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CONVEX HARMONIC MAPPINGS ARE NOT NECESSARILY IN $h^{1/2}$

ALEXANDRU ALEMAN AND MARÍA J. MARTÍN

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Abstract. We construct convex harmonic mappings in the unit disk that do not belong to the harmonic Hardy space $h^{1/2}$. This provides a negative answer to a question raised by P. Duren.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. The theory of injective (univalent) complex-valued harmonic functions in $\mathbb{D}$ has received a lot of attention in recent years (see for example [6] and the bibliography therein). It turns out that the analogy to the classical theory of analytic univalent functions is far from obvious and leads to a number of challenging questions. An important class of univalent harmonic maps consists of maps with a convex range, which are somewhat better understood with the help of geometrical arguments. Nevertheless, this class is much wider than its analytic counterpart and there are a number of difficult problems related to its structure.

The present note is concerned with the growth of convex harmonic mappings, more precisely with their integral means defined for $p > 0$, $0 \le r < 1$, and h harmonic in $\mathbb{D}$, by

$$m_p(r,h) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |h(re^{it})|^p \, dt \right)^{1/p} .$$

The *harmonic Hardy space* h^p is the linear space of (complex-valued) harmonic functions in the unit disk such that

$$\sup_{0<r<1} m_p(r,h) < \infty \, .$$

The object of investigation is the best value of p such that all harmonic convex mappings on $\mathbb{D}$ belong to h^p, and in this context, our paper is the final step in a very interesting line of research which we shall briefly describe below.

Every complex harmonic function F in $\mathbb{D}$ has a unique *canonical representation* of the form $F = H + \overline{G}$, with H and G analytic in $\mathbb{D}$ and $G(0) = 0$. Since affine transformations do not affect the boundedness of integral means, we can restrict our attention to the class C_H^0 consisting of sense-preserving (positive Jacobian) convex

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harmonic mappings F in $\mathbb{D}$ which have a canonical representation $F = H + \overline{G}$ satisfying the normalizations $H(0) = G(0) = G'(0) = 1 - H'(0) = 0$. A beautiful result of Clunie and Sheil-Small [3, Theorem 5.7] states that if $F = H + \overline{G} \in C_H^0$ and $|\lambda| \le 1$, then the analytic functions $H + \lambda G$ are univalent in the unit disk. Hence, by a classical result ([5, Theorem 3.16]), it follows that $C_H^0 \subset h^p$ for all $p < 1/2$. More than that, this result cannot be improved. This follows from a simple inspection of the so-called *half-plane harmonic mapping* $L = H + \overline{G}$, defined for $z \in \mathbb{D}$,

$$(1) \qquad H(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) \quad \text{and} \quad G(z) = \frac{1}{2}\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right).$$

This function maps the unit disk one-to-one onto the half-plane $\mathbb{H} = \{z \in \mathbb{C}\colon Re\{z\} > -1/2\}$, and it plays an extremal role for several problems in C_H^0, such as coefficient bounds or covering theorems (see [1, 6]). Nevertheless, Nowak [9] observed that the half-plane mapping L belongs to $h^{1/2}$, although none of the analytic terms H, G in the canonical representation (1) belong to this space. Due to the extremal role of this function in the family C_H^0, it makes sense to conjecture that this phenomenon extends to the whole class, that is, $C_H^0 \subset h^{1/2}$ (see [6, Sec. 8.5]).

The conjecture is supported by previous results which actually reduce the problem to convex harmonic maps whose image is a half-plane. Indeed, Abu-Muhanna and Schober [1] proved that a function in C_H^0 whose range is not a strip nor a half-plane, must belong to h^1. Later, Grigoryan and Nowak [7] could handle the first case above, by showing that such functions belong to h^1 as well (see also [2,4]). This leaves us with functions in C_H^0 whose range is a half-plane, and using some standard transformations [4,10] we can restrict our attention to harmonic maps F satisfying $F(\mathbb{D}) = \{z \in \mathbb{C}\colon Re\{z\} > -1/2\}$. Unlike the analytic case, the class of such harmonic maps is fairly wide. They are described in [10] (see also [1,4,7]) by the formula

$$(2) \qquad F(z) = \int_{\mathbb{T}} f(z,\eta)\, d\mu(\eta)\,,$$

where $\mathbb{T}$ is the unit circle, μ is a regular Borel probability measure on $\mathbb{T}$ and the kernel f is given by

$$f(z,\eta) = \begin{cases} Re\left\{\frac{z}{1-z}\right\} + i\, Im\left\{\frac{z}{(1-z)^2}\right\}, & \eta = 1, \\ Re\left\{\frac{z}{1-z}\right\} + i\, \left(Im\left\{F_1(z,\eta)\right\} + Im\left\{F_2(z,\eta)\right\}\right), & \eta \neq 1, \end{cases}$$

with

$$(3) \qquad F_1(z,\eta) = \frac{2\eta}{(1-\eta)^2}\log\frac{1-z}{1-\eta z} \quad \text{and} \quad F_2(z,\eta) = \frac{1+\eta}{1-\eta}\frac{z}{1-z}.$$

Note that the function L given in (1) is obtained when μ is the Dirac measure at $\eta = 1$. In fact, it is shown in [7] that every function $f(z,\eta)$, $\eta \in \mathbb{T}$, belongs to $h^{1/2}$, a result which again supports the conjecture, but does not prove it since $h^{1/2}$ is not a locally convex space [11].

In this paper we show that the question whether $C_H^0 \subset h^{1/2}$ has a negative answer. We find a condition which must be satisfied by the measures in the representation formula (2) in order to generate functions in $h^{1/2}$, and then use this condition to construct a counterexample.

To state our main result we use the following notation: Given $z \in \mathbb{T} \setminus \{-1\}$, let $I_+(z)$ be the open arc on $\mathbb{T}$ with endpoints -1 and z that intersects the upper half-plane, and $I_-(z)$ the open arc on $\mathbb{T}$ with endpoints -1 and z that intersects the lower half-plane.

Theorem 1. *Let F be a function of the form (2). If $F \in h^{1/2}$, then both integrals*

$$\int_0^\pi \left(\int_{I_-(e^{-it})} |1-\eta|^{-2} \, d\mu(\eta) \right)^{\frac{1}{2}} dt \quad \text{and} \quad \int_{-\pi}^0 \left(\int_{I_+(e^{-it})} |1-\eta|^{-2} \, d\mu(\eta) \right)^{\frac{1}{2}} dt$$

are finite.

For certain measures μ the condition in the theorem is also sufficient in order to have $F \in h^{1/2}$. We believe that this is the case when

$$\int_{\mathbb{T}} |1-\eta|^{-1} d\mu(\eta) < \infty \,,$$

but we will not pursue this matter here. As mentioned above, the result enables us to construct harmonic mappings F of the form (2) which do not belong to $h^{1/2}$ in a fairly easy way, as the following corollary shows.

Corollary 1. *For $n \geq 2$, let $\eta_n = e^{i/n}$, let δ_n be the Dirac measure at η_n, and let*

$$\mu = K \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \delta_n \,,$$

with

$$K^{-1} = \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \,.$$

Then the corresponding convex harmonic map F given by (2) does not belong to $h^{1/2}$.

Proof. A simple computation shows that for $t < 0$ with $-t$ sufficiently small

$$\int_{I_+(e^{-it})} |1-\eta|^{-2} \, d\mu(\eta) \geq \sum_{n^{-1} > -t} \frac{1}{n \log^2 n} |1-\eta_n|^{-2} \geq c \sum_{n^{-1} > -t} \frac{n}{\log^2 n} \,,$$

for some absolute constant $c > 0$. Then a standard estimate yields

$$\int_{I_+(e^{-it})} |1-\eta|^{-2} \, d\mu(\eta) \geq c_1 \int_2^{-\frac{1}{t}} \frac{x}{\log^2 x} dx \geq \frac{c_2}{t^2 \log^2 |t|} \,,$$

where $c_1, \ c_2 > 0$ are absolute constants, so that

$$\int_{-\pi}^0 \left(\int_{I_+(e^{-it})} |1-\eta|^{-2} \, d\mu(\eta) \right)^{\frac{1}{2}} dt = \infty \,,$$

and the result follows by Theorem 1. ∎

2. Proof of the main result

The radial limits of a harmonic mapping F in the unit disk will be denoted by $\widetilde{F}$, that is,

$$\widetilde{F}(e^{it}) = \lim_{r\to 1^-} F(re^{it}) \,,$$

whenever this limit exists. By the result of Clunie and Sheil-Small [3] mentioned in the introduction, it follows that a convex harmonic mapping F has radial limits a.e., so that an application of Fatou's lemma gives

$$\|F\|_{\frac{1}{2}}^{\frac{1}{2}} \geq \int_{-\pi}^{\pi} |\widetilde{F}(e^{it})|^{1/2}\frac{dt}{2\pi} \geq \int_{-\pi}^{\pi} |Im\{\widetilde{F}(e^{it})\}|^{1/2}\frac{dt}{2\pi} \,. \tag{4}$$

The calculation of the boundary values of the imaginary part of extremal functions of the form (2) is somewhat delicate, and treated separately in the next lemma.

Lemma 1. *Let F be a function of the form (2). If $t \in (0, 2\pi)$ and e^{-it} is not an atom for the measure μ, then $Im\{f(e^{it}, \cdot)\}$ is μ-integrable and*

$$Im\{\widetilde{F}(e^{it})\} = \int_{\mathbb{T}} Im\{f(e^{it}, \eta)\}\, d\mu(\eta) \,.$$

Proof. For fixed $t \in (0, 2\pi)$, we can choose $\varepsilon > 0$ with $|e^{it} - 1| \geq \varepsilon$. Assume that $r > 1 - \frac{\varepsilon}{3(1+\varepsilon)}$. Then for $|\eta - 1| \geq \varepsilon/4$ we have

$$\begin{aligned} \big|Im\{f(re^{it}, \eta)\}\big| &\leq \frac{2}{|1-\eta|^2}\cdot\left|Arg\left\{\frac{1-re^{it}}{1-\eta re^{it}}\right\}\right| + \left|\frac{1+\eta}{1-\eta}\right|\cdot\left|\frac{re^{it}}{1-re^{it}}\right| \\ &\leq \frac{32\pi}{\varepsilon^2} + \frac{8}{\varepsilon}\cdot\frac{1}{r|1-e^{it}| - (1-r)} \\ &\leq \frac{32\pi + 12}{\varepsilon^2}\,. \end{aligned} \tag{5}$$

Moreover,

$$\big|Im\{f(re^{it}, 1)\}\big| \leq \left|\frac{re^{it}}{(1-re^{it})^2}\right| \leq \frac{9}{4\varepsilon^2}\,. \tag{6}$$

Finally, when $0 < |\eta - 1| < \varepsilon/4$ we have

$$\left|\frac{(\eta-1)re^{it}}{1-\eta re^{it}}\right| < \frac{\varepsilon}{4(|1-re^{it}| - |1-\eta|)} < \frac{3}{5}\,,$$

so that

$$\left|\log\frac{1-re^{it}}{1-\eta re^{it}} - \frac{(\eta-1)re^{it}}{1-\eta re^{it}}\right| = \left|\log\left(1 + \frac{(\eta-1)re^{it}}{1-\eta re^{it}}\right) - \frac{(\eta-1)re^{it}}{1-\eta re^{it}}\right| \leq \frac{C}{\varepsilon^2}|\eta-1|^2\,,$$

for some absolute constant $C > 0$. Thus for $0 < |\eta - 1| < \varepsilon/4$

$$\begin{aligned} &\big|Im\{f(re^{it}, \eta)\}\big| \leq \left|\frac{2\eta}{(1-\eta)^2}\,\log\frac{1-re^{it}}{1-\eta re^{it}} + \frac{1+\eta}{1-\eta}\frac{re^{it}}{1-re^{it}}\right| \\ &\leq \left|-\frac{2\eta re^{it}}{(1-\eta)(1-\eta re^{it})} + \frac{1+\eta}{1-\eta}\frac{re^{it}}{1-re^{it}}\right| + \frac{2}{|1-\eta|^2}\cdot\left|\log\frac{1-re^{it}}{1-\eta re^{it}} - \frac{(\eta-1)re^{it}}{1-\eta re^{it}}\right| \\ &\leq \frac{|1+\eta re^{it}|}{|(1-\eta re^{it})(1-re^{it})|} + \frac{2C}{\varepsilon^2}\,, \end{aligned}$$

and from the conditions satisfied by r, t, η, we obtain as above

$$\left|Im\{f(re^{it}, \eta)\}\right| \le \frac{C'}{\varepsilon^2}\,, \tag{7}$$

for some absolute constant $C' > 0$. It is also clear that

$$\lim_{r\to 1^-} f(re^{it}, \eta) = f(e^{it}, \eta)\,,$$

whenever $1 - \eta e^{it} \neq 0$, hence, if e^{-it} is not an atom of μ the above equality holds μ-a.e. The result follows by the estimates (5), (6), (7), and an application of the dominated convergence theorem. □

There is a further reduction based on the cancellation properties of the kernel Imf involved in these formulas, which is easier to see on the real line, or in the half-plane setting.

If $\mathbb{H} = \{z \in \mathbb{C}\colon Re\{z\} > -1/2\}$, then the unit disk is mapped conformally onto $\mathbb{H}$ by the map $w = z/(1-z)$, which extends to a diffeomorphism from $\mathbb{T}\setminus\{1\}$ onto $\partial\mathbb{H}$. This suggests the change of variables

$$-\frac{1}{2} + xi = \frac{e^{it}}{1-e^{it}}\,, \quad x \in \mathbb{R}\,.$$

If μ is a Borel probability measure on $\mathbb{T}$ and u is a bounded μ-measurable function on $\mathbb{T}$, then the change of variables given above yields the equality

$$\int_{\mathbb{T}\setminus\{1\}} u d\mu = \int_{\mathbb{R}} u\left(\frac{2xi-1}{2xi+1}\right) d\nu(x)\,, \tag{8}$$

where ν is the pull-back measure of $\mu|\mathbb{T}\setminus\{1\}$, and is defined on Borel sets $A \subset \mathbb{R}$ by the relation

$$\nu(A) = \mu(B)\,, \quad \text{where} \quad B = \{|\eta| = 1\colon\ -i\cdot\left(\frac{\eta}{1-\eta} + \frac{1}{2}\right) \in A\}\,.$$

In particular, from Lemma 1 we obtain

$$\begin{aligned} Im\{\widetilde{F}(e^{it})\} &= \int_{\mathbb{T}} Im\{f(e^{it}, \eta)\}\, d\mu(\eta) \\ &= Im\{f(e^{it}, 1)\}\mu(\{1\}) + \int_{\mathbb{R}} Im\left\{f\left(e^{it}, \frac{2yi-1}{2yi+1}\right)\right\} d\nu(y)\,, \end{aligned}$$

and by another application of (8) for the arclength measure it follows that

(9)

$$\begin{aligned} \int_{-\pi}^{\pi} |Im\{\widetilde{F}(e^{it})\}|^{1/2}\frac{dt}{2\pi} &\ge -C_0[\mu(\{1\})]^{1/2} \\ &\quad + \frac{2}{\pi}\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}} Im\left\{f\left(\frac{2xi-1}{2xi+1}, \frac{2yi-1}{2yi+1}\right)\right\} d\nu(y)\right|^{1/2} \frac{dx}{1+4x^2}\,, \end{aligned}$$

where $C_0 = \|Im\{f(\cdot, 1)\}\|_{\frac{1}{2}}^{\frac{1}{2}} < \infty$. The advantage of these new variables is that the kernel Imf becomes very simple. We have

$$\frac{2\eta}{(1-\eta)^2} = -\frac{1}{2} - 2y^2\,,\ \frac{1+\eta}{1-\eta} = 2yi\,,\ \frac{1+z}{1-z} = 2xi\,,\ \frac{1-z}{1-\eta z} = \frac{\frac{1}{2}+yi}{i(x+y)}\,.$$

Moreover, for complex numbers $a, b,\ b \neq 0$, with nonnegative real part,

$$Arg\frac{a}{b} = \int_{\tan(Arg b)}^{\tan(Arg a)} \frac{dt}{1+t^2}\,,$$

so that, if $-x$ is not an atom of ν, then

$$\text{(10)}\quad Im\left\{f\left(\frac{2xi-1}{2xi+1}, \frac{2yi-1}{2yi+1}\right)\right\} = \begin{cases} -y + \left(\frac{1}{2}+2y^2\right)\int_{2y}^{\infty}\frac{dt}{1+t^2}\,, & x+y \geq 0, \\ -y - \left(\frac{1}{2}+2y^2\right)\int_{-\infty}^{2y}\frac{dt}{1+t^2}\,, & x+y < 0, \end{cases}$$

ν-a.e. Some direct consequences of (10) are listed below.

Proposition 1. *Let $g(x, y)$ be the right hand side of* (10). *Then:*
(i) for every compact set $K \subset \mathbb{R}$, the function g is bounded on $\mathbb{R} \times K$,
(ii) g is bounded on the sets

$$\Omega_+ = \{(x, y):\ x \geq 0,\ y \geq 0\} \cup \{(x, y):\ x \geq 0,\ x + y < 0\}\,,$$
$$\Omega_- = \{(x, y):\ x \leq 0,\ y \leq 0\} \cup \{(x, y):\ x \leq 0,\ x + y \geq 0\}\,,$$

(iii) we have

$$\lim_{\substack{-x,y\to-\infty \\ x+y\geq 0}} \frac{g(x, y)}{|-\frac{1}{2}+iy|^2} = -\lim_{\substack{-x,y\to+\infty \\ x+y<0}} \frac{g(x, y)}{|-\frac{1}{2}+iy|^2} = 2\pi\,.$$

Proof. (i) is an obvious consequence of (10). (ii) Let $s_+ = \sup_{\Omega_+}|g|$ and let $((x_n, y_n))$ be a sequence in Ω_+ with

$$\lim_{n\to\infty} |g(x_n, y_n)| = s_+\,.$$

If (y_n) has a bounded subsequence, then the result follows by (i). If $y_n \to +\infty$, then $x_n + y_n > 0$ and from

$$\text{(11)}\qquad \frac{1}{1+t^2} = \frac{1}{t^2} + O\left(\frac{1}{t^4}\right),\ |t| \to \infty\,,$$

we see that

$$g(x_n, y_n) = \frac{1}{2}\int_{2y_n}^{\infty} \frac{dt}{1+t^2} + O\left(\frac{1}{y_n}\right),$$

which is impossible, because the right hand side of this equality converges to zero when $n \to \infty$, while $s_+ \geq g(0, 0) > 0$. Similarly, if $y_n \to -\infty$, (11) gives

$$g(x_n, y_n) = -\frac{1}{2}\int_{-\infty}^{2y_n} \frac{dt}{1+t^2} + O\left(\frac{1}{y_n}\right),$$

and the first part of (ii) follows. The proof of the second part is identical. (iii) is another direct consequence of (10) since

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi\,.$$

□

We can now turn to the proof of Theorem 1. Since the right hand side g of (10) satisfies

$$\int_{\mathbb{R}} g(x, y)d\nu(y) = \int_{\mathbb{R}} Im\left\{f\left(\frac{2xi-1}{2xi+1}, \frac{2yi-1}{2yi+1}\right)\right\} d\nu(y)$$

a.e. on $\mathbb{R}$, we can apply (4), (9) and (10) to obtain

$$\|F\|_{\frac{1}{2}}^{\frac{1}{2}} \geq -C_0[\mu(\{1\})]^{1/2} + \frac{2}{\pi}\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}} g(x,y)d\nu(y)\right|^{1/2}\frac{dx}{1+4x^2}\,.$$

Using the fact that the set of atoms of ν is at most countable, we can apply Proposition 1(ii) to conclude that

$$\int_0^{\infty}\left(\left|\int_{[0,\infty)} g(x,y)d\nu(y)\right|^{1/2} + \left|\int_{(-\infty,-x]} g(x,y)d\nu(y)\right|^{1/2}\right)\frac{dx}{1+4x^2} < \infty$$

and

$$\int_{-\infty}^{0}\left(\left|\int_{(-\infty,0]} g(x,y)d\nu(y)\right|^{1/2} + \left|\int_{[-x,\infty)} g(x,y)d\nu(y)\right|^{1/2}\right)\frac{dx}{1+4x^2} < \infty\,,$$

hence, there exists an absolute constant $C_1 > 0$ such that

$$\begin{aligned}
(12) \qquad \|F\|_{\frac{1}{2}}^{\frac{1}{2}} + C_1 &\geq \int_0^{\infty}\left|\int_{(-x,0)} g(x,y)d\nu(y)\right|^{1/2}\frac{dx}{1+4x^2} \\
&\quad + \int_{-\infty}^{0}\left|\int_{(0,-x)} g(x,y)d\nu(y)\right|^{1/2}\frac{dx}{1+4x^2} \\
&= \int_0^{\infty}\left(\int_{(-x,0)} |g(x,y)|d\nu(y)\right)^{1/2}\frac{dx}{1+4x^2} \\
&\quad + \int_{-\infty}^{0}\left(\int_{(0,-x)} |g(x,y)|d\nu(y)\right)^{1/2}\frac{dx}{1+4x^2}\,.
\end{aligned}$$

Now use Proposition 1(iii) to conclude that there is $A > 0$ such that

$$g(x,y) > 5\left|-\frac{1}{2} + yi\right|^2\,,$$

whenever $x > A,\ y < -A,\ x + y \geq 0$, and

$$g(x,y) < -5\left|-\frac{1}{2} + yi\right|^2\,,$$

when $x < -A,\ y > A,\ x + y < 0$. By Proposition 1(i) we have

$$\int_0^{\infty}\left(\int_{[-A,0)} |g(x,y)|d\nu(y)\right)^{1/2}\frac{dx}{1+4x^2} + \int_{-\infty}^{0}\left(\int_{(0,A]} |g(x,y)|d\nu(y)\right)^{1/2}\frac{dx}{1+4x^2} < \infty\,.$$

Then from (12) we obtain that there exists an absolute constant $C_2 > 0$ such that

$$\|F\|_{\frac{1}{2}}^{\frac{1}{2}} + C_2 \geq 5\int_A^{\infty} \left(\int_{(-x,-A)} \left| -\frac{1}{2} + yi \right|^2 d\nu(y) \right)^{1/2} \frac{dx}{1+4x^2}$$
$$+ 5\int_{-\infty}^{-A} \left(\int_{(A,-x)} \left| -\frac{1}{2} + yi \right|^2 d\nu(y) \right)^{1/2} \frac{dx}{1+4x^2},$$

and this clearly implies

$$\|F\|_{\frac{1}{2}}^{\frac{1}{2}} + C_3 \geq 5\int_0^{\infty} \left(\int_{(-x,0)} \left| -\frac{1}{2} + yi \right|^2 d\nu(y) \right)^{1/2} \frac{dx}{1+4x^2} \tag{13}$$
$$+ 5\int_{-\infty}^{0} \left(\int_{(0,-x)} \left| -\frac{1}{2} + yi \right|^2 d\nu(y) \right)^{1/2} \frac{dx}{1+4x^2}, \tag{14}$$

for some positive constant C_3. Then the result follows by the change of variable (8).

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Department of Mathematics, Lund University, P. O. Box 118, S-221 00 Lund, Sweden
E-mail address: `aleman@maths.lth.se`

Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland
E-mail address: `maria.martin@uef.fi`