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Multiplier Transformations and K-Uniformly P-Valent Starlike Functions[1]

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Abstract

Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, which are analytic in the open unit disk $D = \{z : z \in C; |z| < 1\}$.Using the multiplier transformation, the auther introduce new subclasses of k-uniformly p-valent starlike functions and investigate thier inclusion relations and the closure properties of the above classes of functions under integral operators. These results also extend to k-uniformly close-to-convex functions.

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1 Introduction

Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$, which are analytic in the open unit disk $D = \{z : z \in C; |z| < 1\}$. If f and g are analytic in D, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in D such

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that $f(z) = g(w(z))$. A function $f(z) \in A(p)$ is said to be in $UST_p(k,\alpha)$, the class of k-uniformly p-valent starlike functions of order if it satisfies the condition

$$\Re e\{\frac{zf'(z)}{f(z)}\} - \alpha \geq k\left|\frac{zf'(z)}{f(z)} - p\right|, k \geq 0, 0 \leq \alpha < 1. \tag{1.1}$$

Replacing f in (1.1) by $zf'(z)$ we obtain the condition

$$\Re e\{1 + \frac{zf''(z)}{f'(z)}\} - \alpha \geq k\left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, k \geq 0, 0 \leq \alpha < 1. \tag{1.2}$$

required for the function f to be in the subclass $UCV_p(k,\alpha)$ of k-uniformly p-valent convex functions of order α.

Uniformly p-valent starlike and p-valent convex functions were first introduced [4] when $p = 1$, and [1] when $p \geq 1, p \in \mathbb{N}$, and then studied by various authors.

Setting

$$\Omega_{k,\alpha} = \{u + iv, u - \alpha > k\sqrt{(u-p)^2 + v^2}\}$$

with $p(z) = \frac{zf'(z)}{f(z)}$or $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which map D onto the conic domain $\Omega_{k,\alpha}$ such that $p(z) \in \Omega_{k,\alpha}$, we may rewrite the conditions (1.1) or (1.2) in the form

$$p(z) \prec Q_{k,\alpha}(z). \tag{1.3}$$

We note that the explicit forms of function $Q_{k,\alpha}(z)$ for $k = 0$ and $k = 1$ are $Q_{0,\alpha}(z) = \dfrac{1 + (1-2\alpha)z}{1-z}$ and $Q_{1,\alpha}(z) = p + \dfrac{2(p-\alpha)}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$.
For $0 < k < 1$, we obtain

$$\Omega_{k,\alpha} = \left\{u + iv, \left(\frac{(1-k^2)u + (k^2p - \alpha)}{k(p-\alpha)}\right)^2 - \left(\frac{(1-k^2)v}{(p-\alpha)\sqrt{1-k^2}}\right)^2 = 1\right\}$$

and

$$Q_{k,\alpha}(z) = \frac{(p-\alpha)}{1-k^2}\cos\left\{\frac{2}{\pi}\arccos(k)i\log(\frac{1+\sqrt{z}}{1-\sqrt{z}})\right\} - \frac{(k^2p-\alpha)}{1-k^2}.$$

For $k>1$,

$$\Omega_{k,\alpha}=\{u+iv,\left(\frac{(k^2-1)u+(k^2p-\alpha)}{k(p-\alpha)}\right)^2+\left(\frac{(k^2-1)v}{(p-\alpha)\sqrt{k^2-1}}\right)^2=1\}$$

and

$$Q_{k,\alpha}(z)=\frac{(p-\alpha)}{k^2-1}\sin\left\{\frac{\pi}{2K(x)}\int_0^{\frac{u(z)}{\sqrt{x}}}\frac{dt}{\sqrt{1-t^2\sqrt{1-k^2t^2}}}\right\}+\frac{(k^2p-\alpha)}{k^2-1},$$

where $u(z)=\frac{z-\sqrt{x}}{1-\sqrt{xz}}, x\in(0,1)$ and K is such that $k=\cos h\frac{\pi K'(x)}{4K(x)}$.

By virtue of (1.3) and the properties of the domains, we have

$$\Re e(p(z))>\Re e(Q_{k,\alpha}(z))>\frac{kp+\alpha}{k+1}. \tag{1.4}$$

Define $UCC_p(k,\alpha,\beta)$ to be the family of functions $f(z)\in A(p)$ such that

$$\frac{zf'(z)}{g(z)}\prec Q_{k,\alpha}(z).$$

for some $g(z)\in UST_p(k,\beta)$.

Similarly, we define $UQC_p(k,\alpha,\beta)$ to be the family of functions $f(z)\in A(p)$ such that

$$\frac{(zf'(z))'}{g'(z)}\prec Q_{k,\alpha}(z).$$

for some $g(z)\in UCV_p(k,\beta)$.

We note that $UCC_p(0,\alpha,\beta)$ is the class of close-to-convex p-valent functions of order α and type β and $UQC_p(0,\alpha,\beta)$ is the class of quasi-convex p-valent functions of order α and type β.

For any integer n , we define the multiplier transformations $I^{\lambda}_{n,p}$ of functions $f(z)\in A(p)$ by

$$I^{\lambda}_{n,p}f(z)=z^p+\sum_{k=1}^{\infty}(k+p)\left(\frac{\lambda+p}{\lambda+p+k}\right)^n a_{k+p}z^{k+p},\lambda\geq 0.$$

The operators $I^{\lambda}_{n,p}$ are closely related to the Komatu integral operators[5] and the differential and integral operators defined by Sălăgean[9].We also note that $I^0_{0,p}f(z)=zf'(z)$ and $I^{\lambda}_{n,1}f(z)=I^{\lambda}_nf(z)$ the operator defined by Cho and Kim[2].

2 Main Results

In this section, we prove some results on the linear operator $I^{\lambda}_{n+1,p}$.

In order to give our theorems, we need following lemmas.

Lemma 1. *If* $f(z) \in A(p)$, *then*

$$(\lambda + p)I^{\lambda}_{n,p}f\ (z) = z(I^{\lambda}_{n+1,p}f\ (z))' + (\lambda)I^{\lambda}_{n+1,p}f\ (z). \tag{2.1}$$

First is the inclusion theorem.

Theorem 2. *Let* $f(z) \in A(p)$. *If* $I^{\lambda}_{n,p}f\ (z) \in UST_p(k, \alpha)$.
Then $I^{\lambda}_{n+1,p}f\ (z) \in UST_p(k, \alpha)$ *.ma which is due to Eenigenburg, Miller, Mocanu, and Reade [3].*

Lemma 3. *Let* γ, β *be complex constants and* $h(z)$*be univalently convex in the unit disk* D *with* $h(0) = p$

and $\Re e(\beta h(z) + \gamma) > 0$. Let $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ be analytic in D. Then

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} \prec h(z) \text{ implies } g(z) \prec h(z).$$

Proof of Theorem 2. Setting $s(z) = \dfrac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}f\ (z)}$ in (2.1), we can write

$$\frac{(\lambda + p)(I^{\lambda}_{n,p}f\ (z))}{I^{\lambda}_{n+1,p}f\ (z)} = \frac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}f\ (z)} + (\lambda) = s(z) + (\lambda). \tag{2.2}$$

Differentiating (2.2) yields

$$\frac{z(I^{\lambda}_{n,p}f\ (z))'}{I^{\lambda}_{n,p}f\ (z)} = s(z) + \frac{zs'(z)}{s(z) + (\lambda)}. \tag{2.3}$$

From this and the argument given in Section 1, we may write

$$s(z) + \frac{zs'(z)}{s(z) + (\lambda)} \prec Q_{k,\alpha}(z).$$

Therefore, the theorem follows by Lemma A and the condition (1.4) since $Q_{k,\alpha}(z)$ is univalent and convex in D and $\Re e(Q_{k,\alpha}(z)) > \frac{kp+\alpha}{k+1}$.

Theorem 4. *Let* $f(z) \in A(p)$. *If* $I_{n,p}^{\lambda} f\ (z) \in UCV_p(k,\alpha)$ *then* $I_{n+1,p}^{\lambda} f\ (z) \in UCV_p(k,\alpha)$.

Proof. By virtue of (1.1), (1.2) and Theorem 2.1, we have

$$\begin{aligned} I_{n,p}^{\lambda} f\ (z) \in UCV_p(k,\alpha) &\Leftrightarrow z(I_{n,p}^{\lambda} f\ (z))' \in UST_p(k,\alpha) \\ &\Leftrightarrow I_{n,p}^{\lambda} z f\ '(z) \in UST_p(k,\alpha) \\ &\Rightarrow I_{n+1,p}^{\lambda} z f\ '(z) \in UST_p(k,\alpha) \\ &\Leftrightarrow I_{n+1,p}^{\lambda} f\ (z) \in UCV_p(k,\alpha) \end{aligned}$$

and the proof is complete.

We next prove:

Theorem 5. *Let* $f(z) \in A(p)$. *If* $I_{n,p}^{\lambda} f\ (z) \in UCC_p(k,\alpha,\beta)$. *Then* $I_{n+1,p}^{\lambda} f\ (z) \in UCC_p(k,\alpha,\beta)$.

To prove the above theorem, we shall need the following lemma which is due to Miller and Mocanu [6].

Lemma 6. *Let* $h(z)$ *be convex in the unit disk* D *and let* $E \geq 0$. *Suppose* $B(z)$*is analytic in* D *with* $\Re e(B(z)) > 0$. *If* $g(z)$*is analytic in* D *and* $h(0) = g(0)$. *Then*

$$Ez^2 g''(z) + B(z)g(z) \prec h(z) \text{ implies } g(z) \prec h(z).$$

Proof of Theorem 5. Since $I_{n,p}^{\lambda} f\ (z) \in UCC_p(k,\alpha,\beta)$, by definition, we can write

$$\frac{z(I_{n,p}^{\lambda} f\ (z))'}{k(z)} \prec Q_{k,\alpha}(z)$$

for some $k(z) \in UST_p(k,\beta)$. For $g(z)$ such that $I_{n,p}^{\lambda} g(z) = k(z)$, we have

$$\frac{z(I_{n,p}^{\lambda} f\ (z))'}{I_{n,p}^{\lambda} g(z)} \prec Q_{k,\alpha}(z). \tag{2.4}$$

Letting $h(z) = \dfrac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}g(z)}$ and $H(z) = \dfrac{z(I^{\lambda}_{n+1,p}g(z))'}{I^{\lambda}_{n+1,p}g(z)}$, we observe that h and H are analytic in D and $h(0) = H(0) = p$. Now, by Theorem 2.1, $I^{\lambda}_{n+1,p}g(z) \in UST_p(k, \beta)$ and so $\Re e\{H(z)\} > \dfrac{kp+\beta}{k+1}$. Also, note that

$$z(I^{\lambda}_{n+1,p}f\ (z))' = (I^{\lambda}_{n+1,p}g(z))h(z). \tag{2.5}$$

Differentiating both sides of (2.5) yields

$$z\frac{(z(I^{\lambda}_{n+1,p}f\ (z))')'}{I^{\lambda}_{n+1,p}g(z)} = \frac{z(I^{\lambda}_{n+1,p}g(z))'}{I^{\lambda}_{n+1,p}g(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z).$$

Now using the identity (2.1), we obtain

$$\begin{aligned}
\frac{z(I^{\lambda}_{n,p}f\ (z))'}{I^{\lambda}_{n,p}g(z)} &= \frac{I^{\lambda}_{n,p}(zf\ '(z))}{I^{\lambda}_{n,p}g(z)} \\
&= \frac{z(I^{\lambda}_{n+1,p}(zf\ '(z)))' + (\lambda)I^{\lambda}_{n+1,p}(zf\ '(z))}{z(I^{\lambda}_{n+1,p}g(z))' + (\lambda)I^{\lambda}_{n+1,p}g(z)} \\
&= \frac{\frac{z(I^{\lambda}_{n+1,p}(zf\ '(z)))'}{I^{\lambda}_{n+1,p}g(z)} + (\lambda)\frac{I^{\lambda}_{n+1,p}(zf\ '(z))}{I^{\lambda}_{n+1,p}g(z)}}{\frac{z(I^{\lambda}_{n+1,p}g(z))'}{I^{\lambda}_{n+1,p}g(z)} + (\lambda)} \\
&= \frac{H(z)h(z) + zh'(z) + (\lambda)h(z)}{H(z) + (\lambda)} \\
&= h(z) + \frac{zh'(z)}{H(z) + (\lambda)}. \qquad (2.6)
\end{aligned}$$

From (2.4), (2.5), and (2.6), we conclude that

$$h(z) + \frac{zh'(z)}{H(z) + (\lambda)} \prec Q_{k,\alpha}(z).$$

On letting $E = 0$ and $B(z) = \frac{1}{H(z)+(\lambda)}$, we obtain

$$\Re e(B(z)) = \frac{1}{|H(z) + (\lambda)|^2}\Re e(H(z) + (\lambda)) > 0.$$

The above inequality satisfies the conditions required by Lemma B. Hence $h(z) \prec Q_{k,\alpha}(z)$ and so the proof is complete.

Using a similar argument to that in Theorem 4, we can prove

Theorem 7. *Let $f(z) \in A(p)$. If $I^{\lambda}_{n,p}f\ (z) \in UQC_p(k,\alpha,\beta)$ then $I^{\lambda}_{n+1,p}f\ (z) \in UQC_p(k,\alpha,\beta)$.*

Finally, we examine the closure properties of the above classes of functions under the generalized Bernardi-Libera-Livingston integral operator $F_c(f)$ which is defined by

$$F_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt, (c+p \geq 0), f(z) \in A(p). \tag{2.7}$$

Theorem 8. *Let $c > \dfrac{-(kp+\alpha)}{k+1}$. If $I^{\lambda}_{n+1,p}f\ (z) \in UST_p(k,\alpha)$, then $I^{\lambda}_{n+1,p}F_c(f\ (z)) \in UST_p(k,\alpha)$ where F_c is the integral operator defined by (2.7).*

Proof. From (2.1), we have

$$z(I^{\lambda}_{n+1,p}F_cf\ (z))' = (c+p)I^{\lambda}_{n+1,p}f\ (z) - cI^{\lambda}_{n+1,p}F_cf\ (z). \tag{2.8}$$

Substituting $s(z) = \dfrac{z(I^{\lambda}_{n+1,p}F_cf\ (z))'}{I^{\lambda}_{n+1,p}F_cf\ (z)}$ in (2.8), we can write

$$\frac{z(I^{\lambda}_{n+1,p}F_cf\ (z))'}{I^{\lambda}_{n+1,p}F_cf\ (z)} + c = (c+p)\frac{I^{\lambda}_{n+1,p}f\ (z)}{I^{\lambda}_{n+1,p}F_cf\ (z)}. \tag{2.9}$$

Differentiating (2.9) yields

$$s(z) + \frac{zs'(z)}{s(z)+c} = \frac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}f\ (z)}.$$

Applying Lemma A, it follows that $s(z) \prec Q_{k,\alpha}(z)$, that is, $\dfrac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}f\ (z)} \prec Q_{k,\alpha}(z)$.

A similar argument leads to:

Theorem 9. *Let $c > \dfrac{-(kp+\alpha)}{k+1}$. If $I^{\lambda}_{n+1,p}f\ (z) \in UCV_p(k,\alpha)$, then $I^{\lambda}_{n+1,p}F_c(f\ (z)) \in UCV_p(k,\alpha)$.*

Theorem 10. *Let* $c > \frac{-(kp+\alpha)}{k+1}$. *If* $I^{\lambda}_{n+1,p}f\ (z) \in UCC_p(k,\alpha,\beta)$ *then* $I^{\lambda}_{n+1,p}F_c(f\ (z)) \in UCC_p(k,\alpha,\beta)$.

Proof. By definition, there exists a function $k(z) \in UST_p(k,\beta)$. For $g(z)$ such that $I^{\lambda}_{n+1,p}g(z) = k(z)$, we have

$$\frac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}g(z)} \prec Q_{k,\alpha}(z). \tag{2.10}$$

Now from (2.8) we have

$$\frac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}g(z)} = \frac{z(I^{\lambda}_{n+1,p}F_c(zf\ '(z)))' + cI^{\lambda}_{n+1,p}F_c(zf\ '(z))}{z(I^{\lambda}_{n+1,p}F_cg(z))' + (\lambda)I^{\lambda}_{n+1,p}F_cg(z)}$$

$$= \frac{\frac{z(I^{\lambda}_{n+1,p}F_c(zf\ '(z)))'}{I^{\lambda}_{n+1,p}F_cg(z)} + c\frac{I^{\lambda}_{n+1,p}F_c(zf\ '(z))}{I^{\lambda}_{n+1,p}F_cg(z)}}{\frac{z(I^{\lambda}_{n+1,p}F_cg(z))'}{I^{\lambda}_{n+1,p}F_cg(z)} + c}. \tag{2.11}$$

Since $I^{\lambda}_{n+1,p}g(z) \in UST_p(k,\beta)$, by Theorem 8, we have $F_c(I^{\lambda}_{n+1,p}g(z)) \in UST_p(k,\beta)$. Letting $H(z) = \frac{z(I^{\lambda}_{n+1,p}F_cg(z))'}{I^{\lambda}_{n+1,p}F_cg(z)}$, we observe that $\Re e\{H(z)\} > \frac{kp+\beta}{k+1}$. Now, let h be defined by

$$z(I^{\lambda}_{n+1,p}F_cf\ (z))' = (I^{\lambda}_{n+1,p}F_cg(z))h(z). \tag{2.12}$$

Differentiating both sides of (2.12) yields

$$\frac{z(I^{\lambda}_{n+1,p}(zF_cf\ )')'(z)}{(I^{\lambda}_{n+1,p}F_cg)(z)} = \frac{z(I^{\lambda}_{n+1,p}F_cg)'(z)}{(I^{\lambda}_{n+1,p}F_cg)(z)}h(z) + zh'(z) = H(z)h(z) + zh'(z). \tag{2.13}$$

Therefore from (2.11) and (2.13), we obtain

$$\frac{z(I^{\lambda}_{n+1,p}f\ (z))'}{I^{\lambda}_{n+1,p}g(z)} = \frac{H(z)h(z) + zh'(z) + ch(z)}{H(z) + c}$$

that is,

$$h(z) + \frac{zh'(z)}{H(z) + c} \prec Q_{k,\alpha}(z).$$

On letting $B(z) = \dfrac{1}{H(z)+c}$, we note that $\Re e\{B(z)\} > 0$ if $c > -\dfrac{kp+\beta}{k+1}$. Now for $E = 0$ and $B(z)$ as described, we conclude the proof since the required conditions of Lemma B are satisfied. A similar argument yields

Theorem 11. *Let* $c > \dfrac{-(kp+\alpha)}{k+1}$. *If* $I^{\lambda}_{n+1,p}f\ (z) \in UQC_p(k,\alpha,\beta)$, *then* $I^{\lambda}_{n+1,p}F_c\big(f\ (z)\big) \in UQC_p(k,\alpha,\beta)$.

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