r + 3\,r^2\,,$$

and therefore

$$0.5 + r \geq 0.5 + 0.5\,r + 1.5\,r^2$$

or $0 \geq 3\,r^2 - r$. Since r is positive, we conclude that $r \leq 1/3$. □

Problem B.3.15 *Consider the Cramér-Lundberg model (see Section 3.2) with the premium income* $\Pi(t) = t$ *and with the claims flow represented by a Poisson process with intensity* 0.5. *Suppose that claim amounts are equal to* 1 *with probability* 1. *Find the Cramér-Lundberg coefficient.*

SOLUTION We have that the Cramér-Lundberg coefficient r satisfies the equation

$$0.5 + r = 0.5\, e^r\,,$$

which we can write in the form

$$f(r) := 0.5\, e^r - r - 0.5\,.$$

It is not difficult to find an approximate solution to this equation (using Newton's method, say): $r \approx 1.26$. □

Problem B.3.16 *Consider* 50 *independent identical insurance policies. Suppose that the average claim received from a policy during a certain time period is* 100 *with variance* 200*. Also suppose that the equivalence principle is used for premiums calculations and that all premiums income is invested in a non-risky asset with the yield rate of* 0.025 *per specified period. Estimate the probability of solvency and the expected profit.*

SOLUTION The collected premiums are $50 \times 100 = 5000$. At the end of the specified period this accumulates to 5125. Then we compute the probability of solvency:

$$P\big(\{\omega:\ S \le 5125\}\big) \approx \Phi\left(\frac{5125 - 5000}{10\,\sqrt{2}\,\sqrt{50}}\right) \approx 0.89\,,$$

where S is the aggregate claims payment. The expected profit is the difference between the premium income and the expected aggregate claims payment: $5125 - 5000 = 125$. Note that without the investment opportunity, the probability of solvency is 0.5, which is not acceptable. □

Problem B.3.17 *Repeat the previous problem assuming that there is an opportunity to invest in a risky asset with profitability*

$$\rho = \begin{cases} 0.06 & \textit{with probability } 0.5 \\ -0.005 & \textit{with probability } 0.5\,. \end{cases}$$

SOLUTION We have that the collected premiums amount of 5000 accumulates to

$$\begin{cases} 5000\,(1 + 0.06) = 5300 & \text{with probability } 0.5 \\ 5000\,(1 - 0.005) = 4975 & \text{with probability } 0.5\,, \end{cases}$$

therefore the expected profit is

$$0.5\,5300 + 0.5\,4975 - 5000 = 137.5 > 125$$

and the probability of solvency is

$$0.5\,P\big(\{\omega:\ S \le 5300\}\big) + 0.5\,P\big(\{\omega:\ S \le 4975\}\big) \approx 0.5\,\Phi\big(3\big) + 0.5\,\Phi\big(-0.25\big) \approx 0.7\,.$$

Note that the probability of solvency in this case is less than in the previous problem in spite of the fact that the expected profit is higher. This is one of the reasons that insurance companies may have restrictions on proportions of their capital that can be invested in risky assets. □

Problem B.3.18 *Consider an insurance company whose annual aggregate claims payment has an exponential distribution with the average of* 40,000*. Suppose that* *this company operates in the framework of a (B, S)-market, where the profitability of a risky asset is*

$$\rho = \begin{cases} 0.1 & \textit{with probability } 0.5 \\ 0.3 & \textit{with probability } 0.5\,, \end{cases}$$

and the rate of interest is 0.2. Suppose that $S_0 = 10$, and that all premium income is invested in a portfolio. Find an investment strategy $\pi = (\beta, \gamma)$ that minimizes the probability of bankruptcy.

SOLUTION If Π is the collected premiums income, then at time 0 we have a portfolio with

$$(\Pi - 10\,\gamma) + 10\,\gamma = \Pi\,.$$

It time 1 the value of this portfolio is

$$\begin{cases} (\Pi - 10\,\gamma)\,1.2 + 13\,\gamma = 1.2\,\Pi + \gamma & \text{with probability } 0.5 \\ \\ (\Pi - 10\,\gamma)\,1.2 + 9\,\gamma = 1.2\,\Pi - 3\,\gamma & \text{with probability } 0.5\,. \end{cases}$$

Hence the probability of bankruptcy is

$$0.5\,e^{-\lambda(1.2\,\Pi + \gamma)} + 0.5\,e^{-\lambda(1.2\,\Pi - 3\,\gamma)}\,.$$

Minimizing function

$$f(\gamma) := e^{-\lambda\,\gamma} + e^{3\,\lambda\,\gamma}\,,$$

we obtain

$$\gamma = \frac{\ln\big(1/3\big)}{4\,\lambda} \approx -10,986\,.$$

∎

Problem B.3.19 *Find the probability that a newborn individual survives to the age of 30 if the force of mortality is constant $\mu_x \equiv \mu = 0.001$.*

SOLUTION We have (see Section 3.4)

$$p_0(30) = e^{-\int_0^{30} 0.001\,dt} = e^{-0.03} \approx 0.97\,.$$

∎

Problem B.3.20 *Explain why function $(1+x)^{-2}$ cannot be used as the force of mortality.*

SOLUTION By contradiction, suppose

$$\mu_x = \frac{1}{(1+x)^2}\,.$$

Then

$$p_0(t) = e^{-\int_0^t (1+s)^{-2}\,ds} = e^{-\left(1-\frac{1}{1+t}\right)},$$

and therefore

$$\lim_{t\to\infty} p_0(t) = e^{-1} \approx 0.37\,,$$

i.e., a newborn individual survives to any age with the positive probability 0.37. ∎

Problem B.3.21 *Consider the survival function (see Section 3.4)*

$$s(x) = 1 - \frac{x}{100}\,, \qquad 0 \le x \le 100\,.$$

Find the force of mortality and the probability that a newborn individual survives to the age of 20 *but dies before the age of* 40*.*

SOLUTION We have

$$p_x(t) = \frac{1-(x+t)/100}{1-x/100} = \frac{100-x-t}{100-x} = 1 - \frac{t}{100-x}\,.$$

Then

$$-\int_0^t \mu_{x+s}\,ds = \ln\left(1 - \frac{t}{100-x}\right),$$

therefore

$$-\mu_{x+t} = -\left(\frac{1}{100-x}\right)\Big/\left(1-\frac{t}{100-x}\right)$$

and

$$\mu_x = \frac{1}{100-x}\,.$$

Finally, the required probability is

$$1 - \frac{20}{100} - 1 + \frac{40}{100} = 0.2\,.$$

∎

Problem B.3.22 *Consider the Gompertz' model with* $\mu = [\![1.1]\!]^x$*. Find* $p_0(t)$*.*

SOLUTION We have

$$p_x(t) = e^{-\int_0^t [\![1.1]\!]^{x+s}\,ds} = e^{-[\![1.1]\!]^x \frac{[\![1.1]\!]^t - 1}{\ln[\![1.1]\!]}},$$

hence

$$p_0(t) = e^{-\frac{[\![1.1]\!]^t-1}{\ln[\![1.1]\!]}} \approx e^{-10.492\left([\![1.1]\!]^t - 1\right)}.$$

∎

Problem B.3.23 *Consider an insurance company with the initial capital of* 250. *Suppose that the company issues* 40 *independent identical insurance policies and that the average claim amount is* 50 *per policy with standard deviation* 40. *Premiums are calculated according to the Expectation principle with the security loading coefficient* 0.1. *The company has an option of entering a quota share reinsurance contract with retention function* $h(x) = x/2$ *(see* *Section 3.5**). The reinsurance company calculates its premium according to the Expectation principle with the security loading coefficient* 0.15. *Estimate the expected profit and the probability of bankruptcy of the (primary) insurance company in the cases when it purchases the reinsurance contract and when it does not.*

SOLUTION If S is the aggregate claims payment, then

$$E(S) = 40 \times 50 = 2000 \quad \text{and} \quad \sqrt{V(S)} = 40\sqrt{40} \approx 252.98\,.$$

Since $2000\,(0.15 - 0.1) < 250$, then the purchase of the reinsurance contract reduces the probability of bankruptcy of the insurance company. Indeed, we have that the premiums amount is

$$\Pi = 40 \times 50 \times (1 + 0.1) = 2200\,.$$

Therefore, in the case when the reinsurance contract is not purchased, the expected profit is $\Pi - E(S) = 200$ and the probability of bankruptcy is

$$P\big(\{\omega:\ S > 250 + 2000\}\big) \approx 1 - \Phi\left(\frac{250 + 2200 - 2000}{252.98}\right) \approx 0.03764\,.$$

Otherwise, the premium

$$\Pi_1 = 40 \times 50 \times (1 + 0.15) \times 0.5 = 1150$$

is paid to the reinsurance company. Hence the expected profit is $\Pi - 0.5\,E(S) - \Pi_1 = 50$ and the probability of bankruptcy is

$$P\big(\{\omega:\ 0.5\,S > 500 + 1050\}\big) \approx 1 - \Phi\left(\frac{1100}{252.98}\right) \approx 7 \times 10^{-6}\,.$$

∎

Problem B.3.24 *Suppose that annual aggregate claims payments of an insurance company are uniformly distributed in* $[0, 2000]$. *Consider a stop-loss reinsurance contract with the retention level* 1600. *Compute expectations and variances of aggregate claims payments of both insurance and insurance companies.*

SOLUTION Let S and R be the aggregate claims payments of insurance and insurance companies, respectively. Then

$$E(S) = \int_0^{1600} \frac{x}{2000}\,dx + \int_{1600}^{2000} \frac{1600}{2000}\,dx = 960\,,$$

and therefore

$$E(R) = 1000 - E(S) = 40\,.$$

Further

$$V(S) = \int_0^{1600} \frac{x^2}{2000}\,dx + \int_{1600}^{2000} \frac{1600^2}{2000}\,dx \approx 1,194,667\,,$$

and

$$V(R) = \int_{1600}^{2000} \frac{(x-1600)^2}{2000}\,dx \approx 10,666.7\,,$$

so that

$$V(S) \approx 273,066.7 \quad \text{and} \quad V(R) \approx 9066.7\,.$$

Note that variance of the risk process without the reinsurance contract is $2000 \times 2000/12 \approx 333,333 > V(S) + V(R)$. □

Appendix C

Bibliographic Remark

Chapter 1

We introduce the notions of a financial market, of basic and derivative securities; we discuss the probabilistic foundations of financial modelling and general ideas of financial risk management (see [7], [22], [29], [42]).

Quantitative analysis of risks related to contingent claims and maximization of utility functions is described in the framework of the simplest (Cox-Ross-Rubinstein) model of a market (see [11]).

As in the probability theory, where many general ideas and methods are often first explained in a discrete (Bernoulli) case (see [41]), in financial mathematics binomial markets are considered to be a good starting point in studying such fundamental notions as arbitrage, completeness, hedging and optimal investment (see [1], [14], [16], [18], [24], [27], [28], [30], [35], [37], [42]).

Chapter 2

This chapter begins with a comprehensive study of discrete markets. We give proofs of two Fundamental Theorems of financial mathematics, and discuss a methodology for pricing contingent claims in complete and incomplete markets, in markets with constraints and in markets with transaction costs (see [10], [16], [29], [30], [37], [42]).

Next, we study financial risks in the framework of the Black-Scholes model [6], [32]. The celebrated Black-Scholes formula is first derived in the discrete Gaussian setting. Then we demonstrate how the Black-Scholes model, formula and equation can be obtained from the binomial model and the Cox-Ross-Rubinstein formula by limit arguments. [27].

Methods of stochastic analysis are commonly used in the analysis of risks in the Black-Scholes model: for pricing contingent claims with or without taking into account dividends and transaction costs, for various types of hedging, for solving problems of optimal investment, including the case of insider information (see [3], [5], [14], [21], [24], [25], [31], [42]).

Further, we discuss continuous models of bonds markets and pricing of options on these bonds, including computational aspects (see [4], [35], [38], [42]).

One section is devoted to real options that we associate with long-term investment projects. The Bellmann principle is one of the main tools in studying real options (see [8], [13], [23], [26], [36]).

Technical analysis (see [34]) is a very common tool in investigating the qualitative structure of risks. We demonstrate how probabilistic methods can add some quantitative aspects to technical analysis (see [43]).

Handbooks [2], [20] are the standard sources of information on special functions and differential equations that are useful for solving the Bellmann equation, optimal stopping stopping time problem, etc.

Chapter 3

Complex binomial and Poisson models are used for modelling the capital of an insurance company. Actuarial criteria in premium calculations are presented (see [9], [31], [39]).

Probability of bankruptcy is used as a measure of solvency of an insurance company. Various estimates of probability of bankruptcy are given, including the celebrated Cramér-Lundberg estimate [12], [15], [39], [44], [45].

We discuss models that take into account an insurance company's financial investment strategies (see [17], [29], [30], [31]).

Another important type of insurance that is related to combination of risks in insurance and in finance is represented by equity-linked life insurance contracts and by reinsurance with the help of derivative securities. Analysis of such mixed risks requires a combination of modern methods of financial mathematics and actuarial mathematics (see [29], [30], [31], [33]).

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Glossary of Notation

$:=$	equality by definition
a.s.	almost surely
$\emptyset$	the empty set
$\square$	the end of proof
$\{x \in A \mid Z\}$	the subset of A whose elements possess property Z
$A \times B$	the cartesian product of sets A and B
I_A	the indicator function of set A
$f\vert_A$	the restriction of function $f : X \to Y$ to the subset A of X
(a_k), $(a_k)_{k=1}^{\infty}$	the sequence $a_1, \ldots, a_k, \ldots$
$\mathbb{N}, \mathbb{Z}, \mathbb{R}$	the sets of natural numbers, integers and real numbers
$\mathbb{R}^N$	the set of all real N-tupels $(r_1, \ldots, r_n)$
2^A	the set of all subsets of A
$f(x) =_{x \to a} \mathcal{O}(g(x))$	$\vert f(x)\vert \leq \text{const}\ \vert g(x)\vert$ in a neighborhood of a
$o(x)$	a function satisfying $\vert o(x)/x\vert \to 0$ as $x \to 0$
$[\![x]\!]$	the integer part of $x \in \mathbb{R}$
$x \wedge y$	$:= \min\{x, y\}$
$C^n[0, \infty)$	the space of n-times continuously differentiable functions on $[0, \infty)$
$P(A)$	the probability of event A
$P(A\vert B)$	the conditional probability of event A assuming event B
$P(A\vert \mathcal{F})$	the conditional probability of A with respect to a σ-algebra $\mathcal{F}$
$\widetilde{P}$	a martingale probability
$\mathcal{M}\big(S_n/B_n\big)$	the collection of all martingale probabilities
$E(X)$	the expectation of a random variable X
$V(X)$	the variance of a random variable X
$\mathcal{N}(m, \sigma^2)$	a Gaussian (normal) random variable with mean value m and variance σ^2

$E(X\|Y)$	the conditional expectation of a random variable X with respect to a random variable Y
$E(X\|\mathcal{F})$	the conditional expectation of a random variable X with respect to a σ-algebra $\mathcal{F}$
$Cov(X,Y)$	the covariance of X and Y
$(X)^+$	$:= \max\{X, 0\}$
$\mathbb{F}$	a filtration (information flow)
$\langle M, M \rangle$	the quadratic variation of a martingale M
$H * m_n$	a discrete stochastic integral
$(\varphi * w)_t$	a stochastic integral
$\varepsilon_n(U)$	a stochastic exponential
$\mathcal{E}_t(Y)$	a stochastic exponential
SF	the collection of all self-financing portfolios
$\mathcal{M}_0^N$	the collection of all stopping times