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ON A CLASS OF BOUNDED UNIVALENT FUNCTIONS

BY

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Abstract. The object of the present paper is to discuss some properties of the class of strongly starlike functions of order γ and type β, $(0<\gamma\leq 1, 0\leq\beta<1)$ in the open unit disk, such as sharp bounds of coefficients, the radius of convexity. Convolution and integral operators are also discussed for this class.

1. Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta=\{z:|z|<1\}$. A function $f(z)$ belonging to A is said to be strongly starlike of order γ and type β in Δ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\beta\right)\right|<\frac{\pi\gamma}{2}, \qquad (z\in\Delta) \tag{1.2}$$

for some $\beta\,(0\leq\beta<1)$ and $\gamma\,(0<\gamma\leq 1)$, and denoted by $f(z)\in\bar{S}^*_\beta(\gamma)$ ([9]), with $\bar{S}^*_0(1)\equiv S^*$ the class of function that are starlike. Note that $\bar{S}^*_0(\gamma)\equiv\bar{S}^*(\gamma)$, the class of strongly starlike functions which was introducd by Brannan and Kirwan [3]. For additional results on $\bar{S}^*(\gamma)$, see [3], [7].

Nunokawa et al. [9] obtained a sufficient condition for function in A to be in $\bar{S}^*_\beta(\gamma)$. Sharp bounds of the coefficients a_2 and a_3 were given in [2].

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In this paper we determine the sharp bounds of a_4 and the radius of convexity. Also, we state results concerning convolution and integral operator.

2. Preliminary Results

In order to derive the properties of the class $\bar{S}^*_\beta(\gamma)$, we need the following lemmas.

Lemma 1.([10]) *If $p(z)$ is in P, the class of all functions $p(z)$ analytic in Δ for which $Re\{p(z)\} > 0$ and $p(z) = 1 + c_1 z + c_2 z + \cdots$, then*

$$|c_k| \le 2. \tag{2.1}$$

Lemma 2.([5]) *If $p(z)$ is in P, then*

$$\begin{aligned} \left|c_1^2 - c_2\right| &\le 2 \\ \left|c_1^3 - 2c_2c_1 + c_3\right| &\le 2. \end{aligned} \tag{2.2}$$

Lemma 3.([6]) *If $p(z)$ is in P and s is a natural number, then*

$$|c_n - c_{n-1}c_s| \le 2\ , \qquad n \ge 5, \quad n = 1, 2, 3, \ldots. \tag{2.3}$$

Lemma 4.([4, 11]) *If $p(z)$ is analytic in Δ and $p(0) = 1$, then for any complex number η such that $Re\,\eta \ge 0$, we have*

$$\left|q(z) + \frac{1}{\eta} z q'(z)\right| \le \lambda \quad \textit{implies} \quad |q(z) - 1| < \frac{\lambda\,|\eta|}{|1+\eta|}, z \in \Delta.$$

Lemma 5.([2]) *Let $\mu > 0$, $0 \le \beta < 1$ and $0 < \gamma \le 1$. If $f \in A$ satisfies*

$$\left|\left(\frac{zf'(z)}{f(z)} - \beta\right)\left(\frac{f(z)}{z}\right)^{\mu} - (1-\beta)\right| < \frac{(1-\beta)\,(\mu\,(1-\beta)+1)\sin(\pi\gamma/2)}{\left|\mu(1-\beta) + (\mu\,(1-\beta)+1)\,e^{i\pi\gamma/2}\right|},\ z \in \Delta, \tag{2.4}$$

then $f(z) \in \bar{S}^*_\beta(\gamma)$.

Lemma 6.([14]) *If ϕ a convex function and if $g \in A$ is starlike in Δ, then the function $(\phi * gF)\,/\,(\phi * g)$ takes values in the convex hull of $F(\Delta)$ for every function F in A.*

Lemma 7.([7]) *Let $p(z) \in P$. Then we have the following sharp estimates in Δ;*

$$\left(1-|z|^2\right)|p'(z)| \le 2\operatorname{Re} p(z)$$

and

$$\left|\frac{1+|z|^2}{1-|z|^2} - p(z)^{\gamma}\right| \le \frac{1+|z|^2}{1-|z|^2} - \left(\frac{1-|z|}{1+|z|}\right)^{\gamma}, \qquad 0<\gamma\le 1.$$

3. Main Results

Theorem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $\bar{S}^*_\beta(\gamma)$ $(0<\gamma\le 1,$ $0\le\beta<1)$. Then*

$$|a_4| \le \begin{cases} \frac{2}{3}\gamma(1-\beta)\ ,\ 0<\gamma\le\sqrt{2/\left(17-15\beta+6\beta^2\right)}; \\ \frac{2}{9}\gamma(1-\beta)\left[17\alpha^2+1-3\beta\gamma^2(5-2\beta)\right]\ ,\ \sqrt{2/\left(17-15\beta+6\beta^2\right)}\le\gamma\le 1. \end{cases}$$

The result is sharp.

Proof. For $f(z) \in \bar{S}^*_\beta(\gamma)$, there is $p(z) \in P$ such that $zf'(z)/f(z) = (1-\beta)p(z)^\gamma + \beta$. Assume that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then direct calculation gives us

$$\frac{1}{1-\beta}a_2 = \gamma p_1$$

$$2a_3 = \gamma(1-\beta)\left[p_2 + \frac{(3-2\beta)\gamma-1}{2}p_1^2\right]$$

and

$$3a_4 = \gamma(1-\beta)\Big[p_3 + \left(\frac{5-3\beta}{2}\gamma - 1\right)p_2p_1 + \frac{1}{12}\left\{17\gamma^2 - 15\gamma + 4 - 3\beta\gamma((7-2\beta)\gamma-3)\right\}p_1^3\Big]. \qquad (3.1)$$

If $0<\gamma\le\sqrt{2/\left(17-15\beta+6\beta^2\right)}$, then

$$\frac{3}{\gamma(1-\beta)}a_4 = \frac{1}{12}\left[17\gamma^2 - 15\gamma + 4 - 3\beta\gamma\left((5-2\beta)\gamma-3\right)\right]\left(p_3 - 2p_2p_1 + p_1^3\right)$$
$$+\frac{1}{3}\left[1 - \frac{17}{2}\gamma^2 + \frac{3}{2}\beta\gamma^2(5-2\beta)\right](p_{3-}p_2p_1)$$
$$+\frac{1}{12}\left[17\gamma^2 + 15\gamma + 4 - 3\beta\gamma\left((5-2\beta)\gamma+3\right)\right]p_3.$$

By using Lemmas 1-3, we get that $|a_4| \leq 2\gamma(1-\beta)/3$.

If $\sqrt{2/\left(17-15\beta+\gamma\beta^2\right)} \leq \gamma \leq 1$, then

$$\begin{aligned}\frac{3}{(1-\beta)\gamma}a_4 &= \frac{1}{12}\left[17\gamma^2-15\gamma+4-3\beta\gamma\left((5-2\beta)\gamma-3\right)\right]\left(p_3-2p_2p_1+p_1^3\right)\\ &+\frac{1}{3}\left[\frac{17}{2}\gamma^2-1-\frac{3}{2}\beta\gamma^2\left(5-2\beta\right)\right]p_2p_1\\ &+\frac{1}{12}\left[8+15\gamma-17\gamma^2+3\beta\gamma\left((5-2\beta)\gamma-3\right)\right]p_3.\end{aligned}$$

For Lemmas 1 and 2, we have that $|a_4| \leq \frac{2}{9}\gamma(1-\beta)\left[17\gamma^2+1-3\beta\gamma^2(5-2\beta)\right]$. When $0<\gamma<\sqrt{2/\left(17-15\beta+6\beta^2\right)}$, equality holds if and only if $|p_3|=2$ and $|p_1|=|p_2|=0$, that is,

$$\frac{zf'(z)}{f(z)} = (1-\beta)\left[\frac{1+\varepsilon z^3}{1-\varepsilon z^3}\right]^\gamma+\beta,\ \text{where}\ |\varepsilon|=1.$$

If $\sqrt{2/\left(17-15\beta+6\beta^2\right)} \leq \gamma \leq 1$, then inequality becomes equality if and only if $|p_1|=2$. Hence

$$\frac{zf'(z)}{f(z)} = (1-\beta)\left[\frac{1+\varepsilon z}{1-\varepsilon z}\right]^\gamma+\beta\,,\ \text{where}\ |\varepsilon|=1.$$

Theorem 2. *Let* $f(z)=z-cz^n$, $c \geqslant 0$, $\alpha \geqslant 0$, $0 \leq \beta < 1$ *and* $0<\gamma \leq 1$. *Then*

(i) $\mathrm{Re}\left\{\frac{zf(z)}{f(z)}-\beta\right\} > \alpha \Rightarrow f \in \bar{S}^*_\beta\left(\frac{2}{\pi}\sin^{-1}\left(\frac{1-\beta-\alpha}{1-\beta}\right)\right)$.

(ii) $f \in \bar{S}^*_\beta(\gamma) \Rightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-\beta\right\} > \alpha$, *where*

$$\alpha = \left[3-\sqrt{1+8\sin^2\left(\pi\gamma/2\right)}\right]/2\left[1+\sin\left(\pi\gamma/2\right)\right].$$

The result is sharp with extremal function $f(z)=z-\frac{1-\beta-\alpha}{2-\beta-\alpha}z^2$.

Proof. (i) Since

$$Re\left\{\frac{zf'(z)}{f(z)}-\beta\right\} = \mathrm{Re}\left\{\frac{(1-\beta)z-(n-\beta)cz^n}{z-cz^n}\right\} > \alpha\,, \quad (|z|<1)\,, \tag{3.2}$$

choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}-\beta$ is real. Upon clearing the denominator in (3.2) and letting $z \to 1$ through real values, we obtain

$$(1-\beta)-(n-\beta)c \geq \alpha\left(1-c\right).$$

Thus,

$$(n-\alpha-\beta)\,c \leq 1-\beta-\alpha. \tag{3.3}$$

Therefore

$$\begin{aligned}\left|\frac{1}{1-\beta}\left(\frac{zf'(z)}{f(z)}-\beta\right)-1\right| &= \frac{1}{1-\beta}\left|\frac{z-ncz^n}{z-cz^n}-1\right| \\ &\leq \frac{1}{1-\beta}\frac{(n-1)c}{1-c} \\ &\leq \frac{1-\beta-\alpha}{1-\beta}.\end{aligned}$$

The last inequality being equivalent to (3.3), yields

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\beta\right)\right| \leq \sin^{-1}\left(\frac{1-\beta-\alpha}{1-\beta}\right).$$

This proves that $f \in \bar{S}^*_\beta\left(\frac{2}{\pi}\sin^{-1}\left(\frac{1-\beta-\alpha}{1-\beta}\right)\right)$.

To show sharpness, set

$$f(z) = z - tz^n, \quad \left[t = \frac{1-\beta-\alpha}{n-\beta-\alpha}\right].$$

Then

$$\frac{zf'(z)}{f(z)} - \beta = \frac{(1-\beta)-(n-\beta)tz^{n-1}}{1-tz^{n-1}} = M\left(z^{n-1}\right),$$

where $M(z)$ is a Möbius transformation mapping Δ onto the interior of the disk having its diameter on the real axis with end points $\frac{(1-\beta)-(n-\beta)t}{1-t}$ and $\frac{(1-\beta)+(n-\beta)t}{1+t}$. This disk has center $\frac{(1-\beta)-(n-\beta)t^2}{1-t^2}$ and radius $\frac{(n-1)t}{1-t^2}$. Thus,

$$\begin{aligned}\max_{|z|=1}\left|\arg\left(\frac{zf'(z)}{f(z)}-\beta\right)\right| &= \sin^{-1}\left[\frac{(n-1)t}{(1-\beta)-(n-\beta)\,t^2}\right] \\ &= \sin^{-1}\left[\frac{(1-\beta-\alpha)\,(n-\beta-\alpha)}{(1-\beta)n-\alpha^2+\beta^2-\beta}\right].\end{aligned}$$

Since $\frac{(n-\beta-\alpha)}{(1-\beta)n-\alpha^2+\beta^2-\beta} \to \frac{1}{1-\beta}$ as $n \to \infty$, it follows that $f(z) \notin \bar{S}^*_\beta(\eta)$ for any $\eta < \frac{2}{\pi}\sin^{-1}\left(\frac{1-\beta-\alpha}{1-\beta}\right)$ and $n = n(\eta)$ sufficiently large.

(ii) Setting $\alpha_n = \frac{(1-\beta)-(n-\beta)c}{1-c}$ so that $c = \frac{1-\beta-\alpha_n}{n-\beta-\alpha_n}$, we note that

$$\left|\frac{1}{1-\beta}\left(\frac{zf'(z)}{f(z)}-\beta\right)-1\right| \leq \frac{(n-1)c}{(1-\beta)\,(1-c)}. \tag{3.4}$$

Since $(n-\beta-\alpha_n)c = 1-\beta-\alpha_n$, therefore (3.4) equals $\frac{1-\beta-\alpha_n}{1-\beta}$ which is equivalent to

$$\left|\frac{1}{1-\beta}\left(\frac{zf'(z)}{f(z)}-\beta\right)-1\right| \leq \frac{1-\beta-\alpha_n}{1-\beta},$$

or

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\beta-\alpha_n.$$

Then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \beta+\alpha_n,$$

or

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}-\beta\right) \geq \alpha_n.$$

It suffices to show that $\alpha_n \geq \alpha$ for all n. Proceeding as in the proof of (i), we have

$$\begin{aligned}\max_{|z|=1}\left|\arg\left(\frac{zf'(z)}{f(z)}-\beta\right)\right| &= \sin^{-1}\left[\frac{(n-1)c}{(1-\beta)-(n-\beta)\,c^2}\right]\\ &= \sin^{-1}\left[\frac{(1-\beta-\alpha_n)\,(n-\beta-\alpha_n)}{(1-\beta)n-\alpha_n^2+\beta^2-\beta}\right]\\ &\leq \frac{\pi}{2}\gamma. \end{aligned} \tag{3.5}$$

Solving for α_n, we get

$$\alpha_n \geq \frac{(1-2\beta+n)-\sqrt{(n-1)^2+4\,(n+\beta^2-\beta\,(1+n))\sin^2(\pi\gamma/2)}}{2\,(1+\sin(\pi\gamma/2))} = \eta_n. \tag{3.6}$$

Since η_n is an increasing function of n, it follows from (3.6) that $\alpha_n \geq \eta_2 = \alpha$.

Corollary 1. *Let $f(z) = z - cz^n$, $c \geqslant 0$. Then $f(z) \in \bar{S}^*_\beta(\gamma)$ if and only if*

$$c \leq \frac{\sqrt{(n-1)^2+4(n-\beta)(1-\beta)\sin^2(\pi\gamma/2)}-(n-1)}{2(n-\beta)^2\sin^2(\pi\gamma/2)}. \tag{3.7}$$

The result is sharp for all n.

Proof. As in Theorem 2, we see from (3.5) that

$$\sin^{-1}\left[\frac{(n-1)c}{(1-\beta)-(n-\beta)\,c^2}\right] \leq \frac{\pi}{2}\gamma.$$

Thus

$$(n-\beta)\sin(\pi\gamma/2)\,c^2+(n-1)\,c-(1-\beta)\sin(\pi\gamma/2)\le 0\ ,$$

which is equivalent to (3.7).

The next theorem shows that a certain integral operator belongs to the class $\bar{S}^*_\beta(\gamma)$.

Theorem 3. *Let* $\mu>0$, $\operatorname{Re} c>-\mu$ *and* $0<\gamma\le 1$. *If* $f\in A$ *satisfies*

$$\left|\left(\frac{zf'(z)}{f(z)}-\beta\right)\left(\frac{f(z)}{z}\right)^{\mu}-(1-\beta)\right| < (1-\beta)\left|\frac{\mu+c+1}{\mu+c}\right|\frac{(\mu(1-\beta)+1)\sin(\pi\gamma/2)}{|\mu(1-\beta)+(mu(1-\beta)+1)e^{i\pi\gamma/2}|},\qquad z\in\Delta, \tag{3.8}$$

then $I_{\mu,c}(f)$ *defined by*

$$[I_{\mu,c}(f)](z)=\left[\frac{\mu+c}{z^c}\int_0^z f^{\mu}(t)\,t^{c-1}dt\right]^{1/\mu}\ \left([I_{\mu,c}(f)](z)/z\neq 0\ \textit{in}\ \Delta\right)$$

is in $\bar{S}^*_\beta(\gamma)$.

Proof. For convenience, we let $F(z)=[I_{\mu,c}(f)](z)$. Consider the function p defined by

$$p(z)=\frac{1}{1-\beta}\left\{\frac{zF'(z)}{F(z)}-\beta\right\}\left(\frac{F(z)}{z}\right)^{\mu},\quad z\in\Delta.$$

Then we easily see that

$$p(z)+\frac{1}{\mu+c}zp'(z)=\frac{1}{1-\beta}\left\{\frac{zf'(z)}{f(z)}-\beta\right\}\left(\frac{f(z)}{z}\right)^{\mu}.$$

Hence using Lemma 4 and (3.8), we deduce that

$$\left|\left\{\frac{zF'(z)}{F(z)}-\beta\right\}\left(\frac{F(z)}{z}\right)^{\mu}-(1-\beta)\right|<\frac{(1-\beta)\,(\mu(1-\beta)+1)\sin(\pi\gamma/2)}{|\mu(1-\beta)+(\mu(1-\beta)+1)\,e^{i\pi\gamma/2}|}.$$

Therefore, using Lemma 4, it follows that the function F is strongly starlike of order γ and typt β.

The next theorem shows that the class $\bar{S}^*_\beta(\gamma)$ is closed under convolution with convex function.

Theorem 4. *Let φ be a normalized $(\varphi(0) = \varphi'(0) - 1 = 0)$ convex function in Δ and $f \in \bar{S}^*_\beta(\gamma)$. Then $\varphi * f \in \bar{S}^*_\beta(\gamma)$.*

Proof. Let

$$G(z) = \frac{1}{1-\beta}\left\{\frac{zf'(z)}{f(z)} - \beta\right\}.$$

Note that for $F(z) = \varphi * f(z)$, we have $zF'(z) = \varphi * (zf'(z))$. Hence, using a result of Lemma 6, we get

$$\begin{aligned} z\frac{F'(z)}{F(z)} &= \varphi * (zf'(z))/(\varphi * f)(z) \\ &= \frac{\varphi * \{(1-\beta)(Gf)(z) + \beta f(z)\}}{(\varphi * f)(z)} \\ &= (1-\beta)\frac{\varphi * (Gf)(z)}{(\varphi * f)(z)} + \beta. \end{aligned}$$

This implies that

$$\frac{1}{1-\beta}\left[\frac{zF'(z)}{F(z)} - \beta\right] = \frac{\varphi * (Gf)(z)}{(\varphi * f)(z)}$$

lies in the convex hull of $G(\Delta)$. Because $\frac{1}{1-\beta}\left[\frac{zF'(z)}{F(z)} - \beta\right]$ is analytic in Δ and $G(\Delta) \subset \Omega \equiv \{w : |\arg w| < \pi\gamma/2\}$, we see that $\frac{1}{1-\beta}\left[\frac{zF'(z)}{F(z)} - \beta\right]$ lies in Ω, that is, $F = \varphi * f \in \bar{S}^*_\beta(\gamma)$.

Now, we give a sharp radius of convexity for the class $\bar{S}^*_\beta(\gamma)$.

Theorem 5. *Let $f \in \bar{S}^*_\beta(\gamma)$. Then f maps $\{z : |z| < r_\gamma\}$ conformally onto a convex region, where r_γ is the unique root of*

$$\{(1-\beta)(1-r)^\gamma + \beta(1+r)^\gamma\}\left((1-r)^\gamma - \frac{\beta}{1-\beta}(1+r)^\gamma\right) - 2\gamma r(1+r)^{\gamma-1}(1-r)^{\gamma-1} = 0,$$

in $(0,1)$.

This is the best possible radius of convexity.

Proof. For $f(z) \in \bar{S}^*_\beta(\gamma)$, there is $p(z) \in P$ such that

$$\frac{zf'(z)}{f(z)} = (1-\beta)p(z)^\gamma + \beta. \tag{3.9}$$

Taking the logarithmic differentiations in both sides of (3.9), we get

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}=\gamma\frac{zp'(z)/p(z)}{1+\mu p(z)^{-\gamma}},$$

where $\mu=\frac{\beta}{1-\beta}$. Then we have

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geq(1-\beta)\operatorname{Re}\ p(z)^{\gamma}+\beta-\gamma\left|\frac{zp'(z)/p(z)}{1+\mu p(z)^{-\gamma}}\right|.$$

From the first inequality in Lemma 7, we have

$$\left|\frac{zp'(z)}{p(z)}\right|\leq\frac{2\,|z|}{1-|z|^{2}},$$

and from the second inequality in Lemma 7, we see that

$$\operatorname{Re}\ \{\ p(z)^{\gamma}\}\geq\left(\frac{1-|z|}{1+|z|}\right)^{\gamma}.$$

Then

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geq(1-\beta)\left(\frac{1-|z|}{1+|z|}\right)^{\gamma}+\beta-\frac{2\gamma\,|z|}{\left(1-|z|^{2}\right)\left(1-\mu\left(\frac{1+|z|}{1-|z|}\right)^{\gamma}\right)}$$
$$=\frac{[(1-\beta)(1-|z|)^{\gamma}+\beta(1+|z|)^{\gamma}][(1-|z|)^{\gamma}-\mu(1+|z|)^{\gamma}]-2\gamma|z|(1-|z|)^{\gamma-1}(1+|z|)^{\gamma-1}}{(1+|z|)^{\gamma}\left((1-|z|)^{\gamma}-\mu\,(1+|z|)^{\gamma}\right)}.$$

Note that

$$\begin{aligned}&\left((1-|z|)^{\gamma}-\mu\,(1+|z|)^{\gamma}\right)\left[(1-\beta)\,(1-|z|)^{\gamma}+\beta\,(1+|z|)^{\gamma}\right]\\&-2\gamma\,|z|\,(1-|z|)^{\gamma-1}\,(1+|z|)^{\gamma-1}\end{aligned}$$

is decreasing in $0\leq|z|<1$, so that there is a unique root of

$$\begin{aligned}&\left((1-|z|)^{\gamma}-\mu\,(1+|z|)^{\gamma}\right)\left[(1-\beta)\,(1-|z|)^{\gamma}+\beta\,(1+|z|)^{\gamma}\right]\\&-2\gamma\,|z|\,(1-|z|)^{\gamma-1}\,(1+|z|)^{\gamma-1}=0,\end{aligned}$$

in (0.1). When $|z|$ is less than this root, the right hand of the above inequality is positive.

Direct calculation of $\operatorname{Re}\left\{1+\frac{zk''\gamma(z)}{k'\gamma(z)}\right\}$ show that the radius is best possible where $k\gamma(z)$ is defined by $k\gamma\,(0)=k'\gamma\,(0)-1=0$ and $\frac{zk\gamma(z)}{k\gamma(z)}=\left(\frac{1+z}{1-z}\right)^{\gamma}+\beta$.

Remark 1. In the case of $\beta = 0$, we find that
Theorem 1 gives the coefficient bounds of a_4 in the class $\bar{S}^*(\gamma)$ which was found by Rosihan and Singh [13].
Theorem 2 gives Theorems 2 and 3 by Ahuja and Silverman [1].
Theorem 3 gives Theorem 4.1 by Ponnusamy and Singh [12].
Theorem 4 gives Theorem 6 in [7].
Theorem 5 gives Theorem 2 in [7].

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