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Journal of Functional Analysis 258 (2010) 67–98

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Resolvent estimates and decomposable extensions of generalized Cesàro operators

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Received 19 December 2008; accepted 9 October 2009

Available online 21 October 2009

Communicated by N. Kalton

Abstract

We determine the spectrum of generalized Cesàro operators with essentially rational symbols acting on various spaces of analytic functions, including Hardy spaces, weighted Bergman and Dirichlet spaces. Then we show that in all cases these operators are subdecomposable.

Keywords: Cesàro operator; Spaces of analytic functions; Decomposable extension

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the unit disc $\mathbb{D}$ and consider for $g \in H(\mathbb{D})$ the generalized Cesàro operator $\mathcal{C}_g$ defined by

$$\mathcal{C}_g f(z) = \frac{1}{z} \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}).$$

The classical Cesàro operator $\mathcal{C}$, which is usually defined on sequence spaces by its action on a sequence $\boldsymbol{x} = (x_0, x_1, \ldots)$:

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.10.006

$$(\mathcal{C}\boldsymbol{x})_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k,$$

is obtained for $g'(z) = \frac{1}{1-z}$.

The boundedness and compactness of such operators defined on various spaces of analytic functions, like Hardy spaces, weighted Bergman and Dirichlet spaces, or even Bloch-type spaces, have attracted a lot of attention (see for example, [6,4,7,12,16,24,30], or the survey papers [29,3] and the references therein). For the sake of completion we include a list of short definitions of those spaces which are relevant for this work.

For $0 < p < \infty$, the Hardy spaces H^p are defined by

$$H^p = \left\{ f \in H(\mathbb{D}),\ \|f\|_p^p = \lim_{r\to 1^-} \int_0^{2\pi} \left|f\left(re^{it}\right)\right|^p \frac{dt}{2\pi} = \int_0^{2\pi} \left|f\left(e^{it}\right)\right|^p \frac{dt}{2\pi} < \infty \right\},$$

while H^∞ is the space of all bounded analytic functions on the unit disc endowed with the supremum norm. The standard weighted Bergman spaces $L_a^{p,\alpha}$, $\alpha > -1$, $0 < p < \infty$ are defined by

$$L_a^{p,\alpha} = \left\{ f \in H(\mathbb{D}),\ \|f\|_{L_a^{p,\alpha}}^p = \int_{\mathbb{D}} \left|f(z)\right|^p \left(1-|z|\right)^\alpha dA(z) < \infty \right\}.$$

For the unweighted case $\alpha = 0$ we simply write $L_a^{p,0} = L_a^p$. The weighted Dirichlet spaces $D^{p,\alpha}$, $\alpha > p-1$, consist of those analytic functions in $\mathbb{D}$ whose derivative belongs to $L_a^{p,\alpha}$, that is,

$$D^{p,\alpha} = \left\{ f \in H(\mathbb{D}),\ \|f\|_{D^{p,\alpha}}^p = \left|f(0)\right|^p + \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|\right)^\alpha dA(z) < \infty \right\}.$$

Finally, we will also consider the well-known growth classes

$$A^{-\gamma} = \left\{ f \in H(\mathbb{D}):\ \|f\|_{-\gamma} = \sup_{z\in\mathbb{D}} \left(1-|z|\right)^\gamma \left|f(z)\right| < \infty \right\},$$

for $\gamma > 0$, as well as the closed subspace $A_0^{-\gamma}$, $\gamma > 0$ of $A^{-\gamma}$ defined by

$$A_0^{-\gamma} = \left\{ f \in A^{-\gamma}:\ \limsup_{|z|\to 1} \left(1-|z|\right)^\gamma \left|f(z)\right| = 0 \right\}.$$

The description of the spectrum of the operators considered above appears to be a delicate matter, even for the classical example $\mathcal{C}$. Siskakis (see [26,28]) developed a method based on composition semigroups to obtain the spectrum of $\mathcal{C}$ on Hardy spaces and unweighted Bergman spaces, and the result was extended to certain weighted Dirichlet spaces by Galanopoulos in [11]. Dahlner [8] used a different approach essentially based on Hardy-type inequalities and determined the fine spectrum of $\mathcal{C}$ acting on standard weighted Bergman spaces $L_a^{p,\alpha}$, $\alpha > -1$, $1 < p < \infty$. In all cases, it turns out that the spectrum of the Cesàro operator is a closed disc which consists of the origin and the points $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\exp(g/\lambda)$ does not belong to the space in question, where $g(z) = \log \frac{1}{1-z}$. Using Fredholm theory in a quite clever way, Young (see [31,32]) was able to show that this result extends to the case when the derivative of the symbol g is the sum of a rational function with poles on the unit circle and a bounded function. The spaces considered by Young are H^2 and L_a^2, but the result continues to hold for all standard weighted Bergman spaces $L_a^{p,\alpha}$, for $\alpha \geqslant -1$ and $1 < p < \infty$ (see [2]). In connection with the spectra of such operators, we should point out that for the closely related, non-normalized version of $\mathcal{C}_g$, defined by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D},\ g, f \in H(\mathbb{D}),$$

Pommerenke [22] observed that if this operator is bounded on H^2 (or on any other Banach space of analytic functions in the unit disc containing the constants), then its spectrum always contains the set of $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\exp(g/\lambda)$ does not belong to the space. However, very recently, it has been shown in [5] that for general symbols g the spectrum of T_g on a weighted Bergman space $L_a^{p,\alpha}$ can be much larger.

A further development in this direction has been inspired by the pioneering work of Kriete and Trutt [14] who proved in 1971 that the usual Cesàro operator $\mathcal{C}$ is subnormal on H^2. More recently, using the connection between this operator and certain composition semigroups, Miller, Miller and Smith [20] showed that $\mathcal{C}$ is subdecomposable on H^p, $1 < p < \infty$, that is, it has a decomposable extension, and then Miller and Miller [17,18] extended the result to the case of unweighted Bergman spaces L_a^p, for $p \geqslant 2$. Recall that an operator T on the Banach space X is called decomposable if for any pair of open sets $U; V \subset \mathbb{C}$ with $U \cup V = \mathbb{C}$ there exist T-invariant subspaces $M, N \subset X$ such that $\sigma(T|M) \subset U$, $\sigma(T|N) \subset V$ and $M + N = X$. The semigroup technique used in the above papers fails to work for other values of the parameter p. However, the approach developed by Dahlner [8] yields the same result for $\mathcal{C}$ acting on all standard weighted Bergman spaces $L_a^{p,\alpha}$, with $\alpha > -1$, and $1 < p < \infty$. Finally, the nonreflexive case $p = 1$ could not be handled with any of the above methods. The question whether $\mathcal{C}$ on H^1 (or $L_a^{1,\alpha}$) is subdecomposable was answered affirmatively by the second author in [21]. Actually, all results mentioned here state that when acting on the spaces in question, $\mathcal{C}$ has Bishop's property (β). The fact that this property characterizes subdecomposability has been proved by Albrecht and Eschmeier [1] (see also [10]).

Much less is known about the corresponding questions for generalized Cesàro operators. In [19] Miller, Miller and Neumann proved that $\mathcal{C}_g$ has Bishop's property (β) when $g'(z) = \frac{1}{1-z^2}$, $g'(z) = \frac{z^n}{1-z}$, or when $(1-z)g'(z)$ extends analytically in a larger disc and has positive real part in $\mathbb{D}$. Moreover, it is pointed out in [2] that their technique applies to the corresponding operators considered on the unweighted Bergman spaces.

In this paper we consider generalized Cesàro operators with essentially rational symbols, that is, $g' = r + h$, where r is a rational function with simple poles on the unit circle and h is a smooth function, $h''' \in H^\infty$. We give a unified approach to the description of the spectrum of such operators and estimate their resolvent. As a consequence we prove that these generalized Cesàro operators are subdecomposable on all Banach spaces of analytic functions listed in this introduction. Our method relies on certain common properties of these spaces which are briefly discussed below.

(1) Composition with conformal automorphisms of the disc $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ induces bounded operators on these spaces, but they not necessarily uniformly bounded in $a \in \mathbb{D}$. However, there exists a number $\gamma > 0$ depending only on the space such that the weighted composition operators $C^{\gamma}_{\psi_a}$ given by

$$C^{\gamma}_{\psi_a} f = \left(\psi_a'\right)^{\gamma} f \circ \psi_a$$

are uniformly bounded in $a \in \mathbb{D}$.

(2) Composition operators $C_\phi f = f \circ \phi$ induced by polynomials of degree one $\phi : \mathbb{D} \to \mathbb{D}$ are bounded on these spaces and satisfy an estimate of the form

$$\|C_\phi\|_X \leqslant \frac{c}{(1 - |\phi(0)|)^{\gamma}},$$

with the same constant $\gamma > 0$ as in (1). For Hardy and Bergman spaces as well as for the growth classes considered here, the estimate holds for arbitrary analytic self-maps of the disc. This is no longer true for Dirichlet spaces but the simple maps needed for our purposes do satisfy the estimate. A proof will be given in Section 3.

(3) The operator of multiplication by the independent variable $M_\xi f(z) = zf(z)$ is bounded and bounded below all spaces considered here and its spectrum equals the closed unit disc.

(4) The polynomials are dense in all spaces listed above, except the growth classes $A^{-\gamma}$, and all of them contain the functions $z \to \log(1 - \bar{b}z)$ for all $b \in \partial\mathbb{D}$. The results for $A^{-\gamma}$ will be then deduced with help of $A_0^{-\gamma}$.

(5) The norm on these spaces can be localized with help of composition operators. This is a more technical fact which is needed only to deal with the case when the derivative of the symbol has several poles. It can be expressed as follows.

Given distinct points $b_1, \ldots, b_n \in \partial\mathbb{D}$, there exists a finite set $\mathcal{F}$ contained in the set $\mathfrak{F}$ of Riemann maps from $\mathbb{D}$ onto the interior of C^5-Jordan curves contained in $\overline{\mathbb{D}}$ which fix the origin, such that:

(i) If $\phi \in \mathcal{F}$ then $\overline{\phi(\mathbb{D})}$ contains exactly one of the points b_j, $1 \leqslant j \leqslant n$, and the composition operator C_ϕ is bounded on X.
(ii) If $f \in H(\mathbb{D})$ satisfies $f \circ \phi \in X$ for all $\phi \in \mathcal{F}$ then $f \in X$.

These are the only properties needed in our proofs and for this reason our results hold for any Banach space of analytic functions where (1)–(5) are fulfilled. Therefore we have stated our theorems in the general context. We prove that generalized Cesàro operators with essentially rational symbols are bounded on such a space and describe their fine spectrum in our main theorem, Theorem 5.1. In particular, it turns out that the spectrum of such an operator equals a finite union of closed discs having the origin on their boundary and determined by the poles of the rational part of g' and its residua. Moreover, the Fredholm index is defined on the complement of the boundaries of these discs and equals the sum of their characteristic functions. Finally, there is an operator-valued function defined a.e. on $\mathbb{C}$ whose value at λ is a left-inverse of $\lambda I - \mathcal{C}_g$ and whose operator norm is locally integrable on $\mathbb{C} \setminus \{0\}$. This last result easily implies that $\mathcal{C}_g$ has Bishop's property (β), and hence it is subdecomposable.

The paper is organized as follows. In Section 2 we derive a formal expression for the resolvent of $\mathcal{C}_g$ which is then used to obtain information about the range of $\lambda I - \mathcal{C}_g$ when g has poles on the boundary. In Section 3 we explore the assumptions (1)–(3) and develop the basic tools needed for our main estimate which could be seen as a Hardy-type inequality for such Banach spaces of analytic functions. This technical result together with its proof are presented in Section 4. In Section 5 we gather the information obtained in the previous sections and give the proof of the main result mentioned above.

2. General form of the resolvent

The present section is devoted to some preliminary considerations concerning the resolvent of the generalized Cesàro operator

$$\mathcal{C}_g f(z) = \frac{1}{z} \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D}, \tag{2.1}$$

where g is a fixed analytic function in $\mathbb{D}$ with $g(0) = 0$. We consider first $\mathcal{C}_g$ as an operator on the space $H(\mathbb{D})$ of all analytic functions in $\mathbb{D}$ and derive a formula for its resolvent operator, that is, the solution f of the resolvent equation

$$\lambda f - \mathcal{C}_g f = h, \quad h \in H(\mathbb{D}). \tag{2.2}$$

For computational purposes it is convenient to introduce the analytic function u in $\mathbb{D} \setminus (-1, 0]$ which satisfies the equality

$$\frac{z u'(z)}{u(z)} = g'(z).$$

A simple computation shows that

$$u(z) = z^{g'(0)} \exp\left(\int_0^z \frac{g'(\zeta) - g'(0)}{\zeta}\, d\zeta \right), \quad z \in \mathbb{D} \setminus (-1, 0], \tag{2.3}$$

where for the possibly non-integer power above we choose the principal branch, that is, $1^{g'(0)} = 1$. Note that the function u^α extends analytically to $\mathbb{D}$ if and only if $\alpha g'(0)$ is a non-negative integer. Another direct computation shows that if $h = 0$ Eq. (2.2) has a nonzero solution if and only if $\lambda = \frac{g'(0)}{n}$, $n \in \mathbb{Z}^+$. In this case all solutions have the form

$$f(z) = c \frac{u^{\frac{1}{\lambda}}(z)}{z}, \quad c \in \mathbb{C},\ z \in \mathbb{D}.$$

Given an analytic function h in $\mathbb{D}$ we shall denote throughout in what follows by $m_0(h)$ the multiplicity of its zero at the origin ($m_0(h) = 0$ if $h(0) \neq 0$). Also, by ξ we denote the identity function on $\mathbb{D}$, that is, $\xi(z) = z$.

Proposition 2.1. *Let* $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) *If* $g'(0) = 0$, *or* $m_0(h) > Re\,\frac{g'(0)}{\lambda} - 1$ *then Eq.* (2.2) *has the analytic solution* f *in* $\mathbb{D}$ *given by*

$$f(z) = R_\lambda h(z) = \frac{h(z)}{\lambda} + \frac{u(z)^{\frac{1}{\lambda}}}{\lambda^2 z} \int_0^z u(\zeta)^{-\frac{1}{\lambda}} g'(\zeta) h(\zeta)\, d\zeta.$$

(ii) *If* $g'(0) \neq 0$ *and* $Re\,\frac{g'(0)}{\lambda} - 1 \geqslant m_0(h)$, *let* m *be an arbitrary positive integer with* $m > Re\,\frac{g'(0)}{\lambda} - 1$ *if* $\lambda \neq \frac{g'(0)}{n}$, $n \in \mathbb{Z}^+$, *and if* $\lambda = \frac{g'(0)}{n}$ *for some* $n \in \mathbb{Z}^+$, *let* $m = n$. *Then* (2.2) *has the solution*

$$f(z) = \sum_{k=0}^{m-1} \beta_k(h,\lambda)\xi^k + R_\lambda\left(h - \sum_{k=0}^{m-1} \beta_k(h,\lambda)(\lambda I - \mathcal{C}_g)\xi^k\right),$$

where the coefficients $\beta_k(h,\lambda)$ *are the uniquely determined solutions of the linear system*

$$h^{(j)}(0) = \sum_{k=0}^{m-1} \big[(\lambda I - \mathcal{C}_g)\xi^k\big]^{(j)}(0) x_k, \quad 0 \leqslant j \leqslant m-1. \tag{2.4}$$

(iii) *The solution given above is unique if* $\lambda \neq \frac{g'(0)}{n}$, $n \in \mathbb{Z}^+$, *and if* $\lambda = \frac{g'(0)}{n}$ *for some* $n \in \mathbb{Z}^+$ *then all solutions have the form* $f + cu^{\frac{1}{\lambda}}$, $c \in \mathbb{C}$.

Proof. (i) We multiply by the independent variable and differentiate both sides of (2.2) to obtain

$$\lambda z f'(z) + \lambda f(z) - f(z) g'(z) = \frac{d}{dz}\big(zh(z)\big)$$

or equivalently,

$$f'(z) + \left(\frac{1}{z} - \frac{g'(z)}{\lambda z}\right) f(z) = \frac{1}{\lambda z} \frac{d}{dz}\big(zh(z)\big).$$

We then use the integrating factor $z \to z u(z)^{-\frac{1}{\lambda}}$ to get

$$\frac{d}{dz}\big(f(z) z u(z)^{-\frac{1}{\lambda}}\big) = \frac{1}{\lambda} u(z)^{-\frac{1}{\lambda}} \frac{d}{dz}\big(zh(z)\big) \tag{2.5}$$

for all $z \in \mathbb{D} \setminus (-1, 0]$. Using our assumption, we find the particular solution

$$f(z) = \frac{u(z)^{\frac{1}{\lambda}}}{\lambda} \int_0^1 u(tz)^{-1/\lambda}\big(tzh'(tz) + h(tz)\big)\, dt, \tag{2.6}$$

since the integral on the right is convergent in this case. It is also easily seen that the right-hand side defines an analytic function in the whole disc, that is, we have

$$f(z)=\frac{u(z)^{\frac{1}{\lambda}}}{\lambda z}\int_0^z u(\zeta)^{-1/\lambda}\big(\zeta h'(\zeta)+h(\zeta)\big)\,d\zeta,\quad z\in\mathbb{D},$$

because the nonanalytic powers of z cancel out. Recall that $\frac{zu'(z)}{u(z)}=g'(z)$, integrate by parts and use again the assumption to obtain the equality in (i).

(ii) Write

$$(\lambda I-\mathcal{C}_g)\xi^k(z)=z^k\left(\lambda-\frac{g'(0)}{k+1}\right)G_k(z),\quad k\geqslant 0,$$

with $G_k\in H(\mathbb{D})$, $G_k(0)=1$. If m is a nonnegative integer as in the statement, then the matrix with entries

$$\big[(\lambda I-\mathcal{C}_g)\xi^k\big]^{(j)}(0),\quad 0\leqslant k,j\leqslant m-1,$$

is lower triangular with nonzero diagonal entries $k!(\lambda-\frac{g'(0)}{k+1})$. Thus the linear system given in the statement has a unique solution

$$\big(\beta_0(h,\lambda),\dots,\beta_{m-1}(h,\lambda)\big)^t\in\mathbb{C}^m$$

and clearly, the function

$$h_1=h-\sum_{k=0}^{m-1}\beta_k(h,\lambda)(\lambda I-\mathcal{C}_g)\xi^k$$

has a zero of order m at the origin. Then part (i) applies to h_1, that is,

$$\lambda R_\lambda h_1-\mathcal{C}_g R_\lambda h_1=h_1,$$

which shows that f is a solution of (2.2).

(iii) follows by linearity. □

We want to estimate the coefficients $\beta_k(h,\lambda)$ given in part (ii) above. It turns out that this can be done in a fairly general context. As usual, by Banach spaces of analytic functions in $\mathbb{D}$ we mean Banach spaces continuously contained in the locally convex space $H(\mathbb{D})$. Note that if X_1, X_2 are such Banach spaces with $X_1\subset X_2$, the inclusion map from X_1 into X_2 is automatically continuous by the closed graph theorem.

Lemma 2.1. *Let X be a Banach space of analytic functions in $\mathbb{D}$ which contains the polynomials and is contained in the growth class $A^{-\gamma}$ for some $\gamma>0$ and assume that $\mathcal{C}_g$ is bounded on X. Given $\lambda\in\mathbb{C}\setminus\{0\}$, let m be an arbitrary positive integer if $\lambda\neq\frac{g'(0)}{n}$, $n\in\mathbb{Z}^+$, or $m=n$ if $\lambda=\frac{g'(0)}{n}$ for some $n\in\mathbb{Z}^+$. Then for $h\in X$, the solutions $\beta_k(h,\lambda)$, $0\leqslant k\leqslant m-1$ of the linear system* (2.4) *define continuous linear functionals on X and there exists a constant $c_0 > 0$ depending only on X such that*

$$\|\beta_k(\cdot,\lambda)\| \leqslant c_0^k\big(|\lambda| + \|\mathcal{C}_g\|\big)^k\big((k+1)!\big)^\gamma \prod_{j=0}^{k-1} \frac{\max\{1, \|\xi^j\|_X\}}{|\lambda - \frac{g'(0)}{k+1}|}.$$

Proof. The same argument as in the proof of Proposition 2.1 shows that (2.4) has a unique solution. The values $\beta_k(h,\lambda)$, $k \geqslant 0$ can be computed in the following way. Set $g_k = (\lambda I - \mathcal{C}_g)\xi^k$, $h_{-1} = h$ and define h_k, $k \geqslant 0$ inductively by

$$h_k = h_{k-1} - \frac{h_{k-1}^{(k)}(0)}{g_k^{(k)}(0)} g_k = h_{k-1} - \frac{h_{k-1}^{(k)}(0)}{k!(\lambda - \frac{g'(0)}{k+1})} g_k.$$

Clearly, for m as in the statement

$$h_{m-1} = h - \sum_{k=0}^{m-1} \frac{h_{k-1}^{(k)}(0)}{k!(\lambda - \frac{g'(0)}{k+1})} g_k,$$

has a zero of order m at the origin, hence,

$$\beta_k(h,\lambda) = \frac{h_{k-1}^{(k)}(0)}{k!(\lambda - \frac{g'(0)}{k+1})}, \quad k \geqslant 0.$$

For functions $f \in A^{-\gamma}$ we have the standard Cauchy estimate on the circle centered at the origin and of radius $r_k = 1 - \frac{1}{k+1}$,

$$\big|f^{(k)}(0)\big| \leqslant k! r_k^{-k}(1-r_k)^{-\gamma}\|f\|_{-\gamma} \leqslant e k!(k+1)^\gamma \|f\|_{-\gamma}.$$

Using this inequality together with the fact that X is continuously contained in $A^{-\gamma}$ we obtain

$$\big|\beta_k(h,\lambda)\big| \leqslant e(k+1)^\gamma \frac{\|h_{k-1}\|_{-\gamma}}{|\lambda - \frac{g'(0)}{k+1}|} \leqslant c(k+1)^\gamma \frac{\|h_{k-1}\|_X}{|\lambda - \frac{g'(0)}{k+1}|}, \quad k \geqslant 0, \tag{2.7}$$

where $c > 0$ is a constant depending only on X. From the definition of h_k we have that

$$\begin{aligned}\|h_k\|_X &\leqslant \|h_{k-1}\|_X + c(k+1)^\gamma \frac{\|h_{k-1}\|_X}{|\lambda - \frac{g'(0)}{k+1}|}\big(|\lambda| + \|\mathcal{C}_g\|\big)\big\|\xi^k\big\|_X \\ &= \|h_{k-1}\|_X\left(1 + \frac{c(k+1)^\gamma}{|\lambda - \frac{g'(0)}{k+1}|}\big(|\lambda| + \|\mathcal{C}_g\|\big)\big\|\xi^k\big\|_X\right),\end{aligned}$$

and by iterating this inequality we obtain for $k \geqslant 1$

$$\|h_{k-1}\|_X \leqslant \|h\|_X \prod_{j=0}^{k-1}\left(1 + \frac{c(j+1)^\gamma}{|\lambda - \frac{g'(0)}{j+1}|}\big(|\lambda| + \|\mathcal{C}_g\|\big)\big\|\xi^j\big\|_X\right). \tag{2.8}$$

Finally, note that if $g'(0) = 0$, then

$$\inf_{\lambda} \frac{|\lambda| + \|\mathcal{C}_g\|}{|\lambda|} = 1$$

and if $g'(0) \neq 0$,

$$\inf_{j,\lambda} \frac{|\lambda| + \|\mathcal{C}_g\|}{|\lambda - \frac{g'(0)}{j+1}|} = \min\left\{1, \frac{(j+1)\|\mathcal{C}_g\|}{|g'(0)|}\right\} \geqslant \min\{1, c_1(X)^{-1}\},$$

where

$$c_1(X) = \sup_{\|f\|_X = 1} \left|f(0)\right|.$$

Then the result follows from (2.7) and (2.8). □

These general considerations can be applied in order to study the spectrum and resolvent of $\mathcal{C}_g$ on such spaces. A preliminary result is given below. The proof follows directly from Proposition 2.1 and the closed graph theorem, so that it will be omitted. Given a space X as above and $m \in \mathbb{Z}^+$, we denote by X_m the subspace of X consisting of functions with a zero of order at least m at the origin. Note that X_m is closed in X.

Corollary 2.1. *Let X be a Banach space of analytic functions in $\mathbb{D}$ which contains the polynomials, is contained in $A^{-\gamma}$ for some $\gamma > 0$ and assume that $\mathcal{C}_g$ is bounded on X.*

(i) *If $g'(0) = 0$ the point spectrum of $\mathcal{C}_g|X$ is void and if $g'(0) \neq 0$ we have*

$$\sigma_p(\mathcal{C}_g|X) = \left\{\frac{g'(0)}{n}\colon n \in \mathbb{Z}^+,\ u^{\frac{n}{g'(0)}} \in X\right\}.$$

(ii) *If $Y^\lambda = (\lambda I - \mathcal{C}_g)X$ is closed for some $\lambda \neq 0, \frac{g'(0)}{n}$, $n \in \mathbb{Z}^+$, then for every positive integer $m > Re\, \frac{g'(0)}{\lambda} - 1$, the operator R_λ defined in Proposition 2.1(i) is a bounded operator from Y^λ_m into X_m. If $r_{\lambda,m}$ denotes its norm, then the left inverse $T(\lambda) : Y^\lambda \to X$ of $\lambda I - \mathcal{C}_g$ with*

$$\left\|T(\lambda)\right\| \leqslant r_{\lambda,m} + \left(1 + r_{\lambda,m}\|\lambda I - \mathcal{C}_g\|\right) \sum_{k=0}^{m-1} \left\|\beta_k(\cdot,\lambda)\right\| \left\|\xi^k\right\|.$$

Our next application concerns symbols g with a special type of singularity on the unit circle.

Proposition 2.2. *Assume that there exist constants $a, b \in \mathbb{C} \setminus \{0\}$ with $|b| = 1$ such that $g'(z) - a(1 - \bar{b}z)^{-1}$ has a finite limit at b. Let X be a Banach space of analytic functions in $\mathbb{D}$ which is contained in $A^{-\gamma}$ for some $\gamma > 0$. Then for $\lambda \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{Z}^+$ with*

$$Re\, \frac{a}{\lambda} > \gamma, \qquad m > Re\, \frac{g'(0)}{\lambda} - 1,$$

we have that

$$y_{b,\lambda}(h) = \lim_{r\to 1} \int_0^{rb} u(\zeta)^{-\frac{1}{\lambda}} g'(\zeta) h(\zeta)\, d\zeta,$$

defines a continuous linear functional on X_m, which is nonzero if X contains the polynomials. If $\mathcal{C}_g$ is bounded on X then

$$(\lambda I - \mathcal{C}_g)X_m = (\lambda I - \mathcal{C}_g)X \cap X_m \subset \ker y_{b,\lambda}.$$

Proof. It is a simple matter to verify that the assumption on g implies that the function $z \to (1-\bar{b}z)^a u(z)$ is bounded and bounded away from zero near b. From the fact that X is continuously contained in $A^{-\gamma}$ it follows that on the line segment from the origin to b the integrand $u^{-\frac{1}{\lambda}} g'h$ satisfies an inequality of the form

$$\left|u^{-\frac{1}{\lambda}} g'h(tb)\right| \leqslant c(1-t)^{Re\frac{a}{\lambda}-\gamma-1}\|h\|_X,$$

for some fixed constant $c > 0$ and all $t \in [0,1]$. Then by the dominated convergence theorem we obtain that $y_{b,\lambda}$ is a continuous linear functional on X

$$y_{b,\lambda}(h) = \int_0^{b} u(\zeta)^{-\frac{1}{\lambda}} g'(\zeta) h(\zeta)\, d\zeta.$$

Also, by the above argument, we have that $u(tb)^{-\frac{1}{\lambda}} g'(tb)h(tb)\,dt$ is a finite nonzero measure on $[0,1]$ which cannot annihilate all polynomials, hence $y_{b,\lambda} \neq 0$ if X contains these functions. It remains to verify the equalities in the statement. The first one is trivial. By Proposition 2.1(i) we have that if $h \in (\lambda I - \mathcal{C}_g)X \cap X_m$ then $R_\lambda h + cu^{\frac{1}{\lambda}}\xi^{-1} \in X$ for some complex constant c which is nonzero only if $\lambda = \frac{g'(0)}{n}$ for some $n \in \mathbb{Z}^+$. Since X is continuously contained in $A_0^{-\gamma}$ this implies that

$$\lim_{r\to 1}(1-r)^\gamma \left(R_\lambda h(rb) + c\frac{u^{\frac{1}{\lambda}}(rb)}{rb}\right) = 0.$$

We also have $h \in A_0^{-\gamma}$ which yields

$$\lim_{r\to 1}(1-r)^\gamma \left(\frac{u^{\frac{1}{\lambda}}(rb)}{\lambda^2 rb}\int_0^{rb} u(\zeta)^{-\frac{1}{\lambda}} g'(\zeta)h(\zeta)\,d\zeta + c\frac{u^{\frac{1}{\lambda}}(rb)}{rb}\right) = 0.$$

From the fact that $z \to (1-\bar{b}z)^a u(z)$ has a finite nonzero limit at b, and $Re\frac{a}{\lambda} > \gamma$, we obtain that $c = 0$ and

$$\lim_{r\to 1}\int_0^{rb} u(\zeta)^{-\frac{1}{\lambda}} g'(\zeta)h(\zeta)\,d\zeta = 0. \qquad \square$$

3. Weighted conformal invariance and composition operators

In this section we shall consider Banach spaces of analytic functions which satisfy some additional properties related to composition and weighted composition operators. In order to motivate our assumptions, let us recall two common properties of such operators on Hardy and weighted Bergman spaces.

(1) These spaces are invariant under composition with the conformal automorphisms of $\mathbb{D}$ given by $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$, where $a \in \mathbb{D}$ is fixed, but the norms of the composition operators induced by these maps $C_{\psi_a} f = f \circ \psi_a$ are not uniformly bounded in $a \in \mathbb{D}$. However, for a suitable choice of $\gamma > 0$ the weighted composition operators $C_{\psi_a}^{\gamma}$ defined by

$$C_{\psi_a}^{\gamma} f = (\psi_a')^{\gamma} f \circ \psi_a \tag{3.9}$$

become uniformly bounded (actually they are isometries) on the spaces in question. Using a change of variable, one can easily verify from the definition that $\gamma = \frac{1}{p}$ for H^p, and $\gamma = \frac{\alpha+2}{p}$ for $L_a^{p,\alpha}$.

(2) Given any analytic self-map ϕ of the unit disc the composition operator C_ϕ defined by $C_\phi f = f \circ \phi$ is bounded on these spaces and its norm is dominated $(1 - |\phi(0)|)^{-\gamma}$, where γ happens to be the same parameter as in (1) (see for example [25]).

Motivated by these facts we consider Banach spaces of analytic functions in $\mathbb{D}$ which satisfy the weighted conformal invariance property described in (1), as well as a weaker form of (2). More precisely, our Banach spaces X are continuously contained in the locally convex space $H(\mathbb{D})$ and have the following properties depending on a fixed number $\gamma > 0$.

(A1) *The weighted composition operators* $C_{\psi_a}^{\gamma}$ *given by* (3.9) *are uniformly bounded on* X.
(A2) *For every function* $\phi : \mathbb{D} \to \mathbb{D}$ *of the form* $\phi(z) = \rho z + \lambda$, $\lambda, \rho \in \mathbb{C}$, *the composition operator* C_ϕ *is bounded on* X *and there exists a constant* $c > 0$ *such that*

$$\|C_\phi\| \leqslant \frac{c}{(1 - |\phi(0)|)^{\gamma}}.$$

Besides the Hardy and weighted Bergman spaces there are other classical Banach spaces of analytic functions in the unit disc which satisfy these two assumptions.

The simplest additional examples are the Banach spaces $A^{-\gamma}$, $A_0^{-\gamma}$, $\gamma > 0$ which satisfy (A1) and (A2) with the same parameter γ. The verification of this fact is based on a straightforward application of the Schwarz–Pick lemma. For example, if $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and $f \in A^{-\gamma}$, we have

$$\left|\phi'(z)\right|^{\gamma} \left|f\left(\phi(z)\right)\right| \leqslant \|f\|_{-\gamma} \frac{|\phi'(z)|^{\gamma}}{(1 - |\phi(z)|)^{\gamma}} \leqslant \|f\|_{-\gamma} \frac{2^{\gamma}}{(1 - |z|^2)^{\gamma}},$$

and

$$\begin{aligned}\left|f\left(\phi(z)\right)\right| &\leqslant \|f\|_{-\gamma} \frac{2^{\gamma}}{(1 - |\phi(z)|^2)^{\gamma}} \leqslant \|f\|_{-\gamma} \frac{2^{\gamma}(1 - |\phi(0)|^2)^{\gamma}}{(1 - |z|^2)^{\gamma} |1 - \overline{\phi(0)}\phi(z)|^{2\gamma}} \\ &\leqslant \|f\|_{-\gamma} \frac{4^{\gamma}}{(1 - |\phi(0)|)^{\gamma}(1 - |z|^2)^{\gamma}}.\end{aligned}$$

It is also easy to show that $A_0^{-\gamma}$ satisfies (A1) and (A2). In the case of weighted Dirichlet spaces, the situation is more complicated. The proposition below can be deduced from a recent result obtained in [5].

Proposition 3.1. *Let $p \geqslant 1$, $\alpha > -1$ with $\alpha \leqslant p-1$. Then the weighted Dirichlet space $D^{p,\alpha}$ satisfies* (A1) *for some $\gamma > 0$ if and only if $\alpha > p-2$. In this case, $\gamma = \frac{\alpha+2}{p} - 1$ and $D^{p,\alpha}$ satisfies* (A2) *as well.*

Proof. Note that

$$\left(C_{\psi_a}^{\gamma} 1\right)' = \left(\left(\psi_a'\right)^{\gamma}\right)' = \frac{(1-|a|^2)^{\gamma}}{(1-\bar{a}\xi)^{2\gamma}}$$

and use the standard estimates for weighted Bergman space norms (see [13]) to conclude that $C_{\psi_a}^{\gamma} 1$ are uniformly bounded in $D^{p,\alpha}$, if and only if $\gamma \leqslant \frac{\alpha+2}{p} - 1$. If this inequality is strict then

$$\begin{aligned} &\sup\left\{\left|C_{\psi_a}^{\gamma} f(0)\right|:\ f \in D^{p,\alpha},\ \|f\| \leqslant 1\right\} \\ &\quad = \left(1-|a|^2\right)^{\gamma} \sup\left\{\left|f(a)\right|:\ f \in D^{p,\alpha},\ \|f\| \leqslant 1\right\} \to \infty \end{aligned}$$

when $|a| \to 1$, and from the estimate

$$\left|f(a)\right| = O\left(\left(1-|a|\right)^{\frac{\alpha+2}{p}-1}\right), \quad f \in D^{p,\alpha},\ |a| \to 1, \tag{3.10}$$

we see that the operators $C_{\psi_a}^{\gamma}$ cannot be uniformly bounded for $a \in \mathbb{D}$. Now assume that $\gamma = \frac{\alpha+2}{p} - 1 > 0$ and note from above that

$$\sup\left\{\left|C_{\psi_a}^{\gamma} f(0)\right|:\ f \in D^{p,\alpha},\ \|f\| \leqslant 1,\ a \in \mathbb{D}\right\} < \infty.$$

By the weighted conformal invariance of the space $L_a^{p,\alpha}$, we have that

$$\begin{aligned} &\int_{\mathbb{D}} \left|\left(C_{\psi_a}^{\gamma} f\right)'(z)\right|^p \left(1-|z|^2\right)^{\alpha} dA(z) \\ &\quad = \int_{\mathbb{D}} \left|f'(z) + \gamma\left(\psi_a''\left(\psi_a'\right)^{-2}\right)\left(\psi_a(z)\right) f(z)\right|^p \left(1-|z|^2\right)^{\alpha} dA(z) \\ &\quad = \int_{\mathbb{D}} \left|f'(z) + \gamma(1-\bar{a}z)^{-1} f(z)\right|^p \left(1-|z|^2\right)^{\alpha} dA(z). \end{aligned}$$

Using the rotational invariance of $D^{p,\alpha}$, it is easy to verify that the last integral on the right stays bounded when $a \in \mathbb{D}$ and $f \in D^{p,\alpha}$ with $\|f\| \leqslant 1$, if and only if

$$\sup_{\substack{f \in D^{p,\alpha} \\ \|f\| \leqslant 1}} \int_{\mathbb{D}} \frac{|f(z)|^p}{|1-z|^p} \left(1-|z|^2\right)^{\alpha} dA(z) < \infty.$$

Now a special case of Theorems 5.1 and 5.2 in [5] yields the inequality

$$\int_{\mathbb{D}} \frac{|f(z)|^p}{|1-z|^p}\left(1-|z|^2\right)^\alpha dA(z) \leqslant c\left(\left|f(0)\right|^p + \int_{\mathbb{D}} \frac{|f'(z)|^p}{|1-z|^p}\left(1-|z|^2\right)^{\alpha+p} dA(z)\right)$$
$$\leqslant 2^p c\|f\|^p,$$

which gives the desired result. The verification of (A2) with this value of the parameter γ follows directly from the corresponding result for $L_a^{p,\alpha}$ together with the pointwise estimate (3.10). If $\phi:\mathbb{D}\to\mathbb{D}$ has the form $\phi(z)=\rho z+\lambda$ then $|\rho|+|\lambda|\leqslant 1$ and

$$\|f\circ\phi\|^p = \left|f(\lambda)\right|^p + |\rho|^p\int_{\mathbb{D}}\left|f'(\rho z+\lambda)\right|^p\left(1-|z|^2\right)^\alpha dA(z)$$
$$\leqslant \frac{c_1}{(1-|\lambda|)^{\frac{\alpha+2}{p}-1}}\|f\| + \frac{c_2|\rho|^p}{(1-|\lambda|)^{\frac{\alpha+2}{p}}}\|f\| \leqslant \frac{c_3}{(1-|\lambda|)^{\frac{\alpha+2}{p}-1}}\|f\|,$$

and the proof is complete. □

Let us list a number of direct consequences of the two assumptions above.

Remarks 3.1. (1) If (A1) holds then the weighted composition operators $C^\gamma_{\psi_a}$ are also uniformly bounded below on X, since $(C^\gamma_{\psi_a})^2$ equals the identity operator on this space.

(2) Each of the assumptions (A1) or (A2) implies that X is (continuously) contained in $A^{-\gamma}$. If (A1) holds then

$$\sup_{a\in\mathbb{D}}\left(1-|a|^2\right)^\gamma\left|f(a)\right| = \sup_{a\in\mathbb{D}}\left|C^\gamma_{\psi_a}f(0)\right| \leqslant c\sup_{a\in\mathbb{D}}\left\|C^\gamma_{\psi_a}\right\|\|f\|$$

for some positive constant c and all $f\in X$. The proof that (A2) implies that X is contained in $A^{-\gamma}$ is similar and can be found in Lemma 3.1 from [21].

Some deeper properties of such spaces of analytic functions are proved below. Given a function $h\in H(\mathbb{D})$ with $hX\subset X$ we shall denote throughout by M_h the operator of multiplication by h, that is $M_h f = hf$, $f\in X$. A direct application of the closed graph theorem shows that M_h is bounded on X. Also recall that ξ denoted the identity function on $\mathbb{D}$.

Proposition 3.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ which satisfies* (A1) *and assume, in addition, that the operator M_ξ is bounded on X with spectrum $\sigma(M_\xi|X)=\overline{\mathbb{D}}$. Then*:

(i) *For all $\eta\in\mathbb{C}$ with $Re\,\eta>0$, and all $a\in\mathbb{D}$ we have $(1-a\xi)^{-\eta}X\subset X$ and there exists a fixed constant $c>0$ such that*

$$\left\|(1-a\xi)^{-\eta}f\right\|_X \leqslant \frac{(1+|\eta|)c^{Re\,\eta}}{(1-|a|)^{Re\,\eta}}\|f\|_X.$$

(ii) *There exists $c' > 0$ such that*

$$\|M_\xi^n\| \leqslant c'n,$$

for all $n \in \mathbb{Z}^+$.

(iii) *If $h \in H(\mathbb{D})$ is such that $h''' \in A^{-\delta}$ for some $\delta \in (0,1)$ then $hX \subset X$ and there exists $c_\delta > 0$ such that*

$$\|M_h\| \leqslant \sum_{k=0}^{2} |h^{(k)}(0)| + \|h'''\|_{-\delta}.$$

Proof. (i) The first part of the statement follows by basic functional analysis from the assumption that $\sigma(M_\xi|X) = \overline{\mathbb{D}}$. In fact,

$$M_{(1-a\xi)^{-\eta}} = \frac{1}{2\pi i} \int\limits_{|\lambda|=2-|a|} (1-a\lambda)^{-\eta}(\lambda - M_\xi)^{-1}\, d\lambda. \tag{3.11}$$

For $|\lambda| > 1$ let $b = 1/\bar{\lambda}$ and use Remarks 3.1(1) to obtain

$$\|(\lambda-\xi)^{-1} f\|_X \leqslant c_1 \|C^\gamma_{\psi_b}(\lambda-\xi)^{-1} f\|_X$$

for some fixed constant $c_1 > 0$. Note that

$$C^\gamma_{\psi_b}(\lambda-\xi)^{-1} f = \frac{\bar{b}(1-\bar{b}\xi)}{1-|b|^2} C^\gamma_{\psi_b} f,$$

and use (A1) together with the assumption that M_ξ is bounded to obtain

$$\|(\lambda-\xi)^{-1} f\|_X \leqslant c_1^2 \frac{|b|(1+\|M_\xi\|)}{1-|b|^2} \|f\|_X. \tag{3.12}$$

In particular, since $b = \bar{\lambda}^{-1}$, (i) holds for $\eta = 1$. If $Re\,\eta > 1$ we use (3.11) and (3.12) to obtain

$$\|(1-a\xi)^{-\eta} f\|_X \leqslant \frac{c_2}{1-|a|} \int\limits_0^{2\pi} \left|1 - a(2-|a|)e^{it}\right|^{-Re\,\eta} dt \leqslant \frac{c^\eta}{(1-|a|)^{Re\,\eta}},$$

by a standard estimate for integrals (see [9]). Finally, if $Re\,\eta < 1$, we let ℓ be an arbitrary continuous linear functional on X and $f \in X$. We consider the analytic function

$$\ell_f^\eta(a) = \ell\big((1-a\xi)^{-\eta} f\big), \quad a \in \mathbb{D}. \tag{3.13}$$

Since

$$\big(\ell_f^\eta\big)'(a) = -\eta\ell\big((1-a\xi)^{-\eta-1} f\big),$$

we have from above

$$\left|\left(\ell_f^{\eta}\right)'(a)\right| \leqslant \frac{|\eta| c^{1+Re\,\eta}}{(1-|a|)^{1+Re\,\eta}} \|\ell\|_{X^*}\|f\|_X, \quad a \in \mathbb{D}.$$

Using again the well-known estimates for analytic functions (see again [9, Chapter 5]) we obtain

$$\left|\ell\left((1-a\xi)^{-\eta} f\right)\right| \leqslant \frac{|\eta| c_1^{Re\,\eta}}{(1-|a|)^{Re\,\eta}} \|\ell\|_{X^*}\|f\|_X, \quad a \in \mathbb{D},$$

and (i) follows.

To see (ii) note that for $\ell \in X^*$ and $f \in X$

$$\ell_f^1(a) = \sum_{n=0}^{\infty} a^n \ell\left(M_\xi^n f\right).$$

As in the proof of Lemma 2.1 we use the coefficient estimates for A^{-1} to obtain

$$\left|\ell\left(M_\xi^n f\right)\right| \leqslant c' n \left\|\ell_f^1\right\|_{-1},$$

and the result follows from (i). Finally, to prove (iii), note that if $h \in H(\mathbb{D})$ satisfies the condition in the statement then

$$\left|\frac{h^{(n)}(0)}{n!}\right| \leqslant c_1 (n+1)^{\delta-3}$$

and thus

$$M_h = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} M_\xi^n,$$

where, by (ii), the sum converges in $\mathcal{B}(X)$. □

Proposition 3.3. *Let X be a Banach space of analytic functions on $\mathbb{D}$ which satisfies* (A1) *and* (A2). *Given a linear fractional map ϕ with $\phi(\mathbb{D}) \subset \mathbb{D}$, denote by λ_ϕ the center and by ρ_ϕ the radius of the disc $\phi(\mathbb{D})$. Let C_ϕ^γ be the weighted composition operator defined by $C_\phi^\gamma f = (\phi')^\gamma f \circ \phi$. Then there exists a constant $c > 0$ such that*

$$\left\|C_\phi^\gamma\right\| \leqslant c \frac{\rho_\phi^\gamma}{(1-|\lambda_\phi|)^\gamma},$$

for all such linear fractional maps ϕ.

Proof. Write $\phi(z) = \rho_\phi \psi_a(z) + \lambda_\phi$ for some $a \in \mathbb{D}$. Then

$$\left(\phi'\right)^\gamma f \circ \phi = \left(\rho_\phi \psi_a'\right)^\gamma f \circ (\rho_\phi \psi_\lambda + \lambda_\phi),$$

and by (A1) and (A2) we have

$$\left\| C_\phi^\gamma f \right\|_X \leqslant c_1 \rho_\phi^\gamma \left\| f \circ (\rho_\phi \xi + \lambda_\phi) \right\|_X \leqslant c_2 \frac{\rho_\phi^\gamma}{(1 - |\lambda_\phi|)^\gamma} \| f \|_X,$$

for some positive constants c_1, c_2 and all $f \in X$. □

When working with the assumptions (A1), (A2) in the general context, we encounter some technical difficulties which in most concrete cases are easily dealt with. These arise from the fact that our assumptions only regard the norms of certain composition and weighted composition operators and do not give any information about the dependence of such operators on their symbols. This is a crucial matter for our further purposes and is addressed in the next lemma, at the cost of an additional assumption.

Lemma 3.1. *Let X be a Banach space of analytic functions in the unit disc that satisfies* (A1), (A2) *and such that M_ξ is bounded on X with $\sigma(M_\xi | X) = \overline{\mathbb{D}}$. Assume, in addition, that polynomials are dense in X. For some fixed $r \in (1, \infty)$, let (ψ_n), (ϕ_n) be convergent sequences in $H(r\mathbb{D})$ with limits $\psi, \phi \in H(r\mathbb{D})$, such that each ϕ_n is a linear fractional map with $\phi_n(\mathbb{D}) \subset \mathbb{D}$. If $\phi(\mathbb{D}) \subset \mathbb{D}$ then C_ϕ is bounded on X and $(M_{\psi_n} C_{\phi_n})$ converges to $M_\psi C_\phi$ in the strong topology on $\mathcal{B}(X)$.*

Proof. Note that under our assumptions we have

$$M_{\psi_n} = \psi_n(M_\xi) = \frac{1}{2\pi i} \int_{|\zeta| = \frac{1+r}{2}} \psi_n(\zeta)(\zeta - M_\xi)^{-1}\, d\xi$$

which shows that $M_{\psi_n} \to M_\psi$ in the operator norm. It will then suffice to prove that (C_{ϕ_n}) is bounded in $\mathcal{B}(X)$.

Indeed, if this holds, we can use the fact that for every polynomial f we have

$$C_{\phi_n} f = f(M_{\phi_n})1,$$

hence, by the first part of the proof $C_{\phi_n} f \to C_\phi f$, for all polynomials f, and since polynomials are dense in X, the result follows.

To verify the above assertion, note first that if ϕ is constant then by the previous argument we have that $\|M_{\phi_n}\| \leqslant 1 - \delta$ for some fixed $\delta > 0$ and all sufficiently large n. For such n and $f \in X$ we can write

$$C_{\phi_n} f = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} \phi_n^k.$$

For every $\varepsilon > 0$ we have an estimate of the form

$$\frac{|f^{(k)}(0)|}{k!} \leqslant c_\varepsilon (1+\varepsilon)^k \|f\|_X,$$

for some fixed $c_\varepsilon > 0$ and all $k \geqslant 0$, and if we choose $\varepsilon > 0$ such that $(1+\varepsilon)\|M_{\phi_n}\| \leqslant 1-\varepsilon$, we obtain

$$\|C_{\phi_n} f\|_X \leqslant \frac{c_\varepsilon}{\varepsilon}\|f\|_X.$$

If ϕ is not constant then the centers λ_n of the discs $\phi_n(\mathbb{D})$ must satisfy $|\lambda_n| \leqslant 1-\delta$ for some fixed $\delta > 0$ and all n. In addition, (ϕ_n') converges uniformly on compact subsets of $r\mathbb{D}$ to ϕ' and $\phi'(z) \neq 0$, $z \in r\mathbb{D}$, which implies that $((\phi_n')^{-\gamma})$ converges uniformly on compact subsets of $r\mathbb{D}$. Then

$$\|C_{\phi_n}\| = \left\|M_{(\phi_n')^{-\gamma}} C_{\phi_n}^{\gamma}\right\| \leqslant \|M_{(\phi_n')^{-\gamma}}\| \left\|C_{\phi_n}^{\gamma}\right\|,$$

and by the first part of the proof together with Proposition 3.3 we obtain again that (C_{ϕ_n}) is bounded in $\mathcal{B}(X)$. □

A direct consequence of the lemma is the fact that a space X with the properties in the statement is automatically invariant under composition with rotations. Indeed, by (A2), the operators $C_{t,\mu}$, $0 < t < 1$, $|\mu| = 1$, defined by

$$C_{t,\mu} f(z) = f(t\mu z),$$

are uniformly bounded on X, and by Lemma 3.1 $C_{t,\mu}$ has a limit in the strong operator topology on $\mathcal{B}(X)$. This implies that the operators of composition with rotations, $C_\mu f(z) = f(\mu z)$ are uniformly bounded on X. Note also that these operators are uniformly bounded below, since $C_{\overline{\mu}} C_\mu = I$.

4. Basic estimates for Cesàro-like operators

It turns out that the assumptions considered in the previous section provide a powerful tool for the study of certain generalized Cesàro operators. Our approach follows a known idea based on composition semigroups (see [26–28,21]), but the general context considered here requires a number of additional steps. The more technical part is deferred to the present section, more precisely, the following lemma which will play an essential role for our further developments.

Throughout in what follows, for a fixed nonnegative integer N, we will denote by H_N^∞ the Banach algebra

$$H_N^\infty = \{f \in H(\mathbb{D}):\ f^{(N)} \in H^\infty\},$$

with the natural norm

$$\|f\|_{N,\infty} = \sum_{n=0}^{N} \left|f^{(n)}(0)\right| + \left\|f^{(N)}\right\|_\infty.$$

Lemma 4.1. *Let X be a Banach space of analytic functions in the unit disc that satisfies* (A1), (A2) *for some $\gamma > 0$, and such that polynomials are dense in X. Assume in addition that*:

(A3) *M_ξ is bounded and bounded below on X with $\sigma(M_\xi|X) = \overline{\mathbb{D}}$.*

Given $\mu, \nu \in \mathbb{C}$, and a positive integer m with $m > Re\,\mu - 1$, define for $f \in X_m$

$$T_{\mu,\nu} f(z) = z^{\mu-1}(1-z)^{-\nu} \int_0^z \zeta^{-\mu}(1-\zeta)^{\nu-1} f(\zeta)\, d\zeta.$$

Then there exists a constant $c \geqslant 1$ depending only on X such that:

(i) *If $Re\,\nu < \gamma$ and $m > 2\gamma + Re(\mu - \nu) - 1$, then $T_{\mu,\nu} f \in X_m$ whenever $f \in X_m$ and*

$$\|T_{\mu,\nu} f\|_X \leqslant \frac{(1 + |Im(\nu - \mu)|)c^m}{\gamma - Re\,\nu} \|f\|_X.$$

(ii) *If $Re\,\nu > \gamma$ and $m > |Re\,\mu| + \nu + 4$, then $T_{\mu,\nu} f \in X_m$ whenever $f \in X_m$ with*

$$\int_0^1 t^{-\mu}(1-t)^{\nu-1} f(t)\, dt = 0,$$

and

$$\|T_{\mu,\nu} f\|_X \leqslant \bigl(1 + \bigl|Im(\nu - \mu)\bigr|\bigr)^m e^{\pi(|Im\,\mu| + |Im\,\nu|)} \frac{c_0^m}{Re\,\nu - \gamma} \|f\|_X.$$

Proof. (i) Since polynomials are in X and M_ξ is bounded below on this space, we can easily conclude that the backward shift

$$Bf = \frac{f - f(0)}{\xi}$$

is a bounded linear operator on X.

Consider the one-parameter family of functions

$$\varphi_t(z) = \frac{e^{-t} z}{(e^{-t} - 1)z + 1}, \quad z \in \mathbb{D}. \tag{4.14}$$

These functions form a semigroup under composition since

$$\varphi_t(z) = \varphi^{-1}\bigl(e^{-t}\varphi(z)\bigr), \quad z \in \mathbb{D},$$

where φ is the starlike function $\varphi(z) = z(1-z)^{-1}$, $z \in \mathbb{D}$. Using also this equality, we can verify that the functions φ_t satisfy

$$\varphi_t(0) = z, \qquad \lim_{t\to\infty} \varphi_t(z) = 0, \tag{4.15}$$

$$\frac{d}{dt}\varphi_t(z) = -\frac{\varphi(\varphi_t(z))}{\varphi'(\varphi_t(z))} = -\varphi_t(z)\bigl(1-\varphi_t(z)\bigr), \tag{4.16}$$

and

$$\varphi_t'(z) = \frac{d}{dz}\varphi_t(z) = e^t\left(\frac{\varphi_t(z)}{z}\right)^2. \tag{4.17}$$

Now in the expression defining $T_{\mu,\nu}$, we integrate along the path $t \to \varphi_t(z)$, $t \in [0,\infty)$, use (4.16) and (4.17), and obtain

$$\begin{aligned}
T_{\mu,\nu} f(z) &= z^{\mu-1}(1-z)^{-\nu}\int_0^z \zeta^{-\mu+m}(1-\zeta)^{\nu-1}B^m f(\zeta)\,d\zeta \\
&= z^{\mu-1}(1-z)^{-\nu}\int_0^\infty \varphi_t^{-\mu+m+1}(z)\bigl(1-\varphi_t(z)\bigr)^\nu f\bigl(\varphi_t(z)\bigr)\,dt \\
&= z^m\int_0^\infty e^{\nu t}\left(\frac{\varphi_t(z)}{z}\right)^{-\mu+\nu+m+1} f\bigl(\varphi_t(z)\bigr)\,dt \\
&= z^m\int_0^\infty e^{(\nu-\gamma)t}\left(\frac{\varphi_t(z)}{z}\right)^{-\mu+\nu+m+1-2\gamma}\bigl(\varphi_t'(z)\bigr)^\gamma f\bigl(\varphi_t(z)\bigr) \\
&= z^m\int_0^\infty e^{(\nu-\gamma)t}\left(\frac{\varphi_t(z)}{z}\right)^{-\mu+\nu+m+1-2\gamma}\bigl(C^\gamma_{\varphi_t} f\bigr)(z)\,dt.
\end{aligned}$$

Let $\eta = -\mu+\nu+m+1-2\gamma$. By Lemma 3.1 we have that

$$t \to M_{(\varphi_t/\xi)^\eta} C^\gamma_{\varphi_t}$$

is a strongly continuous $\mathcal{B}(X)$-valued function on $[0,\infty)$. Moreover, if $m > 2\gamma + Re(\mu-\nu) - 1$, i.e. $Re\,\eta > 0$, then by Proposition 3.2(i) we have

$$\|M_{(\varphi_t/\xi)^\eta}\| \leqslant \bigl(1+|Im\,\eta|\bigr)c_1^{Re\,\eta}, \quad t \geqslant 0.$$

By Proposition 3.3 and the straightforward computation

$$\lambda_{\varphi_t} = \frac{1-e^{-t}}{2-e^{-t}} \leqslant \frac{1}{2}, \quad \rho_{\varphi_t} \leqslant 1,$$

we obtain

$$\bigl\|C^\gamma_{\varphi_t}\bigr\| \leqslant c_2, \quad t \geqslant 0.$$

Thus

$$T_{\mu,\nu} f = \int_0^\infty e^{(\nu-\gamma)t} M_{(\varphi_t/\xi)^\eta} C_{\varphi_t}^\gamma f \, dt,$$

and

$$\begin{aligned}\|T_{\mu,\nu} f\|_X &\leqslant c_2\big(1+|Im\,\eta|\big)c_1^{Re\,\eta}\|f\|_X \int_0^\infty e^{-t\,Re(\nu-\gamma)}\,dt \\ &= c_2\big(1+|Im\,\eta|\big)c_1^{Re\,\eta}\|f\|_X \frac{1}{Re(\nu-\gamma)}.\end{aligned}$$

The result follows then easily from the assumption that $m > 2\gamma + Re(\mu-\nu) - 1$.

(ii) Assume that $Re\,\nu > \gamma$. For $f \in X$ and $a \in \overline{\mathbb{D}} \setminus \{1\}$, let

$$\begin{aligned}f_a &= f - \frac{\int_1^a (\zeta-a)^{-\mu+m}(1-\zeta)^{\nu-1} f(\zeta)\,d\zeta}{\int_1^a (\zeta-a)^{-\mu+m}(1-\zeta)^{\nu-1}\,d\zeta} \\ &= f - \frac{\Gamma(m+1-\mu+\nu)}{\Gamma(m+1-\mu)\Gamma(\nu)} \int_0^1 t^{m-\mu}(1-t)^{\nu-1} f\big(t+(1-t)a\big)\,dt,\end{aligned}$$

and note that

$$\int_1^a (\zeta-a)^{m-\mu}(1-\zeta)^{\nu-1} f_a(\zeta)\,d\zeta = 0.$$

Now for $a \in \overline{\mathbb{D}} \setminus \{1\}$ and $f \in X$ let

$$\begin{aligned}F_a(z) &= (1-a)^{2\gamma+1}(z-a)^{m+\mu+3}(1-z)^{-\nu} \int_a^z (\zeta-a)^{m-\mu}(1-\zeta)^{\nu-1} f_a(\zeta)\,d\zeta \\ &= (1-a)^{2\gamma+1}(z-a)^{m+\mu+3}(1-z)^{-\nu} \int_1^z (\zeta-a)^{m-\mu}(1-\zeta)^{\nu-1} f_a(\zeta)\,d\zeta. \qquad (4.18)\end{aligned}$$

We claim that if f is a polynomial then F_a belongs to X and the X-valued function $a \to F_a$ is analytic in $\mathbb{D}$ and extends continuously to $\overline{\mathbb{D}}$, with

$$\sup_{a\in\partial\mathbb{D}\setminus\{1\}} \|F_a\|_X \leqslant \big(1+\big|Im(\nu-\mu)\big|\big)^m e^{\pi(|Im\,\mu|+|Im\,\nu|)} \frac{c_0^m}{Re\,\nu-\gamma}\|f\|_X, \qquad (4.19)$$

where $c_0 \geqslant 1$ depends only on X.

If we assume the claim for the moment, we can proceed as follows. Since $F_0 = M_\xi^{m+4} T_{\mu,\nu} M_\xi^m f_0$, we can apply the maximum principle and (4.19) to obtain that

$$\left\| M_\xi^{m+4} T_{\mu,\nu} M_\xi^m f_0 \right\|_X = \|F_0\|_X \leqslant \sup_{a\in\partial\mathbb{D}\setminus\{1\}} \|F_a\|_X$$
$$\leqslant \left(1 + \left|Im(\nu-\mu)\right|\right)^m e^{\pi(|Im\,\mu|+|Im\,\nu|)} \frac{c_0^m}{Re\,\nu - \gamma} \|f\|_X,$$

for every polynomial f. The same argument as in the proof of Proposition 2.2 (see also the estimate of $\|f_a\|_X$ below) shows that

$$y_{\mu,\nu}(f) = \int_0^1 t^{-\mu}(1-t)^{\nu-1} f(t)\,dt$$

defines a continuous linear functional on X_m. Now if $f \in \ker y_{\mu,\nu}$, and (f_n) is a sequence of polynomials which converges to f in X, it follows directly by the assumptions and that $\xi^m(B^m f_n)_0 \to f$ in X. From the last inequality above we deduce that $(M_\xi^{m+4} T_{\mu,\nu} M_\xi^m (B^m f_n)_0)$ is a Cauchy sequence in X which converges pointwise to $M_\xi^{m+4} T_{\mu,\nu} f$, and (ii) follows.

To complete the proof it remains to verify the claim. This part of the argument is more involved.

We begin by estimating $\|f_a\|_X$. From the fact that X is continuously contained in $A^{-\gamma}$ together with the elementary computation

$$2\,Re(1-a) - |1-a|^2 = 1 - |a|^2 \geqslant 0,$$

we obtain

$$\left| \int_0^1 t^{m-\mu}(1-t)^{\nu-1} f\big(t + (1-t)a\big)\,dt \right|$$
$$\leqslant c_1 \|f\|_X \int_0^1 \frac{t^{m-Re\,\mu}(1-t)^{Re\,\nu-1}}{(1-|1+(1-t)(a-1)|^2)^\gamma}\,dt$$
$$\leqslant c_1 \|f\|_X |1-a|^{-2\gamma} \int_0^1 t^{m-Re\,\mu-\gamma}(1-t)^{Re\,\nu-1-\gamma}\,dt$$
$$= c_1|1-a|^{-2\gamma} \frac{\Gamma(m+1-Re\,\mu-\gamma)\Gamma(Re\,\nu-\gamma)}{\Gamma(m+1-2\gamma+Re(\nu-\mu))} \|f\|_X.$$

If $\varepsilon, \delta > 0$, then by Stirling's formula we have for all w with $Re\,w > \varepsilon + \delta$

$$\left| \frac{\Gamma(w)}{\Gamma(Re\,w - \delta)} \right| \leqslant k_1 |w|^\delta e^{\frac{\pi}{2}|Im\,\delta| + k_2\,Re\,w} \big(1 + |Im\,w|\big)^{Re\,w},$$

and for all w with $Re\, w > \delta$

$$\left|\frac{\Gamma(w)}{\Gamma(Re\, w-\delta)}\right| \geqslant k_3 e^{-\frac{\pi}{2}|Im\,\delta|},$$

where $k_1, k_2 > 0$ depend only on ε and δ and $k_3 > 0$ only on δ. Thus if $m > Re\,\mu + \gamma$, then

$$\begin{aligned}&\left|\frac{\Gamma(m+1-\mu+\nu)\Gamma(m+1-Re\,\mu-\gamma)\Gamma(Re\,\nu-\gamma)}{\Gamma(m+1+Re(\nu-\mu)-2\gamma)\Gamma(m+1-\mu)\Gamma(\nu)}\right|\\ &\quad\leqslant k_4|m+1-\mu+\nu|^{2\gamma}\bigl(1+\bigl|Im(\nu-\mu)\bigr|\bigr)^{Re(m+1-\mu+\nu)}\\ &\qquad\times\exp\biggl(\frac{\pi}{2}\bigl(\bigl|Im(\nu-\mu)\bigr|+|Im\,\mu|+|Im\,\nu|\bigr)+k_5\,Re(m+1-\mu+\nu)\biggr),\end{aligned}$$

where $k_4, k_5 > 0$ depend only on γ, that is, only on X. If we now use also the assumption that $m > Re(\nu-\mu)+1$, it follows that there exists $c_2 \geqslant 1$ depending only on X such that the right-hand side of the above inequality does not exceed

$$c_2^m\bigl(1+\bigl|Im(\nu-\mu)\bigr|\bigr)^m e^{\pi(|Im\,\mu|+|Im\,\nu|)}.$$

This gives an estimate of the form

$$\|f_a\|_X \leqslant |1-a|^{-2\gamma} c_3^m \exp\pi\bigl(|Im\,\mu|+|Im\,\nu|\bigr)\|f\|_X, \tag{4.20}$$

where $c_3 \geqslant 1$ depends only on X. It is also clear that the X-valued function $a \to (1-a)^{2\gamma+1} f_a$ is analytic in $\mathbb{D}$ and extends continuously to $\overline{\mathbb{D}}$.

We now turn to the actual proof of the claim. In the first equality in (4.18) we integrate along the line segment from a to z and conclude that F_a is analytic in $\mathbb{D}$ for all $a \in \overline{\mathbb{D}} \setminus \{1\}$. If we integrate in the second equality in (4.18) along the line segment from 1 to z we obtain

$$\begin{aligned}F_a(z) &= -(1-a)^{2\gamma+1}(z-a)^{m+\mu+3}\\ &\quad\times\int_0^1 t^{\nu-1}\bigl(1+t(z-1)-a\bigr)^{m-\mu} f_a\bigl(1+t(z-1)\bigr)\,d\zeta.\end{aligned} \tag{4.21}$$

Since $Re\,\nu > \gamma > 0$ and $m > Re\,\mu + 4$, it follows that if f is a polynomial, $F_a^{(4)}(z)$ is bounded for

$$(a,z) \in \bigl(\overline{\mathbb{D}} \setminus \{1\}\bigr) \times \left(\mathbb{D} \setminus \left\{\frac{a-t}{1-t}\colon\, t<1\right\}\right).$$

But then $\|F_a\|_{4,\infty}$ is bounded. Now Proposition 3.2(iii) implies that $F_a \in X$, but even more than that is true: If $0 < r < 1$ and

$$C_r F_a(z) = F_a(rz),$$

then $\lim_{r\to 1^-} C_r F_a = F_a$ in H_3^∞, uniformly in $a \in \overline{\mathbb{D}}$. By Proposition 3.2(iii) it follows that $\lim_{r\to 1^-} C_r F_a = F_a$ in X, uniformly in $a \in \overline{\mathbb{D}}$. It is also easy to verify that $a \to C_r F_a$ is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, hence, so is $a \to F_a$. Then by a repeated application of Proposition 3.2(iii) and Lemma 3.1 we obtain (recall that $Re\,\mu < m$)

$$\| M_{(z-a)^{\mu+m+3}} \| \leqslant c_4 m^3 2^m, \quad |a| = 1,$$

and that $t \to M_{(1+t(\xi-1)-a)^{m-\mu}}$ is a continuous $\mathcal{B}(X)$-valued function on [0, 1] with

$$\| M_{(1+t(\xi-1)-a)^{m-\mu}} \| \leqslant c_5 m^3 2^m.$$

Moreover, if $\phi_t(z) = 1 + t(z-1)$, then $t \to C_{\phi_t}$ is a strongly continuous $\mathcal{B}(X)$-valued function on [0, 1), with

$$\|C_{\phi_t}\| \leqslant c_6 t^{-\gamma}, \quad t \in [0, 1),$$

where this last estimate is obtained directly from (A2). Thus

$$F_a = -(1-a)^{2\gamma+1} M_{(z-a)^{m+\mu+3}} \int_0^1 t^{\nu-1} M_{(1+t(\xi-1)-a)^{m-\mu}} C_{\phi_t} f_a \, dt.$$

Then the claim follows from these estimates, (4.20) and (4.21), and the proof of the lemma is complete. □

The additional assumption that polynomials are dense in our space X excludes from the list of examples the growth classes $A^{-\gamma}$, $\gamma > 0$, this condition is fulfilled only by the "little oh" version $A_0^{-\gamma}$. There is however a simple way to deal with the larger spaces as well.

Given a Banach space X of analytic functions in $\mathbb{D}$, we let $\tilde{X} \subset H(\mathbb{D})$ be the space of functions that are pointwise limits of bounded sequences in X. For $f \in \tilde{X}$ we set

$$\|f\|_{\tilde{X}} = \inf\Big\{ \liminf_{n\to\infty} \|f_n\|_X \colon f_n \in X, \ \lim_{n\to\infty} f_n(z) = f(z), \ z \in \mathbb{D} \Big\}.$$

It is quite easy to show that $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ is a Banach space of analytic functions in $\mathbb{D}$, and that the inclusion map from X into $\tilde{X}$ is contractive. Moreover, if the linear span of functionals of evaluation at points of $\mathbb{D}$ is dense in X^* then the inclusion map above is isometric, and $\tilde{X}$ is isometrically isomorphic to X^{**}. As a concrete example we have

$$\widetilde{A_0^{-\gamma}} = A^{-\gamma}$$

for all $\gamma > 0$. More generally, the conclusion holds true for any Banach space X of analytic functions in $\mathbb{D}$ which satisfies the assumptions (A1)–(A3) and contains the polynomials as a dense subspace.

Indeed, this follows from the fact that the composition operators C_t, $0 < t < 1$, with $C_t f(z) = f(tz)$ satisfy

$$\lim_{t \to 1^-} C_t f = f, \quad f \in X.$$

Then C_t^* also converges strongly to the identity on X^* when $t \to 1^-$, and using Cauchy's formula, we can verify that for every $y \in X^*$, $C_t^* y$ belongs to the closed linear span of the functionals of evaluation at points of $\mathbb{D}$.

This leads to the following result.

Corollary 4.1. *If X satisfies* (A1)–(A3) *then the conclusion of Lemma* 4.1 *continues to hold when X is replaced by $\tilde{X}$.*

Our first application concerns the boundedness of generalized Cesàro operators whose symbol has an essentially rational derivative.

Theorem 4.1. *Let X be a Banach space of analytic functions in the unit disc that satisfies* (A1)–(A3). *Assume in addition that*:

(A4) *Polynomials are dense in X and* $\log(1 - \bar{b}\xi) \in X$ *for all* $b \in \partial\mathbb{D}$.

If $g \in H(\mathbb{D})$ with $g(0) = 0$ has a derivative of the form

$$g'(z) = \sum_{k=1}^{n} \frac{a_k}{1 - \bar{b}_k z} + h(z),$$

where $a_1, \dots, a_n \in \mathbb{C}$, $b_1, \dots, b_n \in \partial\mathbb{D}$, and $h \in H(\mathbb{D})$ with $hX \subset X$, then the generalized Cesàro operator $\mathcal{C}_g$ is bounded on both spaces X and $\tilde{X}$. If $a_1 = \cdots = a_n = 0$ then $\mathcal{C}_g$ is compact on any of these spaces.

Proof. For $0 \leqslant t < 1$ let $\phi_t(z) = tz$, $z \in \mathbb{D}$. It is a simple matter to show that the composition operators C_{ϕ_t}, $0 < t < 1$ are compact on X. For example, this follows immediately from Proposition 3.2(iii). Moreover, by Lemma 3.1 we have that the $\mathcal{B}(X)$-valued function $t \to C_{\phi_t}$ is strongly continuous on [0, 1], and if $\tilde{h}$ is the primitive of h which vanishes at the origin then

$$\mathcal{C}_{\tilde{h}} f = \int_0^1 C_{\phi_t} M_h f \, dt = \lim_{t \to 1^-} \int_0^t C_{\phi_t} M_h f \, dt,$$

where the limit is taken in $\mathcal{B}(X)$. Clearly, this implies that $\mathcal{C}_{\tilde{h}}$ is compact on X. Finally, by the definition of $\tilde{X}$, any compact operator on X extends to a compact operator from $\tilde{X}$ into X.

To prove the theorem it remains to show that $\mathcal{C}_{g_b}$ defines a bounded operator on X, where

$$g_b(z) = b \log(1 - \bar{b}z), \quad |b| = 1,$$

since the boundedness of this operator on $\tilde{X}$ is again immediate by definition. Moreover, it will be sufficient to prove that for some fixed nonnegative integer m, the operators $C_{g_b} M_\xi^m$ are bounded on X. Indeed, using the density of polynomials in X together with the fact that M_ξ is bounded below, we conclude that the backward shift

$$Bf = \frac{f - f(0)}{\xi}$$

is a bounded linear operator on X. Then every $f \in X$ can be written as

$$f = P_m(f) + M_\xi^m B^m f,$$

where $P_m(f)$ is the Taylor polynomial of degree m of f. Clearly, $P_m = I - M_\xi^m B^m$ is bounded on X, and by assumption (A3), it follows that $\mathcal{C}_{g_b} P_m$ are bounded linear operators on X. Thus

$$\mathcal{C}_{g_b} = \mathcal{C}_{g_b} P_m + \mathcal{C}_{g_b} M_\xi^m B^m$$

is then bounded on X. Now this last assertion follows directly from Lemma 4.1. Indeed, as pointed out in the previous section, the operators of composition with rotations are bounded and invertible on X, so that we can restrict our attention to the case when $b = 1$. But then $\mathcal{C}_{g_1} = T_{0,0}$ which is bounded on X_m by the result mentioned above. □

5. Spectra of generalized Cesàro operators

For all concrete examples of Banach spaces of analytic functions in $\mathbb{D}$ considered in this paper, it turns out that the results concerning the Cesàro operator obtained in the previous section, can be extended with little effort to all generalized Cesàro operators of the form given in Theorem 4.1. This is due to a localization technique which is explained in the proposition below.

We denote by $\mathfrak{F}$ the set of Riemann maps from $\mathbb{D}$ onto the interior of C^5-Jordan curves contained in $\overline{\mathbb{D}}$, which fix the origin. As it is well known (see [23, p. 49]), if $\phi \in \mathfrak{F}$ then $\phi \in H_4^\infty$ and ϕ' has a continuous, zero-free extension to $\overline{\mathbb{D}}$.

Proposition 5.1. *If X is one of the spaces H^p, $L_a^{p,\alpha}$, $D^{p,\alpha}$, $A^{-\gamma}$, $A_0^{-\gamma}$ then it satisfies for every positive integer n:*

(A5(n)) *Given distinct points $b_1, \dots, b_n \in \partial\mathbb{D}$, there exists a finite set $\mathcal{F} \subset \mathfrak{F}$ such that:*
- (i) *If $\phi \in \mathcal{F}$ then $\overline{\phi(\mathbb{D})}$ contains exactly one of the points b_j, $1 \leqslant j \leqslant n$, and the composition operator C_ϕ is bounded on X.*
- (ii) *If $f \in H(\mathbb{D})$ satisfies $f \circ \phi \in X$ for all $\phi \in \mathcal{F}$ then $f \in X$.*

Proof. Consider any set of open arcs $I_j \subset \partial\mathbb{D}$, $1 \leqslant j \leqslant n$ such that

$$I_j \cap \{b_1, \dots, b_n\} = \{b_j\}, \quad 1 \leqslant j \leqslant n, \quad \bigcup_{j=1}^n I_j = \partial\mathbb{D}.$$

Construct C^5-Jordan curves Υ_j with $I_j \subset \Upsilon_j$, and $\Upsilon_j \setminus \overline{I_j} \subset \mathbb{D}$, and such that the interior of Υ_j contains the set

$$S_j = \left\{z\colon |z| < \frac{1}{2}\right\} \cup \left\{z \in \mathbb{D} \setminus \{0\}\colon z/|z| \in I_j\right\}.$$

Let ϕ_j be a conformal map from $\mathbb{D}$ onto the interior of Υ_j, with $\phi_j(0)=0$. Then for all $z\in\mathbb{D}$ we have $1-|z|\leqslant 1-|\phi_j(z)|$. If $|z|\to 1$ in $\phi_j^{-1}(S_j)$ then $|\phi_j(z)|\to 1$, which immediately implies that

$$1-|z|\geqslant c_j'\bigl(1-\bigl|\phi_j(z)\bigr|\bigr),$$

for some constant $c_j'>0$ and all $z\in\phi_j^{-1}(S_j)$. Now (A5(n)) can be proved with the following argument. For example, if $f\in L_a^{p,\alpha}$

$$\begin{aligned}\|f\circ\phi_j\|_{p,\alpha}^p &\geqslant c_j''\int\limits_{\phi_j^{-1}(S_j)}\bigl|f\circ\phi_j(z)\bigr|^p\bigl|\phi_j'(z)\bigr|^2\bigl(1-\bigl|\phi_j(z)\bigr|\bigr)^\alpha\,dA(z)\\ &=c_j''\int\limits_{S_j}\bigl|f(z)\bigr|^p\bigl(1-|z|\bigr)^\alpha\,dA(z),\end{aligned}$$

and since $\bigcup_j S_j=\mathbb{D}$, this implies (A5(n)). The cases when $X=A^{-\gamma}$, or $X=A_0^{-\gamma}$ are similar. If $X=D^{p,\alpha}$ we have as above

$$\|f\circ\phi_j\|_{D^{p,\alpha}}^p\geqslant\bigl|f(0)\bigr|^p+c_j'''\int\limits_{\phi_j^{-1}(S_j)}\bigl|f'\circ\phi_j(z)\bigr|^p\bigl|\phi_j'(z)\bigr|^2\bigl(1-\bigl|\phi_j(z)\bigr|\bigr)^\alpha\,dA(z)$$

and the result follows with the same argument. Finally, the reasoning can be applied to the Hardy spaces as well, since for $f\in H^p$

$$\|f\|_p^p\sim\bigl|f(0)\bigr|^p+\int\limits_{\mathbb{D}}\bigl|f(z)\bigr|^{p-2}\bigl|f'(z)\bigr|^2\bigl(1-|z|\bigr)\,dA(z).\qquad\Box$$

Note that (A5(1)) is satisfied by any Banach space of analytic functions with $\mathcal{F}=\{\xi\}$. Moreover, for any finite family $\mathcal{F}$ of conformal maps with the properties in (A5(n)), we have by the closed graph theorem that

$$f\to\max_{\phi\in\mathcal{F}}\|f\|_X,$$

defines an equivalent norm on X.

We can now turn to the main result of this paper, the description of the fine spectrum together with resolvent estimates for generalized Cesàro operators $\mathcal{C}_g$ whose symbol g has an essentially rational derivative. More precisely, we focus on symbols $g\in H(\mathbb{D})$ with $g(0)=0$ and

$$g'(z)=\sum_{k=1}^n\frac{a_k}{1-\bar{b}_kz}+h(z),\tag{5.22}$$

where $h \in H_3^\infty$. Note that if all coefficients $a_1, \dots, a_n$ are zero the corresponding generalized Cesàro operator $\mathcal{C}_g$ is compact by Theorem 4.1, so that its spectrum consists only of eigenvalues and is essentially described in Section 2. More precisely, in this case we have

$$\sigma(\mathcal{C}_g) = \left\{ \frac{g'(0)}{n} \colon n \in \mathbb{Z}^+ \right\} \cup \{0\}.$$

For this reason we shall assume that all coefficients $a_1, \dots, a_n$ are nonzero and that the points $b_1, \dots, b_n \in \partial\mathbb{D}$ are distinct.

Theorem 5.1. *Let X be a Banach space of analytic functions in the unit disc that satisfies* (A1)–(A5(n)) *with some constant $\gamma > 0$. Let $g \in H(\mathbb{D})$ with $g(0) = 0$ such that g' has the form* (5.22) *with $a_1, \dots, a_n \neq 0$ and $b_1, \dots, b_n \in \partial\mathbb{D}$ distinct.*

(i) *The point spectrum $\sigma_p(\mathcal{C}_g)$ is void if $g'(0) = 0$ and if $g'(0) \neq 0$ then*

$$\sigma_p(\mathcal{C}_g) = \left\{ \frac{g'(0)}{k} \colon k \in \mathbb{Z}^+,\ Re\,\frac{ka_j}{g'(0)} < \gamma,\ (1-\xi)^{-\frac{ka_j}{g'(0)}} \in X,\ 1 \leqslant j \leqslant n \right\}.$$

(ii) $\sigma(\mathcal{C}) = \sigma_p(\mathcal{C}) \cup (\bigcup_{j=1}^n \{\lambda \in \mathbb{C}\colon |\lambda - \frac{a_j}{2\gamma}| \leqslant \frac{|a_j|}{2\gamma}\})$.

(iii) $\sigma_e(\mathcal{C}) = \{\lambda \in \mathbb{C}\colon |\lambda - \frac{a_j}{2\gamma}| = \frac{|a_j|}{2\gamma}\}$ *and for $\lambda \in \mathbb{C} \setminus \sigma_e(\mathcal{C}_g)$, the Fredholm index of $\lambda I - \mathcal{C}_g$ is given by*

$$\operatorname{ind}(\lambda I - \mathcal{C}_g) = \sum_{j=1}^n \chi_{\Delta_j}(\lambda),$$

where $\Delta_j = \{\zeta \in \mathbb{C}\colon |\zeta - \frac{a_j}{2\gamma}| < \frac{|a_j|}{2\gamma}\}$, and χ_{Δ_j} denotes its characteristic function.

(iv) *If $T(\lambda) : (\lambda I - \mathcal{C}_g)X \to X$ denotes the left inverse of $\lambda I - \mathcal{C}_g$, $\lambda \in \mathbb{C} \setminus (\sigma_e(\mathcal{C}_g) \cup \sigma_p(\mathcal{C}_g))$ then $\lambda \to \|T(\lambda)\|$ is locally integrable on $\mathbb{C} \setminus \{0\}$.*

Proof. We will use repeatedly the following simple observations. If $\mathcal{F}$ is the family of conformal maps given by (A5(n)) corresponding to the points $b_1, \dots, b_n$, then based on the invariance under composition with rotations, we can assume without loss of generality that every $\phi \in \mathcal{F}$ satisfies

$$\phi^{-1}\bigl(\{b_1, \dots, b_n\}\bigr) = \{1\}.$$

If $\phi \in \mathcal{F}$, $\phi(1) = b_j$ then the function $\frac{1-\overline{b_j}\phi}{1-\xi}$ has a continuous, zero-free extension to $\overline{\mathbb{D}}$, and satisfies

$$\frac{1-\overline{b_j}\phi(z)}{1-z} = \int_0^1 \phi'\bigl(1 + t(z-1)\bigr)\,dt, \qquad \left|\arg \frac{1-\overline{b_j}\phi(z)}{1-z}\right| < \pi.$$

Then any complex power of this function belongs to H_3^∞. Thus,

$$\left\| \left(\frac{1-\overline{b_j}\phi}{1-\xi} \right)^\tau \right\|_{3,\infty}, \quad \left\| (1-\overline{b_k}\phi)^\tau \right\|_{3,\infty}, \quad 1 \leqslant k \leqslant n, \quad \left\| (\phi/\xi)^\tau \right\|_{3,\infty},$$

are locally bounded functions of $\tau \in \mathbb{C}$.

Finally, the function u given by (2.3) is

$$u(z) = z^{g'(0)} h_1(z) \prod_{j=1}^{n} (1-\overline{b_j}z)^{-a_j}, \tag{5.23}$$

where $h_1(z) = \exp(\int_0^z \frac{h(\zeta)-h(0)}{\zeta}\,d\zeta)$. By the assumption that $h \in H_3^\infty$ it follows that $h_1^\tau \in H_3^\infty$ for any complex τ, and that $\tau \to \|h_1^\tau\|_{3,\infty}$ is locally bounded on $\mathbb{C}$.

(i) From (A5(n)) and the above considerations it follows immediately that if $g'(0) \neq 0$, and $k \in \mathbb{Z}^+$ then $u^{\frac{k}{g'(0)}} \in X$ if and only if $(1-\xi)^{-\frac{ka_j}{g'(0)}} \in X$, $1 \leqslant j \leqslant n$. Moreover, such an integer k must satisfy $Re\,\frac{ka_j}{g'(0)} < \gamma$, because the assumptions in the theorem imply that X is contained in $A_0^{-\gamma}$. Indeed, since X is continuously contained in $A^{-\gamma}$ there exists $c_1 > 0$ such that for every $f \in X$ and every polynomial q

$$\limsup_{|z|\to 1} \big(1-|z|\big)^\gamma \big|f(z)\big| = \limsup_{|z|\to 1} \big(1-|z|\big)^\gamma \big|f(z)-q(z)\big| \leqslant c_1 \|f-q\|_X,$$

and the claim follows by (A4). The result now follows from Corollary 2.1(i).

In order to prove the remaining assertions we are going to relate the operator R_λ, $\lambda \neq 0$, defined in Proposition 2.1, to the Cesàro-like operators considered in the previous section. For $m > Re\,\frac{1}{\lambda} - 1$, $f \in X_m$ and $\phi \in \mathcal{F}$ with $\phi(1) = b_j$, we have $f \circ \phi \in X_m$ and

$$\begin{aligned}
\lambda^2 (R_\lambda f) \circ \phi(z) &= \lambda f \circ \phi(z) + \frac{(u\circ\phi)^{\frac{1}{\lambda}}}{\phi(z)} \int_0^{\phi(z)} u^{-\frac{1}{\lambda}}(\zeta) g'(\zeta) f(\zeta)\,d\zeta \\
&= \lambda f \circ \phi(z) + \frac{(u\circ\phi)^{\frac{1}{\lambda}}}{\phi(z)} \int_0^{z} (u\circ\phi)^{-\frac{1}{\lambda}}(\zeta) (g\circ\phi)'(\zeta) f\circ\phi(\zeta)\,d\zeta \\
&= \lambda f \circ \phi(z) + z^{\frac{g'(0)}{\lambda}-1} (1-z)^{-\frac{a_j}{\lambda}} v_\lambda(z) \int_0^{z} \zeta^{-\frac{g'(0)}{\lambda}} (1-\zeta)^{\frac{a_j}{\lambda}-1} w_\lambda(\zeta) f\circ\phi(\zeta)\,d\zeta \\
&= \lambda f \circ \phi(z) + v_\lambda^\phi(z) T_{\mu,\nu_j}\big(w_\lambda^\phi f\circ\phi\big)(z),
\end{aligned}$$

where $\mu = \frac{g'(0)}{\lambda}$, $\nu_j = \frac{a_j}{\lambda}$, and

$$v_\lambda^\phi(z) = \frac{(u\circ\phi)^{\frac{1}{\lambda}} (1-z)^{\frac{a_j}{\lambda}}}{z^{\frac{g'(0)}{\lambda}-1}\phi(z)}, \qquad w_\lambda^\phi(z) = \frac{(u\circ\phi)^{-\frac{1}{\lambda}} (1-z)^{-\frac{a_j}{\lambda}}}{z^{\frac{g'(0)}{\lambda}}} (1-z)(g\circ\phi)'(z).$$

By the remarks made at the beginning of the proof, $v_\lambda^\phi, w_\lambda^\phi \in H_3^\infty$ and $\|v_\lambda^\phi\|_{3,\infty}, \|w_\lambda^\phi\|_{3,\infty}$ are locally bounded functions of $\lambda \in \mathbb{C} \setminus \{0\}$. The last equality above can be then written as

$$\lambda^2 (R_\lambda f) \circ \phi = \lambda C_\phi f + M_{v_\lambda^\phi} T_{\mu,\nu_j} M_{w_\lambda^\phi} C_\phi f. \tag{5.24}$$

(ii)–(iv) If $|\lambda - \frac{a_j}{2\gamma}| > \frac{|a_j|}{2\gamma}$ for $1 \leqslant j \leqslant n$, then

$$Re\,\nu_j = Re\,\frac{a_j}{\lambda} < \gamma, \quad 1 \leqslant j \leqslant n,$$

and by Lemma 4.1(i), $T_{\mu,\nu}$ defines a bounded operator on X_m for sufficiently large m. Then by (5.24) and (A5(n)) it follows that R_λ is bounded on X_m.

Assume that $|\lambda - \frac{a_j}{2\gamma}| < \frac{|a_j|}{2\gamma}$ for $j \in J$ with $\emptyset \neq J \subset \{1, \ldots, n\}$, and recall from Proposition 2.2 that

$$(\lambda I - \mathcal{C}_g) X_m \subset \bigcap_{j \in J} \ker y_{b_j,\lambda}.$$

If $\phi \in \mathcal{F}$ with $\phi(1) = b_j$, $j \notin J$ then by the previous argument, $(R_\lambda f) \circ \phi \in X$, whenever $f \in X_m$ with m sufficiently large. If $\phi \in \mathcal{F}$ with $\phi(1) = b_j$, $j \in J$ then by and a direct computation based on a change of variable we see that $f \in \ker y_{b_j,\lambda}$ if and only if

$$\int_0^1 t^{-\mu}(1-t)^{\nu-1} w_\lambda^\phi(t) f \circ \phi(t)\, dt = 0.$$

Then Lemma 4.1(ii) shows that for sufficiently large m and $f \in \bigcap_{j \in J} \ker y_{b_j,\lambda}$ we have $(R_\lambda f) \circ \phi \in X$ for all such $\phi \in \mathcal{F}$. Thus if m is large enough, then

$$(\lambda I - \mathcal{C}_g) X_m \subset \bigcap_{j \in J} \ker y_{b_j,\lambda},$$

and by (i) $\lambda I - \mathcal{C}_g$ is injective in this case. For such m we conclude that if

$$\lambda \notin \bigcup_{j=1}^n \partial \Delta_j,$$

then $(\lambda I - \mathcal{C}_g)|X_m$ is Fredholm with

$$\operatorname{ind}(\lambda I - \mathcal{C}_g) = \sum_{j=1}^n \chi_{\Delta_j}(\lambda).$$

Then by a well-known result (see for example [15, p. 285, Proposition 3.7.1]) the same holds true for $\lambda I - \mathcal{C}_g$, which proves (ii) and (iii).

Finally, from the assumptions in Lemma 4.1 we see that if λ lies in a compact $K \subset \mathbb{C} \setminus \{0\}$ then we can choose a fixed integer $m(K)$ such that the above considerations hold for all $\lambda \in K$ and $m = m(K)$. Then this lemma combined with (5.24) and (A5(n)) gives an estimate of the form

$$\|R_\lambda f\| \leqslant \max_{1\leqslant j\leqslant n} \frac{c(K,X)}{Re\,\nu_j - \gamma} \leqslant \frac{c'(K,X)}{\operatorname{dist}(\lambda, \bigcup_{j=1}^n \partial\Delta_j)} = \frac{c'(K,X)}{\operatorname{dist}(\lambda, \sigma_e(\mathcal{C}_g))}, \tag{5.25}$$

for all $f \in (\lambda I - \mathcal{C}_g)X_{m(K)}$. We now apply Corollary 2.1(ii) with

$$m = m(K), \qquad r_{\lambda,m} = \frac{c'(K,X)}{\operatorname{dist}(\lambda, \sigma_e(\mathcal{C}_g))},$$

and Lemma 2.1 to conclude that if $\lambda \in \mathbb{C} \setminus \sigma_e(\mathcal{C}_g)$ and $\lambda \neq \frac{g'(0)}{k}$, $k \in \mathbb{Z}^+$, then the left inverse $T(\lambda) : (\lambda I - \mathcal{C}_g)X \to X$ of $\lambda I - \mathcal{C}_g$ satisfies

$$\|T(\lambda)\| \leqslant \frac{c''(K,X)}{\operatorname{dist}(\lambda, \sigma_e(\mathcal{C}_g))} \prod_{j=0}^{m(K)-1} \frac{1}{|\lambda - \frac{g'(0)}{j+1}|}.$$

Now general operator theory shows that $\|T(\lambda)\|$ is continuous on $\mathbb{C} \setminus \sigma_e(\mathcal{C}_g)$, and (iv) follows. □

A similar result holds for the space $\tilde{X}$. The proof is identical to the one above.

Corollary 5.1. *Let X be a Banach space of analytic functions in the unit disc that satisfies* (A1)–(A5(n)) *with some constant $\gamma > 0$. Then the conclusions in Theorem* 5.1(ii)–(iv) *continue to hold when X is replaced by $\tilde{X}$. Moreover, the point spectrum $\sigma_p(\mathcal{C}_g)$ is void if $g'(0) = 0$ and if $g'(0) \neq 0$ then*

$$\sigma_p(\mathcal{C}_g|\tilde{X}) = \left\{ \frac{g'(0)}{k} \colon k \in \mathbb{Z}^+,\ Re\,\frac{ka_j}{g'(0)} \leqslant \gamma,\ (1-\xi)^{-\frac{ka_j}{g'(0)}} \in \tilde{X},\ 1 \leqslant j \leqslant n \right\}.$$

We should also point out that for $n = 1$ the above results hold for every space X that satisfies (A1)–(A4), since the assumption (A5(1)) is automatically fulfilled.

Finally, we can use the resolvent estimate (iv) to show that all generalized Cesàro operators considered in this section have decomposable extensions. This is a consequence of the following general result.

Proposition 5.2. *Let X be a Banach space $S \in \mathcal{B}(X)$. Assume that for almost every $\lambda \in \mathbb{C}$ there exists $T(\lambda) \in \mathcal{B}((\lambda I - S)X, X)$ such that $\|T(\cdot)\|$ is locally integrable on $\mathbb{C} \setminus \{0\}$ and $T(\lambda)(\lambda I - S) = I$ a.e. Then S has a decomposable extension.*

Proof. According to the Albrecht–Eschmeier criterion (see [15, p. 138]) it will be sufficient to show that S has Bishop's property (β), i.e. for every open set $U \subset \mathbb{C}$, a sequence (f_n) of

X-valued analytic functions in U with the property that

$$z \to (S - z) f_n(z), \tag{5.26}$$

converges to zero uniformly on compact subsets of U, converges itself to zero uniformly on compact subsets of U.

A compact subset K of the open set $U \subset \mathbb{C}$ can be covered by a finite unions of discs D_j, $1 \leqslant j \leqslant N$, centered at $z_j \in K$, of radius $r_j > 0$ such that

$$\{|z - z_j| \leqslant 3r_j\} \subset U, \qquad \{2r_j \leqslant |z - z_j| \leqslant 3r_j\} \subset U \setminus \{0\}.$$

If $f : U \to X$ is analytic, then $\|f\|$ is subharmonic in U, hence it satisfies for $1 \leqslant j \leqslant N$

$$\begin{aligned} \sup_{z \in D_j} \|f(z)\| &\leqslant \frac{4}{5\pi r_j^2} \int\limits_{2r_j<|\zeta - z_j|<3r_j} \|f(\zeta)\| \, dA(\zeta) \\ &\leqslant \frac{4}{5\pi r_j^2} \int\limits_{2r_j<|\zeta - z_j|<3r_j} \|T(\zeta)\| \|(S-\zeta) f(\zeta)\| \, dA(\zeta) \\ &\leqslant \sup_{2r_j \leqslant |z-z_j| \leqslant 3r_j} \|(S-z) f(z)\| \frac{4}{5\pi r_j^2} \int\limits_{2r_j<|\zeta - z_j|<3r_j} \|T(\zeta)\| \, dA(\zeta). \end{aligned}$$

Thus if (f_n) has the property that the sequence given in (5.26) converges to zero uniformly on K, the above estimate shows that (f_n) converges uniformly to zero on K, and the result follows. □

Corollary 5.2. *If X is a Banach space of analytic functions in the unit disc that satisfies* (A1)–(A5(n))*, then every generalized Cesàro operator of the form* (5.22) *is subdecomposable on X and $\tilde{X}$.*

Acknowledgment

We would like to thank the referee for useful suggestions and for carefully reading the manuscript.

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