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Subordination results on harmonic k-uniformly convex mappings and related classes

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ABSTRACT

The main purpose of this paper is to establish some connections between certain classes of harmonic univalent functions by applying a number of convolution properties involving hypergeometric functions. To be more precise, we investigate such connections with harmonic k-uniformly convex and a related class $k-ST$ mappings in the plane.

1. Introduction

A continuous complex-valued function $f=u+\mathrm{i}v$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f=h+\overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)|>|g'(z)|, z\in D$.

Denote by S_H the class of functions $f=h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disc $\Delta=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\overline{g}\in S_H$ we may express the analytic functions h and g as

$$h(z)=z+\sum_{m=2}^{\infty}a_m z^m, \qquad g(z)=\sum_{m=1}^{\infty}b_m z^m, \qquad |b_1|<1. \tag{1.1}$$

Let A be the class of all functions f, whose Maclaurin's series is of the form

$$f(z)=z+\sum_{m=2}^{\infty}a_m z^m, \tag{1.2}$$

which are *analytic* in Δ.

Let S be the subclass of A, consisting of univalent functions.

Note that the family S_H of orientation-preserving, normalized harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f=h+\overline{g}$ is identically zero.

Let K, S^*, K_H and S_H^* denote the respective subclasses of S and S_H where the images of $f(u)$ are convex and starlike.

A domain D is called convex in the direction γ $(0\le\gamma<\pi)$ if every line is parallel to the line through 0 and $e^{\mathrm{i}\gamma}$ has a connected intersection with D. Such a domain is close-to-convex. The convex domains are those that are convex in every direction.

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A function $f \in A$ is said to be uniformly convex in Δ if it has the property that for every circular arc γ contained in the open unit disc Δ, with center ζ also in Δ, the image arc $f(\gamma)$ is a convex arc. This class was introduced by Goodman [1]. In another paper Goodman [2] introduced the uniform starlike functions.

The class UCV describes geometrically the domain of values of the expression $1+\frac{zf''(z)}{f'(z)}, z \in \Delta$ to lie in a parabolic region

$$\Omega = \left\{\omega \in \mathbb{C} : (\mathrm{Im}\,(\omega))^2 < 2\mathrm{Re}\,(\omega) - 1\right\}.$$

Rønning [3], and Ma and Minda [4] gave a one variable characterization for $f \in UCV$, as

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta. \tag{1.3}$$

Using the Alexander relation, Rønning [3] defined a new class called S_p consisting of functions $f \in A$ satisfying

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \Delta. \tag{1.4}$$

Kanas and Wiśniowska [5] defined the class $k-UCV$ as

$$k-UCV := \left\{f \in A : \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, (0 \le k < \infty)\right\}. \tag{1.5}$$

Note that the class $k-UCV$ is an extension of the class UCV studied by Goodman [1,2]. In another paper, Kanas and Wiśniowska [6] extended the class S_p as

$$k-ST := \left\{f \in A : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, (0 \le k < \infty)\right\}. \tag{1.6}$$

The various properties of the classes $k-UCV$ and $k-ST$ were extensively studied by Kanas and Srivastava [7].

We note that

$$f \in k-UCV \Longleftrightarrow zf' \in k-ST.$$

A function $f(z)$ is subordinate to $F(z)$ in the disc Δ if there exists an analytic function $w(z)$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$ for $|z|<1$. This is written as $f(z) \prec F(z)$.

Setting $\Omega_k = \{u+\mathrm{i}v, u > k\sqrt{(u-1)^2+v^2}\}$ with $p(z)=\frac{zf'(z)}{f(z)}$ or $p(z)=1+\frac{zf''(z)}{f'(z)}$ and considering the functions $Q_k(z)$ which map Δ onto conic domains Ω_k such that $p(z) \in \Omega_k$, we may rewrite conditions (1.6) or (1.5) in the form

$$p(z) \prec Q_k(z) = 1 + \sum P_n(k) z^n. \tag{1.7}$$

We note that for the function $Q_k(z)$ for $k=0$ and $k=1$ we can explicitly write

$$Q_0(z) = \frac{1+z}{1-z}, \quad \text{and} \quad Q_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

For $0<k<1$ we obtain

$$Q_k(z) = \frac{1}{1-k^2}\cos\left\{\frac{2}{\pi}\arccos(k)\mathrm{i}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right\} - \frac{k^2}{1-k^2},$$

for $k>1$

$$Q_k(z) = \frac{1}{k^2-1}\sin\left\{\frac{\pi}{2K(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\right\} + \frac{k^2}{k^2-1},$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}$ $(0<t<1)$ and K is such that $k=\cos h\frac{\pi K'(t)}{4K(t)}$, $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is the complementary integral of $K(t)$.

By virtue of (1.7) and the properties of the domains, we have

$$\mathrm{Re}\,p(z) > \mathrm{Re}\,(Q_k(z)) > \frac{k}{k+1}.$$

We now introduce the following subclasses of harmonic functions in terms of subordination.

Let $f = h + \overline{g} \in S_H$ such that

$$\varphi(z) = \frac{h(z)-g(z)}{1-b_1},$$

$$\psi = \frac{h(z)-\mathrm{e}^{\mathrm{i}\theta}g(z)}{1-\mathrm{e}^{\mathrm{i}\theta}b_1}, \quad 0 \le \theta < 2\pi.$$

We can now construct the harmonic classes $k-UCV_H$ and $k-ST_H$ as follows

$$k-ST_H=\left\{f\in S_H,\ \frac{z\varphi'(z)}{\varphi(z)}\prec Q_k\right\}$$

$$k-UCV_H=\left\{f\in S_H,\ \frac{\left(z\psi'(z)\right)'}{\psi'(z)}\prec Q_k\right\}.$$

We define the Hadamard product (or convolution) of f and g by

$$(f*g)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad z\in U.$$

For non zero complex parameters $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_q(\beta_j\neq 0,-1,\ldots;j=1,2,\ldots,q)$ the *generalized hypergeometric function* ${}_pF_q(z)$ is defined by

$${}_pF_q(z)={}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z):=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!}\quad (p\le q+1;\ p,q\in N_0:=\mathbb{N}\cup\{0\};\ z\in U),$$

where $\mathbb{N}$ denotes the set of all positive integers and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n=\begin{cases}1, & n=0\\ x(x+1)(x+2)\cdots(x+n-1), & n\in\mathbb{N}.\end{cases}$$

The notation ${}_pF_q$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial, and others; for example see [8,9].

For complex values of $\alpha_1,\ldots,\alpha_p$ and $\beta_1,\ldots,\beta_q(\beta_j\neq 0,-1,\ldots;j=1,2,\ldots,q)$, let $H(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q):A\to A$ be a linear operator defined by

$$\begin{aligned}\left[\left(H(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q)\right)(f)\right](z)&=z_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)*f(z)\\&=z+\sum_{n=2}^{\infty}\Gamma_na_nz^n,\end{aligned}$$

where

$$\Gamma_n=\frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(n-1)!(\beta_1)_{n-1}\ldots(\beta_q)_{n-1}}. \tag{1.8}$$

For notational simplicity, we use a shorter notation $H_{p,q}\binom{\alpha_1}{\beta_1}$ for $H(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q)$ in what follows. The linear operator $H_{p,q}\binom{\alpha_1}{\beta_1}$ called the Dziok–Srivastava operator (see [10]), includes (as its special cases) various other linear operators introduced and studied by Bernardi [11], Carlson and Shaffer [12], Libera [13], Livingston [14], Owa [15], Ruscheweyh [16] and Srivastava–Owa [17].

In [18] we define the Dziok–Srivastava operator of the harmonic functions of the form (1.1) such that

$$H_{p,q}\binom{\alpha_1}{\beta_1}(f)=H_{p,q}\binom{\alpha_1}{\beta_1}(h)+\overline{H_{p,q}\binom{\alpha_1}{\beta_1}(g)},$$

using the Dziok–Srivastava operator of harmonic functions we introduce the following classes:

$$k-ST_H\binom{\alpha_1}{\beta_1}=\left\{f\in S_H,H_{p,q}\binom{\alpha_1}{\beta_1}(f)\in k-ST_H\right\}$$

$$k-UCV_H\binom{\alpha_1}{\beta_1}=\left\{f\in S_H,H_{p,q}\binom{\alpha_1}{\beta_1}(f)\in k-UCV_H\right\}.$$

For $p=q+1,\alpha_2=\beta_1,\ldots,\alpha_p=\beta_q$,

$$k-ST_H\binom{1}{\alpha_2}=k-ST_H$$

$$k-UCV_H\binom{1}{\alpha_2}=k-UCV_H.$$

The main object of this paper is to establish some important connections between the classes $k-ST_H$, $k-UCV_H$, $k-ST_H\binom{\alpha_1}{\beta_1}$ and $k-UCV_H\binom{\alpha_1}{\beta_1}$ by applying differential subordination.

2. Connections with the class $k - ST$ of harmonic starlike functions

In order to establish connections between harmonic $k - ST$ starlike mappings and analytic $k - ST$ starlike mappings, we need the following results.

Lemma 1 ([19]). *If f is of the form (1.1) with*

$$\sum_{m=2}^{\infty} m\,|a_m| + \sum_{m=1}^{\infty} m\,|b_m| \le 1,$$

then $f \in S_H^*$. The result is sharp.

Lemma 2 ([6,7]). *A function $f(z) \in S$ is in $k - ST$ if*

$$\sum_{m=2}^{\infty} (m(k+1)-k)\,|a_m| \le 1,$$

where $k \ge 0$.

Lemma 3 ([20]). *A function $f = h + \overline{g}$ is harmonic convex if and if the analytic functions $h(z) - e^{i\gamma} g(z), 0 \le \gamma < 2\pi$, are convex in the direction $\frac{\gamma}{2}$ and f is suitably normalized.*

Theorem 1. *If $f \in S_H$ is of the form (1.1) with*

$$\sum_{m=2}^{\infty} (m(k+1)-k)\,|a_m| + \sum_{m=1}^{\infty} (m(k+1)-k)\,|b_m| \le 1, \tag{2.1}$$

then $f \in k - ST_H$. The result is sharp.

Proof. From the definition of $k - ST_H$, we need only to prove that $\phi(z) \in k - ST$, where $\phi(z)$ is given by (1.2) such that

$$\phi(z) = z + \sum_{m=2}^{\infty} \left(\frac{a_m - b_m}{1 - b_1}\right) z^m.$$

Using Lemma 2, we have

$$\sum_{m=2}^{\infty} (m(1+k)-k) \left|\frac{a_m - b_m}{1 - b_1}\right| \le \sum_{m=2}^{\infty} (m(1+k)-k) \left(\frac{|a_m| + |b_m|}{1 - |b_1|}\right) \le 1$$

if and only if (2.1) holds and hence we have the result.

The harmonic function

$$f(z) = z + \sum_{m=2}^{\infty} \frac{1}{(m(1+k)-k)} x_m z^m + \sum_{m=1}^{\infty} \frac{1}{(m(1+k)-k)} \overline{y_m}\,\overline{z^m} \quad \left(\text{where } \sum_{m=2}^{\infty} |x_m| + \sum_{m=1}^{\infty} |y_m| = 1\right)$$

shows that the coefficient bound given by (2.1) is sharp. □

Remark 1. If $k = 0$, then we have the coefficient bound given in [19] utilizing a different approach.

Corollary 1. $k - ST_H \subseteq S_H^*$.

Theorem 2. *Let $f = h + \overline{g} \in S_H$ with*

$$\sum_{m=2}^{\infty} (m(1+k)-k)\,|\Gamma_m a_m| + \sum_{m=1}^{\infty} (m(1+k)-k)\,|\Gamma_m b_m| \le 1 \tag{2.2}$$

where Γ_m is defined by (1.7), then $f \in k - ST_H \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$. The result is sharp.

Proof. The result follows immediately using the following lemma.

Lemma 4 ([21]). *If the parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are positive real numbers and if*

$$\sum_{m=2}^{\infty} (m(1+k)-k)\, \Gamma_m\, |a_m| \leq 1$$

then $f \in k - ST\binom{\alpha_1}{\beta_1}$.

The function

$$f(z) = z + \frac{1+\delta}{(m(1+k)-k)\,\Gamma_m}\overline{z^m}, \quad \delta > 0$$

shows that the upper bound in (2.2) cannot be improved. □

For the inclusion relations between the classes of $k - ST_H\binom{\alpha_1}{\beta_1}$ we need the following result [21].

Lemma 5. *Let h be convex univalent in Δ with $h(0) = 1$ and $\mathrm{Re}(\lambda h(z)+\mu) > 0 (\lambda, \mu \in \mathbb{C})$. If p is analytic in Δ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\lambda p(z)+\mu} \prec h(z) \quad (z \in \Delta)$$

implies

$$p(z) \prec h(z) \quad (z \in \Delta).$$

Theorem 3. *Let $f = h + \overline{g} \in S_H$. Then $k - ST\binom{\alpha_1+1}{\beta_1} \subseteq k - ST\binom{\alpha_1}{\beta_1}$.*

Proof. Let $f \in k - ST\binom{\alpha_1+1}{\beta_1}$, then

$$H_{p,q}\binom{\alpha_1+1}{\beta_1}\left(\frac{h-g}{1-b_1}\right) \in k - ST$$

and

$$\left|H_{p,q}\binom{\alpha_1+1}{\beta_1}(h)\right| > \left|H_{p,q}\binom{\alpha_1+1}{\beta_1}(g)\right|.$$

Using Lemma 5, we have

$$H_{p,q}\binom{\alpha_1}{\beta_1}\left(\frac{h-g}{1-b_1}\right) \in k - ST.$$

Since

$$\begin{aligned}\left|\alpha_1 H_{p,q}\binom{\alpha_1+1}{\beta_1}(h)\right| &= \left|z\left(H_{p,q}\binom{\alpha_1}{\beta_1} * h\right)'\right| \\ &= \left|z\left\{\frac{1}{z}\left(H_{p,q}\binom{\alpha_1}{\beta_1} * h'\right)\right\}\right|\end{aligned}$$

this implies

$$\left|H_{p,q}\binom{\alpha_1}{\beta_1}(h)\right| > \left|H_{p,q}\binom{\alpha_1}{\beta_1}(g)\right|$$

or

$$H_{p,q}\binom{\alpha_1}{\beta_1}(h) + H_{p,q}\binom{\alpha_1}{\beta_1}(g) \in k - ST_H$$

and we have the result. □

Theorem 4. *Suppose that $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ and (in the case $p = q + 1$)*

$$\mathrm{Re}\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + 1 + \sum_{j=1}^{p} |\alpha_j|$$

where $P_1 = P_1(k)$ is the coefficient of z in $Q_k(z)$. If $f \in k - ST_H$ and for some k $(0 \le k < \infty)$, then the hypergeometric inequality

$$\frac{(k+2)\left|\alpha_1 \cdots \alpha_p\right|}{\mathrm{Re}(\beta_1)\cdots \mathrm{Re}(\beta_q)_{p+1}} F_{q+1}\Big(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|, P_1+1; 1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q), 2; 1\Big) + {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,\left|\alpha_p\right|, P_1; \mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_q), 1; 1) < 2$$

holds true then $f \in k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$.

Proof. The result follows immediately using the following lemma.

Lemma 6 ([22]). *Suppose that $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ and (in the case $p = q + 1$)*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ is the coefficient of z in $Q_k(z)$. If $f \in k - ST$ and for some k $(0 \le k < \infty)$, then the hypergeometric inequality

$$\frac{(k+2)\left|\alpha_1 \cdots \alpha_p\right|}{\mathrm{Re}(\beta_1)\cdots \mathrm{Re}(\beta_q)_{p+1}} F_{q+1}\Big(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|, P_1+1; 1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q), 2; 1\Big) + {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,\left|\alpha_p\right|, P_1; \mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_q), 1; 1) < 2$$

holds true then $f \in k - ST\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$. □

Corollary 2. $k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \subseteq k - ST_H$.

Theorem 5. *Let $f = h + \overline{g} \in S_H$ and let $F_c(f) = \frac{1+c}{z^c}\int_0^c t^{c-1} f(t)\mathrm{d}t$. If $f \in k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$ then $F_c(f) \in k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$.*

Proof. If $f \in k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$, then $\frac{f-g}{1-b_1} \in k - ST\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$. Using Lemma 5, we have $F_c\left(\frac{f-g}{1-b_1}\right) \in k - ST\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$. Since $\left|H_{p,q}\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}F_c(h)\right| > \left|H_{p,q}\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}F_c(g)\right|$, then $F_c(f) \in k - ST_H\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$. □

3. Connection with harmonic uniformly convex mappings

In this section we shall look at the analogous results involving connections between various classes of k-uniformly convex harmonic mappings by applying differential subordination theory.

Lemma 7 ([19]). *If $f = h + \overline{g}$ with*

$$\sum_{m=2}^{\infty} m^2 |a_m| + \sum_{m=1}^{\infty} m^2 |b_m| \le 1,$$

then $f \in K_H$. The result is sharp.

Lemma 8 ([23]). *Let $k \ge 0$, If f is of the form (1.2) with*

$$\sum_{m=2}^{\infty} m[m(1+k) - k]\Gamma_m |a_m| \le 1,$$

then $f \in k - UCV$, where Γ_m is defined by (1.7).

Theorem 6. *If $f \in S_H$ is of the form (1.1) with*

$$\sum_{m=2}^{\infty} m[m(1+k) - k]|a_m| + \sum_{m=1}^{\infty} m[m(1+k) - k]|b_m| \le 1, \tag{3.1}$$

then $f \in UCV_H$. The result is sharp.

Proof. From the definition of the class $k-UCV_H$ and the coefficient bound of $k-UCV$ given in Lemma 7, we have the result. The function

$$f(z) = z + \frac{(1+\delta)}{m\left[m(1+k)-k\right]}\overline{z}^m, \quad \delta > 0$$

shows that the upper bound in (3.1) cannot be improved. □

Remark 2. If $k = 1$, then we have the coefficient bound given in [19] utilizing a different approach.

Corollary 3. *$k - UCV_H \subseteq K_H$.*

Theorem 7. *If $f = h + \overline{g} \in S_H$ with*

$$\sum_{m=2}^{\infty} m\left[m(1+k)-k\right]|\Gamma_m a_m| + \sum_{m=1}^{\infty} m\left[m(1+k)-k\right]|\Gamma_m b_m| \le 1, \tag{3.2}$$

then $f \in k - UCV_H\binom{\alpha_1}{\beta_1}$, where Γ_m is defined by (1.7). The function

$$f(z) = z + \frac{(1+\delta)}{m\left[m(1+k)-k\right]\Gamma_m}\overline{z}^m, \quad \delta > 0$$

shows that the result is sharp.

Proof. Using the following lemma [22] we have the result. □

Lemma 9. *If the parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are positive real numbers and if*

$$\sum_{m=2}^{\infty} m\left[m(1+k)-k\right]\Gamma_m |a_m| \le 1,$$

then $f \in k - UCV\binom{\alpha_1}{\beta_1}$. The function

$$f(z) = z + \frac{1+\delta}{m\left[m(1+k)-k\right]\Gamma_m}\overline{z^m}, \quad \delta > 0$$

shows that the upper bound in (3.2) cannot be improved.

Theorem 8. *Let $f = h + \overline{g} \in S_H$. Then $k - UCV\binom{\alpha_1+1}{\beta_1} \subseteq k - UCV\binom{\alpha_1}{\beta_1}$.*

Applying the following lemma [22].

Lemma 10. *Suppose that $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ and (in the case $p = q + 1$)*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ is the coefficient of z in $Q_k(z)$. If $f \in k - UCV$, and for some k $(0 \le k < \infty)$, the hypergeometric inequality

$${}_{p+1}F_{q+1}\left(1+|\alpha_1|, \ldots, 1+\left|\alpha_p\right|, P_1+1; 1+\mathrm{Re}(\beta_1), \ldots, \mathrm{Re}\left(\beta_q\right), 2; 1\right) < \frac{\mathrm{Re}(\beta_1), \ldots, \mathrm{Re}(\beta_q)}{(k+2)\left|\alpha_1\cdots\alpha_p\right| P_1},$$

holds true, then we have

$$f \in k - UCV\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}.$$

Theorem 9. *Suppose that $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ and (in) the case $p = q + 1$*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ *is the coefficient of* z *in* $Q_k(z)$. *If* $f \in k - ST_H$ *and for some* k $(0 \le k < \infty)$, *then hypergeometric inequality*

$$\frac{(k+2)\left|\alpha_1\cdots\alpha_p\right|}{\mathrm{Re}(\beta_1)\cdots\mathrm{Re}(\beta_q)_{p+1}}F_{q+1}\Big(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|,P_1+1;1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q),2;1\Big) + {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,\left|\alpha_p\right|,P_1;\mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_q),1;1) < 2$$

holds true then $f \in k - UCV_H\binom{\alpha_1}{\beta_1}$.

Theorem 10. *Let* $f = h + \overline{g} \in S_H$ *and let* $F_c(f) = \frac{1+c}{z^c}\int_0^c t^{c-1}f(t)\mathrm{d}t$. *If* $f \in k - ST_H\binom{\alpha_1}{\beta_1}$ *then* $F_c(f) \in k - UCV_H\binom{\alpha_1}{\beta_1}$.

4. Connections between $k - ST_H(\frac{\alpha_1}{\beta_1})$ and $k - UCV(\frac{\alpha_1}{\beta_1})$

Next we find connections between the classes $k-ST_H$, $k-ST_H\binom{\alpha_1}{\beta_1}$, $k-UCV_H$ and $k-UCV_H\binom{\alpha_1}{\beta_1}$ we need the following results [22].

Lemma 11. *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ *and (in the case* $p = q + 1$*)*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ *is the coefficient of* z *in* $Q_k(z)$. *If* $f \in k - UCV$, *and for some* k $(0 \le k < \infty)$, *the hypergeometric inequality*

$$\frac{(k+2)\left|\alpha_1\cdots\alpha_p\right|}{\mathrm{Re}(\beta_1)\cdots\mathrm{Re}(\beta_q)_{p+1}}F_{q+1}\Big(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|,P_1+1;1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q),3;1\Big) + {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,\left|\alpha_p\right|,P_1;\mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_q),2;1) < 2$$

holds true, then $f \in k - ST\binom{\alpha_1}{\beta_1}$.

Lemma 12. *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ *and (in the case* $p = q + 1$*)*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > p_1 + 1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ *is the coefficient of* z *in* $Q_k(z)$. *If* $f \in k - ST$, *and for some* k $(0 \le k < \infty)$, *the hypergeometric inequality*

$${}_{p+2}F_{q+2}(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|,P_1+1,3;1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q),2,2;1) < \frac{\mathrm{Re}(\beta_1)\cdots\mathrm{Re}(\beta_q)}{2P_1(k+2)\left|\alpha_1\cdots\alpha_p\right|},$$

holds true, then $f \in k - UCV\binom{\alpha_1}{\beta_1}$.

Applying Lemmas 11 and 12, we get

Theorem 11. *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ *and (in the case* $p = q + 1$*)*

$$\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}\left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ *is the coefficient of* z *in* $Q_k(z)$. *If* $f \in k - UCV_H$, *and for some* k $(0 \le k < \infty)$, *the hypergeometric inequality*

$$\frac{(k+2)\left|\alpha_1\cdots\alpha_p\right|}{\mathrm{Re}(\beta_1)\cdots\mathrm{Re}(\beta_q)_{p+1}}F_{q+1}\Big(1+|\alpha_1|,\ldots,1+\left|\alpha_p\right|,P_1+1;1+\mathrm{Re}(\beta_1),\ldots,1+\mathrm{Re}(\beta_q),3;1\Big) + {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,\left|\alpha_p\right|,P_1;\mathrm{Re}(\beta_1),\ldots,\mathrm{Re}(\beta_q),2;1) < 2$$

holds true, then $f \in k - ST_H\binom{\alpha_1}{\beta_1}$.

Theorem 12. *Suppose that* $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(j = 1, 2, \ldots, p)$, $\mathrm{Re}(\beta_j) > 0$ $(j = 1, 2, \ldots, q)$ *and (in the case* $p = q + 1$*)*

$$\mathrm{Re}\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + 1 + \sum_{j=1}^{p} \left|\alpha_j\right|,$$

where $P_1 = P_1(k)$ *is the coefficient of* z *in* $Q_k(z)$*. If* $f \in k - ST_H$*, and for some* k $(0 \le k < \infty)$*, the hypergeometric inequality*

$${}_{p+2}F_{q+2}(1 + |\alpha_1|, \ldots, 1 + \left|\alpha_p\right|, P_1 + 1, 3; 1 + \mathrm{Re}(\beta_1), \ldots, 1 + \mathrm{Re}(\beta_q), 2, 2; 1) < \frac{\mathrm{Re}(\beta_1) \cdots \mathrm{Re}(\beta_q)}{2P_1(k+2)\left|\alpha_1 \cdots \alpha_p\right|},$$

holds true, then $f \in k - UCV_H \binom{\alpha_1}{\beta_1}$.

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