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STARLIKENESS OF A CROSS-PRODUCT OF BESSEL FUNCTIONS

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Dedicated to the memory of Professor Lee Lorch

Abstract. In this paper a necessary and sufficient condition is deduced for the close-to-convexity of a cross product of Bessel and modified Bessel functions of the first kind and their derivatives by using a result of Shah and Trimble about transcendental entire functions with univalent derivatives, the newly discovered power series and infinite product representation of this cross-product, as well as a slightly modified version of a result of Lorch on the monotonicity of the zeros of the cross product with respect to the order.

1. **Introduction and the Main Results**

Let J_ν and I_ν denote the Bessel and modified Bessel functions of the first kind. Motivated by their appearance as eigenvalues in the clamped plate problem for the ball, Ashbaugh and Benguria have conjectured that the positive zeros of the function $z \longmapsto J_\nu(z)I'_\nu(z) - J'_\nu(z)I_\nu(z)$ increase with ν on $\left[-\frac{1}{2},\infty\right)$. Lorch [11] verified this conjecture and presented some other properties of the zeros of the above cross product of Bessel and modified Bessel functions. In this paper we point out that actually the above monotonicity property is valid on $(-1,\infty)$. We are also interested on an application of the above monotonicity property of the zeros of the cross product of Bessel and modified Bessel functions. Namely, our aim is to find a necessary and sufficient condition on the parameter ν such that the normalized form of the above cross product maps the open unit disk into a starlike domain and all of its derivatives are close-to-convex, and hence univalent. It is worth to mention that geometric properties, like univalence, starlikeness, spirallikeness and convexity of Bessel functions were studied in the sixties by Brown [8], and also by Kreyszig and Todd [9]. Note that some other geometric properties of Bessel functions of the first kind were studied later by others, see for example the papers [3, 4, 5, 6, 7, 13, 14] and the references therein. In order to prove our main results we use a result of Shah and Trimble [12] about transcendental entire functions with univalent derivatives and power series and infinite product representations for the above cross product of Bessel and modified Bessel functions of the first kind. Our first main result is the following theorem.

Theorem 1. *The function*

$$z \longmapsto 2^{2\nu} z^{-\frac{\nu}{2}+\frac{3}{4}}\Gamma(\nu+1)\Gamma(\nu+2)\left(J_{\nu+1}(\sqrt[4]{z})I_\nu(\sqrt[4]{z}) + J_\nu(\sqrt[4]{z})I_{\nu+1}(\sqrt[4]{z})\right)$$

is starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex (and hence univalent) there if and only if $\nu \geq \nu^*$, where $\nu^* \simeq -0.9427\ldots$ is the unique root of the next equation on $(-1,\infty)$

$$(\nu-1)J_\nu(1)I_{\nu+1}(1) + (\nu-1)J_{\nu+1}(1)I_\nu(1) = J_\nu(1)I_\nu(1).$$

Motivated by the above result we also consider simply the product $J_\nu(z)I_\nu(z)$ in the next theorem.

Theorem 2. *The function*

$$z \longmapsto 2^{2\nu} z^{-\frac{\nu}{2}+1}\Gamma^2(\nu+1)J_\nu(\sqrt[4]{z})I_\nu(\sqrt[4]{z})$$

is starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex (and hence univalent) there if and only if $\nu \geq \nu^\star$, where $\nu^\star \simeq -0.4336\ldots$ is the unique root of the next equation on $(-1,\infty)$

$$J_{\nu+1}(1)I_\nu(1) - J_\nu(1)I_{\nu+1}(1) = (\nu+1)J_\nu(1)I_\nu(1).$$

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We note that if consider the particular cases

$$J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\cos z,\ J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\sin z$$

and

$$I_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\cosh z,\ I_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\sinh z,$$

then by using the above theorems in particular for $\nu=-\frac{1}{2}$ we obtain that the functions

$$z\longmapsto\frac{1}{2}z^{\frac{3}{4}}\left(\sin\sqrt[4]{z}\cosh\sqrt[4]{z}+\cos\sqrt[4]{z}\sinh\sqrt[4]{z}\right)\quad\text{and}\quad z\longmapsto z\cos\sqrt[4]{z}\cosh\sqrt[4]{z}$$

are starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex (and hence univalent) there.

In order to prove our main results we will need some preliminary results. We note that although Lemma 2, 3 and 4 were deduced to prove Theorem 1, they are also of independent interest and can be applied to solve other problems related to Bessel and modified Bessel functions of the first kind.

The next result of Shah and Trimble [12, Theorem 2] is one of the key tools in the proof of the main results.

Lemma 1. *Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk and $f:\mathbb{D}\to\mathbb{C}$ be an entire function of the form*

$$f(z)=z\prod_{n\geq1}\left(1-\frac{z}{z_n}\right),$$

where all z_n have the same argument and satisfy $|z_n|>1$. Then f is starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex there if and only if the following inequality is valid

$$\sum_{n\geq1}\frac{1}{|z_n|-1}\leq1.$$

It is worth to mention that by using the known recurrence relations $zJ_\nu'(z)-\nu J_\nu(z)=-zJ_{\nu+1}(z)$ and $zI_\nu'(z)-\nu I_\nu(z)=zI_{\nu+1}(z)$ the cross product $J_\nu(z)I_\nu'(z)-J_\nu'(z)I_\nu(z)$ actually can be rewritten as $J_{\nu+1}(z)I_\nu(z)+J_\nu(z)I_{\nu+1}(z)$. In the sequel we will also use the following power series and infinite product representations of the above cross product of Bessel and modified Bessel functions. These results complement the well-known results on Bessel and modified Bessel functions of the first kind and may be of independent interest.

Lemma 2. *If $\nu>-1$ and $z\in\mathbb{C}$, then we have the next power series representation*

$$J_{\nu+1}(z)I_\nu(z)+J_\nu(z)I_{\nu+1}(z)=2\sum_{n\geq0}\frac{(-1)^n\left(\frac{z}{2}\right)^{2\nu+4n+1}}{n!\Gamma(\nu+n+1)\Gamma(\nu+2n+2)}.$$

Lemma 3. *If $\nu>-1$ and $z\in\mathbb{C}$, then we have the next Hadamard factorization*

$$2^{2\nu}z^{-2\nu-1}\Gamma(\nu+1)\Gamma(\nu+2)\left(J_{\nu+1}(z)I_\nu(z)+J_\nu(z)I_{\nu+1}(z)\right)=\prod_{n\geq1}\left(1-\frac{z^4}{\gamma_{\nu,n}^4}\right),$$

where $\gamma_{\nu,n}$ is the nth positive zero of the function $z\longmapsto J_{\nu+1}(z)I_\nu(z)+J_\nu(z)I_{\nu+1}(z)$. Moreover, the zeros $\gamma_{\nu,n}$ satisfy the interlacing inequalities $j_{\nu,n}<\gamma_{\nu,n}<j_{\nu,n+1}$ and $j_{\nu,n}<\gamma_{\nu,n}<j_{\nu+1,n}$ for $n\in\mathbb{N}$ and $\nu>-1$, where $j_{\nu,n}$ stands for the nth positive zero of the Bessel function J_ν.

It is worth to mention that the inequalities $j_{\nu,n}<\gamma_{\nu,n}<j_{\nu+1,n}$ were proved for $\nu\geq-\frac{1}{2}$ by Lorch [11]. To prove our main result in Theorem 1 we will need also the following result. Note that this result was proved also earlier by Lorch [11] for the case $\nu\geq-\frac{1}{2}$. Our proof is just a slight modification of the proof made by Lorch [11].

Lemma 4. *The positive zeros of $z\longmapsto J_\nu(z)I_\nu'(z)-J_\nu'(z)I_\nu(z)$ increase with ν on $(-1,\infty)$.*

2. **Proofs of the Preliminary and Main Results**

In this section our aim is to present the proof of the preliminary and main results.

Proof of Lemma 2. Let us consider the product of Bessel and modified Bessel functions of the first kind

$$\Pi_{\mu,\nu}(z) = \left(\frac{2}{z}\right)^{\mu+\nu} J_\mu(z) I_\nu(z),$$

where $\mu, \nu > -1$ and $z \in \mathbb{C}$. We start with the series representations for both Bessel functions in the manner of Watson [15, p. 147]. Thus

$$\Pi_{\mu,\nu}(z) = \frac{1}{\Gamma(\nu+1)} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n!\Gamma(\mu+n+1)} {}_2F_1(-n, -\mu-n; \nu+1; -1), \tag{2.1}$$

where ${}_2F_1(a, b; c; \cdot)$ stands for the Gaussian hypergeometric function. Indeed, we have

$$\begin{aligned} J_\mu(x) I_\nu(z) &= \left(\frac{z}{2}\right)^{\mu+\nu} \sum_{m\geq 0} \sum_{k\geq 0} \frac{(-1)^m \left(\frac{z}{2}\right)^{2(m+k)}}{m!k!\Gamma(\mu+m+1)\Gamma(\nu+k+1)} \\ &= \left(\frac{z}{2}\right)^{\mu+\nu} \sum_{n\geq 0} \left\{ \sum_{m=0}^{n} \frac{(-1)^m}{(m-n)!m!\Gamma(\mu+m-n+1)\Gamma(\nu+m+1)} \right\} \left(\frac{z}{2}\right)^{2n} \\ &= \left(\frac{z}{2}\right)^{\mu+\nu} \sum_{n\geq 0} \frac{1}{n!} \left\{ \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\Gamma(\mu+n-m+1)\Gamma(\nu+m+1)} \right\} \left(\frac{z}{2}\right)^{2n}. \end{aligned}$$

As to the proof of (2.1), it follows immediately from

$$\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\Gamma(\mu+n-m+1)\,\Gamma(\nu+m+1)} = \frac{{}_2F_1(-n, -\mu-n; \nu+1; -1)}{\Gamma(\nu+1)\Gamma(\mu+n+1)}.$$

Now, consider the Jacobi polynomial (or hypergeometric polynomial) [2, p. 99]

$$P_n^{(\alpha,\beta)}(z) = \begin{cases} \dfrac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \dfrac{1-z}{2}\right) \\ \dfrac{(1+\alpha)_n}{n!} \left(\dfrac{1+z}{2}\right)^n {}_2F_1\left(-n, -n-\beta; \alpha+1; \dfrac{z-1}{z+1}\right) \end{cases},$$

where the latter expression [1, p. 779] is obtained from the first definition via the Pfaff transform of the hypergeometric term. This in turn implies that

$$\Pi_{\mu,\nu}(z) = \sum_{n\geq 0} \frac{(-1)^n P_n^{(\nu,\mu)}(0)}{2^n \Gamma(\mu+n+1)\Gamma(\nu+n+1)} z^{2n}.$$

Now, we are looking for a closed form in the case of the symmetric sum

$$\Pi_{\nu+1,\nu}(z) + \Pi_{\nu,\nu+1}(z) = \sum_{n\geq 0} \frac{(-1)^n \left[P_n^{(\nu,\nu+1)}(0) + P_n^{(\nu+1,\nu)}(0)\right]}{2^n \Gamma(\nu+n+1)\Gamma(\nu+n+2)} z^{2n}. \tag{2.2}$$

By using the recurrence relation [1, p. 782]

$$(1-z)P_n^{(\alpha+1,\beta)}(z) + (1+z)P_n^{(\alpha,\beta+1)}(z) = 2P_n^{(\alpha,\beta)}(z)$$

and the initial value

$$P_n^{(\nu,\nu)}(0) = \frac{\sqrt{\pi}\,\Gamma(\nu+n+1)}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right) n!\,\Gamma(\nu+\frac{n}{2}+1)},$$

together with

$$\frac{1}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)} = \begin{cases} \dfrac{(-1)^k \prod_{j=1}^{k}(2j-1)}{\sqrt{\pi}\,2^k}, & \text{if } n=2k,\ k\in\mathbb{N} \\ 0, & \text{if } n=2k-1,\ k\in\mathbb{N} \end{cases},$$

for all $k \in \mathbb{N}_0$ we conclude that

$$P_n^{(\nu+1,\nu)}(0) + P_n^{(\nu,\nu+1)}(0) = 2\,P_n^{(\nu,\nu)}(0) = \begin{cases} \dfrac{2\sqrt{\pi}\,(-1)^k\,(2k-1)!!\,\Gamma(\nu+2k+1)}{(2k)!\,2^k\,\Gamma\big(\nu+k+1\big)}, & \text{if} \quad n = 2k \\ 0, & \text{if} \quad n = 2k-1 \end{cases}.$$

This in conjunction with (2.2) gives

$$\begin{aligned} \Pi_{\nu+1,\nu}(z) + \Pi_{\nu,\nu+1}(z) &= 2\sum_{n\geq 0} \frac{(-1)^n\,(2n-1)!!\Gamma(\nu+2n+1)z^{4n}}{2^{3n}\,\Gamma(\nu+2n+1)\Gamma(\nu+2n+2)\,(2n)!\,\Gamma\,(\nu+n+1)} \\ &= 2\sum_{n\geq 0} \frac{(-1)^n}{\Gamma(\nu+n+1)\Gamma(\nu+2n+2)\,n!}\left(\frac{z}{2}\right)^{4n}. \end{aligned}$$

Thus we have

$$J_{\nu+1}(z)I_\nu(z) + J_\nu(z)I_{\nu+1}(z) = 2\left(\frac{z}{2}\right)^{2\nu+1}\sum_{n\geq 0}\frac{(-1)^n\left(\frac{z}{2}\right)^{4n}}{n!\Gamma(\nu+n+1)\Gamma(\nu+2n+2)}.$$

□

Proof of Lemma 3. By following Lorch's approach [11] we first show that for $\nu > -1$ the zeros $\gamma_{\nu,n}$ exist. By using the discussion in the introduction after Lemma 1 it is clear that $\gamma_{\nu,n}$ is also the nth positive zero of the function

$$\varphi_\nu(z) = \frac{J_{\nu+1}(z)}{J_\nu(z)} + \frac{I_{\nu+1}(z)}{I_\nu(z)}.$$

In view of the Mittag-Leffler expansions

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n\geq 1}\frac{2z}{j_{\nu,n}^2 - z^2}, \quad \frac{I_{\nu+1}(z)}{I_\nu(z)} = \sum_{n\geq 1}\frac{2z}{j_{\nu,n}^2 + z^2}$$

it follows that

$$\varphi_\nu'(z) = \sum_{n\geq 1}\frac{4j_{\nu,n}^2(3j_{\nu,n}^4 + z^4)}{(j_{\nu,n}^4 - z^4)^2} > 0$$

for $z \in \Delta = (0, j_{\nu,1}) \cup (j_{\nu,1}, j_{\nu,2}) \cup \ldots \cup (j_{\nu,n}, j_{\nu,n+1}) \cup \ldots$ and $\nu > -1$. On the other hand we have the following limits $\lim_{z\searrow 0}\varphi_\nu(z) = 0$, $\lim_{z\nearrow j_{\nu,1}}\varphi_\nu(z) = \infty$, $\lim_{z\searrow j_{\nu,1}}\varphi_\nu(z) = -\infty$, $\lim_{z\nearrow j_{\nu,2}}\varphi_\nu(z) = \infty$, ..., which together with the above monotonicity property implies that $j_{\nu,n} < \gamma_{\nu,n} < j_{\nu,n+1}$ for $n \in \mathbb{N}$ and $\nu > -1$, that is, the zeros $\gamma_{\nu,n}$ and $j_{\nu,n}$ interlace. Moreover, since $\varphi_\nu(j_{\nu+1,n}) = I_{\nu+1}(j_{\nu+1,n})/I_\nu(j_{\nu+1,n}) > 0$ for $n \in \mathbb{N}$ and $\nu > -1$, it follows that $\gamma_{\nu,n} < j_{\nu+1,n}$ for $n \in \mathbb{N}$ and $\nu > -1$. With these we proved the existence of the zeros and also their bounds.

Now, let us focus on the infinite product. We will show that for $\nu > -1$ and $z \in \mathbb{C}$ we have

$$2^{2\nu}z^{-\nu-\frac{1}{2}}\Gamma(\nu+1)\Gamma(\nu+2)\left(J_{\nu+1}(\sqrt{z})I_\nu(\sqrt{z}) + J_\nu(\sqrt{z})I_{\nu+1}(\sqrt{z})\right) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\gamma_{\nu,n}^4}\right). \tag{2.3}$$

This is equivalent to the first statement of Lemma 3. In view of Lemma 2 we have

$$\frac{2^{2\nu}}{z^{\nu+\frac{1}{2}}}\Gamma(\nu+1)\Gamma(\nu+2)\left(J_{\nu+1}(\sqrt{z})I_\nu(\sqrt{z}) + J_\nu(\sqrt{z})I_{\nu+1}(\sqrt{z})\right) = 1 + \sum_{n\geq 1}\frac{(-1)^n\Gamma(\nu+1)\Gamma(\nu+2)z^{2n}}{n!2^{4n}\Gamma(\nu+n+1)\Gamma(\nu+2n+2)}.$$

This is an entire function of growth order $\frac{1}{4}$ since

$$\lim_{n\to\infty}\frac{n\log n}{\log\Gamma(n+1) + \log\Gamma(\nu+n+1) + \log\Gamma(\nu+2n+2) + \log\frac{2^{4n}}{\Gamma(\nu+1)\Gamma(\nu+2)}} = \frac{1}{4}.$$

Here we used the limit $\frac{\log\Gamma(an+b)}{n\log n} \to a$, as $n \to \infty$, where $a, b > 0$. To see this just observe that

$$\lim_{x\to\infty}\frac{\log\Gamma(ax+b)}{x\log x} = a\lim_{x\to\infty}\frac{\psi(ax+b)}{1+\log x} = a\lim_{x\to\infty}\frac{\log(ax+b) - \frac{1}{2(ax+b)} + \mathcal{O}(x^{-2})}{\log x} = a,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma function. Now, applying the Hadamard theorem [10, p. 26] it follows that (2.3) is indeed valid. □

Proof of Lemma 4. We know that the function $\nu \mapsto \varphi_\nu(z)$ is decreasing on $(-1,\infty)$ for $z>0$ fixed, see [11]. Note that this can be verified also by looking at the Mittag-Leffler expansions in the above proof and by using the fact that the zeros $j_{\nu,n}$ are increasing on $(-1,\infty)$ as ν increases for each $n\in\mathbb{N}$ fixed. Hence, for $\varepsilon>0$ and $\nu>-1$ we have that $\varphi_{\nu+\varepsilon}(\gamma_{\nu,n})<\varphi_\nu(\gamma_{\nu,n})=0$ for some fixed $n\in\mathbb{N}$. On the other hand, from the proof of Lemma 3 we know that φ_ν is increasing on Δ for $\nu>-1$ and consequently we have that the equation $\varphi_{\nu+\varepsilon}(\gamma_{\nu+\varepsilon,n})=0$ together with above inequality yield $\gamma_{\nu+\varepsilon,n}>\gamma_{\nu,n}$, which completes the proof. □

Proof of Theorem 1. By using Lemma 3 we get that

$$2^{2\nu}z^{-\frac{\nu}{2}+\frac{3}{4}}\Gamma(\nu+1)\Gamma(\nu+2)\left(J_{\nu+1}(\sqrt[4]{z})I_\nu(\sqrt[4]{z})+J_\nu(\sqrt[4]{z})I_{\nu+1}(\sqrt[4]{z})\right)=z\prod_{n\geq1}\left(1-\frac{z}{\gamma_{\nu,n}^4}\right).$$

On the other hand, by taking the logarithmic derivative of both sides of the relation

$$2^{2\nu}z^{-2\nu-1}\Gamma(\nu+1)\Gamma(\nu+2)\Phi_\nu(z)=\prod_{n\geq1}\left(1-\frac{z^4}{\gamma_{\nu,n}^4}\right),$$

where

$$\Phi_\nu(z)=J_{\nu+1}(z)I_\nu(z)+J_\nu(z)I_{\nu+1}(z),$$

we obtain that

$$\frac{1}{4}\left(2\nu+1-\frac{z\Phi_\nu'(z)}{\Phi_\nu(z)}\right)=\sum_{n\geq1}\frac{z^4}{\gamma_{\nu,n}^4-z^4}.$$

Now, by using the well-known recurrence relations $zJ_{\nu+1}'(z)=zJ_\nu(z)-(\nu+1)J_{\nu+1}(z)$, $zI_{\nu+1}'(z)=zI_\nu(z)-(\nu+1)I_{\nu+1}(z)$, $zJ_\nu'(z)=\nu J_\nu(z)-zJ_{\nu+1}(z)$ and $zI_\nu'(z)=\nu I_\nu(z)+zI_{\nu+1}(z)$, it follows that

$$\Phi_\nu'(z)=2J_\nu(z)I_\nu(z)-\frac{1}{z}J_\nu(z)I_{\nu+1}(z)-\frac{1}{z}J_{\nu+1}(z)I_\nu(z).$$

Since in view of Lemma 4 the function $\nu\mapsto\gamma_{\nu,n}$ is increasing on $(-1,\infty)$ for each $n\in\mathbb{N}$, it follows that for $n\in\{2,3,\dots\}$ and $\nu\geq\nu^*$ we have $\gamma_{\nu,n}>\dots>\gamma_{\nu,1}\geq\gamma_{\nu^*,1}\simeq1.1639\ldots>1$. Moreover, the above monotonicity property of the zeros $\gamma_{\nu,n}$ implies that

$$\nu\longmapsto\sum_{n\geq1}\frac{1}{\gamma_{\nu,n}^4-1}$$

is decreasing on $(-1,\infty)$ and consequently

$$\sum_{n\geq1}\frac{1}{\gamma_{\nu,n}^4-1}\leq1$$

if and only if $\nu\geq\nu^*$, where ν^* is the unique root of the equation

$$\sum_{n\geq1}\frac{1}{\gamma_{\nu,n}^4-1}\leq1\iff(\nu-1)J_\nu(1)I_{\nu+1}(1)+(\nu-1)J_{\nu+1}(1)I_\nu(1)=J_\nu(1)I_\nu(1).$$

Thus, applying Lemma 1 the proof of this theorem is complete. □

Proof of Theorem 2. In view of the well-known infinite products

$$2^\nu\Gamma(\nu+1)z^{-\nu}J_\nu(z)=\prod_{n\geq1}\left(1-\frac{z^2}{j_{\nu,n}^2}\right),\quad 2^\nu\Gamma(\nu+1)z^{-\nu}I_\nu(z)=\prod_{n\geq1}\left(1+\frac{z^2}{j_{\nu,n}^2}\right)$$

it follows that

$$2^{2\nu}z^{-\frac{\nu}{2}+1}\Gamma^2(\nu+1)J_\nu(\sqrt[4]{z})I_\nu(\sqrt[4]{z})=z\prod_{n\geq1}\left(1-\frac{z}{j_{\nu,n}^4}\right).$$

Taking the logarithmic derivative of both sides of the next expression

$$2^{2\nu}z^{-2\nu}\Gamma^2(\nu+1)J_\nu(z)I_\nu(z)=\prod_{n\geq1}\left(1-\frac{z^4}{j_{\nu,n}^4}\right)$$

and using the recurrence relations $zJ_\nu'(z)-\nu J_\nu(z)=-zJ_{\nu+1}(z)$ and $zI_\nu'(z)-\nu I_\nu(z)=zI_{\nu+1}(z)$ it follows that

$$-\frac{1}{4}\left(4\nu+\frac{zI_{\nu+1}(z)}{I_\nu(z)}-\frac{zI_{\nu+1}(z)}{I_\nu(z)}\right)=\sum_{n\geq1}\frac{z^4}{j_{\nu,n}^4-z^4}.$$

On the other hand, it is well-known that the $\nu \mapsto j_{\nu,n}$ is increasing on $(-1,\infty)$ for each $n \in \mathbb{N}$ fixed, and thus $j_{\nu,n} > \ldots > j_{\nu,1} \geq j_{-\frac{1}{2},1} = \frac{\pi}{2} > 1$ for each $\nu \geq -\frac{1}{2}$ and $n \in \{2,3,\ldots\}$. Moreover, the function

$$\nu \longmapsto \sum_{n\geq 1} \frac{1}{j_{\nu,n}^4 - 1}$$

is decreasing on $(-1,\infty)$, which implies that

$$\sum_{n\geq 1} \frac{1}{j_{\nu,n}^4 - 1} \leq 1$$

if and only if $\nu \geq \nu^{\star}$, where $\nu^{\star}$ is the unique root of the equation

$$\sum_{n\geq 1} \frac{1}{j_{\nu,n}^4 - 1} = 1 \iff J_{\nu+1}(1)I_{\nu}(1) - J_{\nu}(1)I_{\nu+1}(1) = (\nu+1)J_{\nu}(1)I_{\nu}(1).$$

Thus, applying Lemma 1 the proof is complete. □

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