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Multiplier family of harmonic univalent functions

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ABSTRACT

The aim of this paper is to study a multiplier family of harmonic univalent functions using the sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers. By specializing $\{c_n\}$ and $\{d_n\}$, the generalized Bernardi–Libera–Livingston integral operator is modified for such functions and the closure of the multiplier family under the modified integral operator is determined. Also, convolution products, closure properties, distortion theorems, convex combinations and neighborhoods for such functions are given.

1. Introduction

Let S_H denote the class of functions f which are complex-valued, harmonic, univalent, sense-preserving in the open unit disk U normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \ |b_1| < 1 \tag{1}$$

are analytic in U. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ in U. Clunie and Sheill-Small [2] studied the class S_H with some geometric subclasses of S_H.

Let $HP(\beta)$ denote the subclass of S_H satisfying $\text{Re}\{h'(z) + g'(z)\} > \beta, \ 0 \leqslant \beta < 1$, which was studied by Yalçin et al. [7]; they also denote by $HP^*(\beta)$ the subclass of $HP(\beta)$ such that the functions h and g in $f = h + \bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = -\sum_{n=1}^{\infty} |b_n| z^n, \ |b_1| < 1. \tag{2}$$

It is known [7] that

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leqslant 1 - \beta, \quad (a_1 = 1 \text{ and } 0 \leqslant \beta < 1). \tag{3}$$

Then $f \in HP(\beta)$. Condition (3) is also necessary if $f \in HP^*(\beta)$.

A function $f = h + \bar{g}$, where h and g are given by (2), is said to be in the multiplier family $FH_\beta(\{c_n\}, \{d_n\})$ if there exist sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers such that

$$\sum_{n=2}^{\infty} c_n|a_n| + \sum_{n=1}^{\infty} d_n|b_n| \leqslant 1 - \beta, \quad d_1|b_1| < 1. \tag{4}$$

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When $\beta = 0$, we get the multiplier family in [1]. The multipliers $\{c_n\}$ and $\{d_n\}$ provide a transition from harmonic convex functions to the class $HP^*(\beta)$. In this paper, the generalized integral operator is modified for such functions. Also, convolution products, closure properties, distortion theorems, convex combinations, and neighborhoods of functions in the class $FH_\beta(\{c_n\},\{d_n\})$ are determined.

2. Main results

Let $n \leqslant c_n$ and $n \leqslant d_n$. Then by (3), we have $FH_\beta(\{c_n\},\{d_n\}) \subset HP^*(\beta)$. Consequently, the functions in $FH_\beta(\{c_n\},\{d_n\})$ are sense-preserving, harmonic, and univalent in U. We observe that if $f_1(z) = z - \sum_{n=2}^{\infty}|a_{1_n}|z^n - \sum_{n=1}^{\infty}|b_{1_n}|\bar{z}^n$ and $f_2(z) = z - \sum_{n=2}^{\infty}|a_{2_n}|z^n - \sum_{n=1}^{\infty}|b_{2_n}|\bar{z}^n$ are in $FH_\beta(\{c_n\}\{d_n\})$ and $0 \leqslant \lambda \leqslant 1$, then so is the linear combination $\lambda f_1 + (1-\lambda)f_2$ by using (4). Therefore, $FH_\beta(\{c_n\},\{d_n\})$ is a convex family.

Now, for $f = h + \bar{g}$ given by (1), we define the modified generalized Bernardi–Libera–Livingston integral operator of f as

$$J_c(f(z)) = J_c(h(z)) + \overline{J_c(g(z))}, \quad c > -1, \tag{5}$$

where

$$J_c(h(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt$$

and

$$J_c(g(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt.$$

Putting $g = 0$ in (5), we get the definition of the generalized Bernardi–Libera–Livingston integral operator on analytic functions, (see [4,5]).

In the next theorem, we study the closure of the class $FH_\beta(\{c_n\},\{d_n\})$ under the modified generalized Bernardi–Libera–Livingston integral operator.

Theorem 1. *Consider the increasing sequences of positive numbers $\{c_n\}$ and $\{d_n\}$ such that $n \leqslant c_n$ and $n \leqslant d_n$ for $n \geqslant 2$. If $f \in FH_\beta(\{c_n\},\{d_n\})$, then $J_c(f(z)) \in FH_\beta(\{c_n\},\{d_n\})$.*

Proof. From (5), we obtain

$$\begin{aligned} J_c(f(z)) &= J_c(h(z)) + \overline{J_c(g(z))} = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \frac{c+1}{\bar{z}^c}\int_0^z \overline{t^{c-1}g(t)}dt \\ &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left[t - \sum_{n=2}^{\infty}|a_n|t^n\right]dt - \frac{c+1}{\bar{z}^c}\int_0^z t^{c-1}\left[\sum_{n=1}^{\infty}|b_n|t^n\right]dt = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\bar{z}^n, \end{aligned}$$

where $|A_n| = \frac{c+1}{c+n}|a_n|,\ |B_n| = \frac{c+1}{c+n}|b_n|$.

Let

$$\sum_{n=2}^{\infty}[c_n|A_n| + d_n|B_n|] = \sum_{n=2}^{\infty}\left[c_n\frac{c+1}{c+n}|a_n| + d_n\frac{c+1}{c+n}|b_n|\right] \leqslant \sum_{n=2}^{\infty}[c_n|a_n| + d_n|b_n|] \leqslant 1 - d_1|b_1| - \beta.$$

Then $J_c f(z) \in FH_\beta(\{c_n\},\{d_n\})$.

If $h,\ g,\ H,\ G$ are of the form (1) and as we know if $f = h + \bar{g}$ and $F = H + \overline{G}$, then the convolution of f and F is defined to be the function

$$f * F(z) = z + \sum_{n=2}^{\infty}a_nA_nz^n + \overline{\sum_{n=1}^{\infty}b_nB_nz^n},$$

while the integral convolution is defined by

$$f \diamondsuit F(z) = z + \sum_{n=2}^{\infty}\frac{a_nA_n}{n}z^n + \overline{\sum_{n=1}^{\infty}\frac{b_nB_n}{n}z^n}.$$

Let P_H^0 denote the class of functions F complex and harmonic in U, $P = H + \overline{G}$ such that $\mathrm{Re}\,P(z) > 0,\ z \in U$ and

$$H(z) = 1 + \sum_{n=1}^{\infty}A_nz^n, \quad G(z) = \sum_{n=2}^{\infty}B_nz^n.$$

It is known [3, Theorem 3] that the sharp inequalities $|A_n| \leqslant n+1,\ |B_n| \leqslant n-1$ are true. □

Theorem 2.

(i) *If* $f = h + \bar{g} \in FH_\beta(\{c_n\},\{d_n\})$, *then for* $\frac{3}{2} \leqslant |A_1| \leqslant 2,\ \frac{1}{A_1} f \diamondsuit P \in FH_\beta(\{c_n\},\{d_n\})$.
(ii) *If f satisfies the condition*

$$\sum_{n=2}^{\infty} n\{c_n|a_n| + d_n|b_n|\} \leqslant 1 - d_1|b_1| - \beta, \tag{6}$$

then $\frac{1}{A_1} f * P \in FH_\beta(\{c_n\},\{d_n\})$.

Proof. We justify the case (ii). Since

$$\sum_{n=2}^{\infty}\left\{c_n\left|\frac{A_n a_n}{A_1}\right| + d_n\left|\frac{B_n b_n}{A_1}\right|\right\} \leqslant \sum_{n=2}^{\infty} n\left\{c_n|a_n|\frac{n+1}{|A_1|n} + d_n|b_n|\frac{n+1}{|A_1|n}\right\} \leqslant \sum_{n=2}^{\infty} n\{c_n|a_n| + d_n|b_n|\} \leqslant 1 - d_1|b_1| - \beta,$$

therefore $\frac{1}{A_1} f * P \in FH_\beta(\{c_n\},\{d_n\})$.

Let S denote the class of analytic univalent functions of the form $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$. It is well known that the sharp inequality $|A_n| \leqslant n$ is true. □

Theorem 3. *Let* $f \in FH_\beta(\{c_n\},\{d_n\})$ *satisfy condition* (6), *where* $n \leqslant c_n$ *and* $n \leqslant d_n$ *for* $n \geqslant 2,\ 0 \leqslant \beta < 1$. *If* $F \in S$ *then for* $|\epsilon| \leqslant 1,\ f * (F + \epsilon\overline{F}) \in HP^*(\beta)$.

Proof. Since

$$\sum_{n=2}^{\infty} n(|a_n A_n| + |b_n B_n|) \leqslant \sum_{n=2}^{\infty} n(c_n|a_n| + d_n|b_n|) \leqslant 1 - d_1|b_1| - \beta,$$

it follows that $f * (F + \epsilon\overline{F}) \in HP^*(\beta)$. □

We shall prove the following result for the closure of the function in the class $FH_\beta(\{c_n\},\{d_n\})$.

Theorem 4. *Let the function* $f_j(z)$ $(j = 1, 2, \ldots, m)$, *defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n_j}| z^n - \sum_{n=1}^{\infty} |b_{n_j}| \bar{z}^n$$

for $z \in U$, *be in the class* $FH_\beta(\{c_n\},\{d_n\})$. *Then the function* $f = h + \bar{g}$ *belongs to the class* $FH_\beta(\{c_n\},\{d_n\})$, *where* $h,\ g$ *are of the form* (2) *and* $a_n = \frac{1}{m}\sum_{j=1}^{\infty} a_{n_j},\ b_n = \frac{1}{m}\sum_{j=1}^{m} b_{n_j}$.

Proof. Since $f_j(z) \in FH_\beta(\{c_n\},\{d_n\})$, therefore

$$\sum_{n=2}^{\infty} c_n|a_{n_j}| + \sum_{n=1} d_n|b_{n_j}| \leqslant 1 - \beta \quad (j = 1, 2, \ldots, m).$$

Hence

$$\sum_{n=2}^{\infty} c_n|a_n| + \sum_{n=1}^{\infty} d_n|b_n| = \sum_{n=2}^{\infty} c_n\left(\frac{1}{m}\left|\sum_{j=1}^{m} a_{n_j}\right|\right) + \sum_{n=1}^{\infty} d_n\left(\frac{1}{m}\left|\sum_{j=1}^{m} b_{n_j}\right|\right) \leqslant \sum_{n=2}^{\infty} c_n|a_{n_j}| + \sum_{n=1}^{\infty} d_n|b_{n_j}| \leqslant 1 - \beta. \qquad \square$$

Next we obtain distortion bounds for functions in $FH_\beta(\{c_n\},\{d_n\})$.

Theorem 5. *Consider the increasing sequences of positive numbers* $\{c_n\}$ *and* $\{d_n\}$ *so that* $c_2 \leqslant d_2,\ n \leqslant c_n$, *and* $n \leqslant d_n$ *for all* $n \geqslant 2$. *If* $f \in FH_\beta(\{c_n\},\{d_n\})$, *then*

$$(1 - |b_1|)r - \frac{1}{c_2}(1 - d_1|b_1| - \beta)r^2 \leqslant |f(z)| \leqslant (1 + |b_1|)r + \frac{1}{c_2}(1 - d_1|b_1| - \beta)r^2.$$

Proof. Let $f \in FH_\beta(\{c_n\},\{d_n\})$. Then taking the absolute value of f, we obtain

$$\begin{aligned}|f(z)| &= \left|z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\bar{z}^n\right| \leqslant (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n = (1 + |b_1|)r + \frac{1}{c_2}\sum_{n=2}^{\infty} c_2(|a_n| + |b_n|)r^n \\ &\leqslant (1 + |b_1|)r + \frac{1}{c_2}\sum_{n=2}^{\infty}(c_2|a_n| + d_2|b_n|)r^n \leqslant (1 + |b_1|)r + \frac{1}{c_2}\sum_{n=2}^{\infty}(c_n|a_n| + d_n|b_n|)r^2 \\ &\leqslant (1 + |b_1|)r + \frac{1}{c_2}(1 - d_1|b_1| - \beta)r^2\end{aligned}$$

and

$$|f(z)| \geqslant (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n = (1-|b_1|)r - \frac{1}{c_2}\sum_{n=2}^{\infty}c_2(|a_n|+|b_n|)r^n \geqslant (1-|b_1|)r - \frac{1}{c_2}\sum_{n=2}^{\infty}(c_2|a_n|+d_2|b_n|)r^n$$
$$\geqslant (1-|b_1|)r - \frac{1}{c_2}\sum_{n=2}^{\infty}(c_n|a_n|+d_n|b_n|)r^2 \geqslant (1-|b_1|)r - \frac{1}{c_2}(1-d_1|b_1|-\beta)r^n.$$

The functions

$$f(z) = z + |b_1|\bar{z} + \frac{1}{c_2}(1-d_1|b_1|-\beta)\bar{z}^2$$

and

$$f(z) = z - |b_1|\bar{z} - \frac{1}{c_2}(1-d_1|b_1|-\beta)\bar{z}^2$$

for $d_1|b_1| \leqslant 1-\beta$ show that the bounds given in Theorem 2 are sharp. □

The following result follows from the left hand inequality in Theorem 2:

Corollary 6. *Let f be as in Theorem 2. Then*

$$\{w : |w| < \frac{1}{c_2}(c_2 - 1 - (c_2 - d_1)|b_1| + \beta\} \subset f(U).$$

It follows that the class $FH_\beta(\{c_n\},\{d_n\})$ is uniformly bounded. Hence it is normal.

Theorem 7. *If $n \leqslant c_n$ and $n \leqslant d_n$ for $n \geqslant 2$, $0 \leqslant \beta < 1$, then the family $FH_\beta(\{c_n\},\{d_n\})$ is closed under convex combinations.*

Proof. Consider $f_{n_i}(z) = z - \sum_{n=2}^{\infty}|a_{i_n}|z^n - \sum_{n=1}^{\infty}|b_{i_n}|\bar{z}^n$, where $\sum_{i=1}^{\infty}t_i = 1$ and $0 \leqslant t_i < 1$. If $f_{n_i} \in FH_\beta(\{c_n\},\{d_n\})$, then

$$\sum_{n=2}^{\infty}c_n|a_{i_n}| + \sum_{n=1}^{\infty}d_n|b_{i_n}| \leqslant 1-\beta. \tag{7}$$

On the other hand, we have

$$\sum_{i=1}^{\infty}t_i f_{n_i}(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i_n}|\right)z^n - \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i_n}|\right)\bar{z}^n.$$

From this and (7), we obtain

$$\sum_{n=2}^{\infty}\frac{c_n}{1-\beta}\left|\sum_{i=1}^{\infty}t_i|a_{i_n}|\right| + \sum_{n=1}^{\infty}\frac{d_n}{1-\beta}\left|\sum_{i=1}^{\infty}t_i|b_{i_n}|\right| = \sum_{i=1}^{\infty}t_i\left\{\sum_{n=2}^{\infty}\frac{c_n}{1-\beta}|a_{i_n}| + \sum_{n=1}^{\infty}\frac{d_n}{1-\beta}|b_{i_n}|\right\} \leqslant \sum_{i=1}^{\infty}t_i = 1.$$

Hence, by an application of (4), $\sum_{i=1}^{\infty}t_i f_i \in FH_\beta(\{c_n\},\{d_n\})$.

Now following Ruscheweyh [6], we call the set

$$N_\delta(f) = \left\{F : F(z) = z - \sum_{n=2}^{\infty}|A_n|z^n - \sum_{n=1}^{\infty}|B_n|\bar{z}^n\right.$$

and

$$\left.\sum_{n=1}^{\infty}n(|a_n - A_n| + |b_n - B_n|) \leqslant (1-\beta)\delta, \delta > 0\right\}$$

the δ-neighborhood of $f = h + \bar{g}$ in $FH_\beta(\{nc_n\},\{nd_n\})$. □

Our next result generates the functions in a neighborhood of $FH_\beta(\{nc_n\},\{nd_n\})$ are in the class $HP^*(\beta)$.

Theorem 8. *Consider the increasing sequences $\{c_n\}$ and $\{d_n\}$ so that $n \leqslant c_n$ and $n \leqslant d_n$ for $n \geqslant 2$. If $\delta \leqslant \frac{1}{c_2}[c_2 - 1 - (c_2 - d_1)|b_1| + \beta]$, $\delta > 0$, then $N_\delta(FH_\beta(\{nc_n\},\{nd_n\}) \subset HP^*(\beta)$.*

Proof. Let $f = h + \bar{g} \in FH_\beta(\{nc_n\},\{nd_n\})$ and $F = H + \overline{G} \in N_\delta(f)$, where $H(z) = z - \sum_{n=2}^{\infty}|A_n|z^n$ and $G(z) = -\sum_{n=1}^{\infty}|B_n|z^n$. We need to show that $F \in HP^*(\beta)$.

It suffices to show that F satisfies the condition

$$\sum_{n=1}^{\infty}n(|A_n| + |B_n|) + |B_1| \leqslant 1-\beta.$$

Now,

$$\begin{aligned}\sum_{n=2}^{\infty} n(|A_n| + |B_n|) + |B_1| &= \sum_{n=2}^{\infty} n|A_n - a_n + a_n| + |B_n - b_n + b_n| + |B_1 - b_1 + b_1| \\ &\leqslant \sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - b_n|) + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |B_1 - b_1| + |b_1| \\ &= \sum_{n=2}^{\infty} n[|A_n - a_n| + |B_n - b_n|] + |B_1 - b_1| + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |b_1| \\ &\leqslant (1-\beta)\delta + |b_1| + \sum_{n=2}^{\infty}(n|a_n| + n|b_n|) \leqslant (1-\beta)\delta + |b_1| + \frac{1}{c_2}\sum_{n=2}^{\infty}(nc_n|a_n| + nd_n|b_n|) \\ &\leqslant (1-\beta)\delta + |b_1| + \frac{1}{c_2}(1 - d_1|b_1| - \beta).\end{aligned}$$

This last expression is never greater than one provided that

$$\delta \leqslant 1 - |b_1| - \frac{1}{c_2}(1 - d_1|b_1| - \beta) = \frac{c_2 - 1 - (c_2 - d_1)|b_1| + \beta}{c_2(1-\beta)}. \qquad \square$$

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