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A remark on averaging operators on homogeneous spaces

By

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Abstract. To any sequence $g_1, \dots, g_d, g_1^{-1}, \dots, g_d^{-1}$ in a group G one can associate an averaging operator acting on functions on a homogeneous space of G. For a connected compact Lie group G we obtain a lower bound for the L^2-norm of this operator in the orthocomplement to the constants.

1. Introduction. Let G be a connected compact Lie group, $M = G/H$ a homogeneous space of G, $g_1, \dots, g_d$ elements of G, and S the corresponding symmetric sequence, i.e.,

$$S = \{g_1, \dots, g_d, g_1^{-1}, \dots, g_d^{-1}\}.$$

The space $L^2(M)$ is defined with respect to a G-invariant finite Borel measure on M. Following A. Lubotzky, R. Phillips and P. Sarnak [4], we consider the averaging operator $T_S : L^2(M) \to L^2(M)$ acting by

$$(T_S f)(x) = \frac{1}{2d} \sum_{j=1}^{d} \left\{ f(g_j x) + f(g_j^{-1} x) \right\}.$$

This is a self-adjoint operator of norm 1, for which every constant function is a fixed vector. Thus, the orthocomplement $L_0^2(M)$ to the subspace of constant functions is invariant under T_S. The operator discrepancy of S is defined by

$$\delta_S = \| T_S|_{L_0^2(M)} \|.$$

Clearly, $\delta_S \leqq 1$. For the two-dimensional sphere the authors of [4] established a lower bound for δ_S. In the present note, we generalize their proof and obtain the same estimate for arbitrary homogeneous spaces of compact Lie groups. Let Γ denote the subgroup of G generated by $g_1, \dots, g_d$.

Theorem. *Let G be a connected compact Lie group, $H \subset G$ a closed subgroup, $H \neq G$, and $M = G/H$ the associated homogeneous space. Then:*

(i) *for any $g_1, \dots, g_d \in G$ one has*

$$\delta_S \geqq \frac{\sqrt{2d-1}}{d};$$

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(ii) *equality is possible only if* Γ *is free and* $g_1, \dots, g_d$ *are free generators of* Γ*;*
(iii) *if* Γ *is amenable then* $\delta_S = 1$.

For $M = S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, the theorem is due to of A. Lubotzky, R. Phillips and P. Sarnak, see Theorem 1.2 of [4]. Their proof is based on an elementary calculation of the trace of any power of T_S in the space of harmonic polynomials of fixed degree on S^2. Y. Colin de Verdière noted that an extension of this method to other homogeneous spaces might be of interest, see [1, p. 89]. We do so using known facts about the asymptotic behaviour of group characters rather than finding an explicit trace formula.

2. Spectral measure of (Γ, S)**.** There is a general notion of spectral measure associated to a homogeneous graph, see [1]. In particular, this measure is defined for the Cayley graph of a group Γ with a symmetric system of generators S. In this case we call it the spectral measure of (Γ, S). We recall briefly the basic facts about this measure which we need in the sequel. The details are found in [1].

Let Γ be a group with a symmetric system of generators $S = \{g_1, \dots, g_d, g_1^{-1}, \dots, g_d^{-1}\}$ and W_s the set of all words $a_{i_1}a_{i_2}\cdots a_{i_s}$ of length s in $2d$ formal variables $a_1, a_2, \dots, a_{2d}$. The substitution $a_1 \to g_1, \dots, a_d \to g_d, a_{d+1} \to g_1^{-1}, \dots, a_{2d} \to g_d^{-1}$ gives rise to a mapping $W_s \to \Gamma$. We sometimes denote a word from W_s and its image in Γ by the same letter. Let m_s the number of elements of $g \in W_s$ such that $g = e$ in Γ.

The spectral measure $\mu = \mu_{\Gamma,S}$ of (Γ, S) is a probability measure on the real line. The moments of μ are given by

$$\int_{-\infty}^{\infty} t^s d\mu(t) = m_s, \qquad s \geqq 0.$$

The support of μ is contained in $[-2d, 2d]$. More precisely, let A be the self-adjoint operator in $l^2(\Gamma)$ defined by $(Af)(\gamma) = \sum_{j=1}^{d}\left\{f(g_j\gamma) + f(g_j^{-1}\gamma)\right\}$. Then

$$\operatorname{supp}(\mu) = \operatorname{spectrum}(A). \tag{1}$$

The norm of A is related to the numbers m_s by

$$\|A\| = \overline{\lim_{s\to\infty}}(m_s)^{\frac{1}{s}}. \tag{2}$$

Let $\nu_{\Gamma,S} = \|A\|$. If $\varphi : \Gamma \to \Gamma^*$ is a group epimorphism then $S^* = \varphi(S)$ is a symmetric system of $2d$ generators of Γ^*. Since the numbers m_s for (Γ^*, S^*) are greater than or equal to the corresponding numbers for (Γ, S), we see from (2) that

$$\nu_{\Gamma^*,S^*} \geqq \nu_{\Gamma,S}. \tag{3}$$

The following theorem gives important information about the number $\nu_{\Gamma,S}$.

Theorem (H. Kesten [3]). *For any group* Γ *with a symmetric system* S *of* $2d$ *generators one has:*

(i) $2\sqrt{2d-1} \leqq \nu_{\Gamma,S} \leqq 2d$;
(ii) $\nu_{\Gamma,S} = 2\sqrt{2d-1}$ *if and only if* S *is a free system of generators of* Γ*;*
(iii) $\nu_{\Gamma,S} = 2d$ *if and only if* Γ *is amenable.*

3. Convergence lemmas. Let $\hat{G}$ denote the dual space of a compact group G. The points of $\hat{G}$ are equivalence classes of irreducible unitary representations of G. The space $\hat{G}$ has discrete topology. For each $\alpha \in \hat{G}$ we fix a representative of α. Let R_α and $\mathscr{H}_\alpha$ be the corresponding representation and the representation space, respectively. As usual, χ_α is the character and d_α the degree of α. Note that if G is a connected, simple, compact Lie group then $\alpha \to \infty$ in $\hat{G}$ if and only if $d_\alpha \to \infty$ in $\mathbb{R}^+$.

Let $\mathbb{R}[G]$ be the group ring of G. A unitary representation R extends to $\mathbb{R}[G]$. Namely, for $q = \sum c_g \cdot g \in \mathbb{R}[G]$ we put $R(q) = \sum c_g \cdot R(g)$.

Recall that the spectral measure of a self-adjoint operator D in a finite-dimensional Hilbert space of dimension N is defined by

$$\mu_D = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j},$$

where $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_N$ are the eigenvalues of D. Here, we apply this definition to $D = R_\alpha(q)$, where $q = g_1 + \cdots + g_d + g_1^{-1} + \cdots + g_d^{-1} \in \mathbb{R}[G]$. To simplify the notation, we write $\mu_\alpha = \mu_{R_\alpha(q)}$. Note that $\operatorname{supp}(\mu_\alpha) \subset [-2d, 2d]$.

Lemma 1. *Let G be a connected, simple, compact Lie group, Z_G the center of G and $g \in G$, $g \notin Z_G$. Then*

$$\lim_{\alpha \to \infty} \frac{\chi_\alpha(g)}{d_\alpha} = 0.$$

Proof. Let $n = \dim G$, $r = \operatorname{rank} G$. The assertion is a consequence of the following fact first proved by D. L. Ragozin [6]. Convoluting a continuous central measure on G with itself n times we get a measure from $L^1(G)$. The lemma is stated explicitly and proved by D. Rider [7]. Recently, K. E. Hare obtained the upper bound $|\chi_\alpha(g)|/d_\alpha \leqq c(g) d_\alpha^{-2/(n-r)}$, see [2]. □

Lemma 2. *Let G be a connected, simple, compact Lie group with $Z_G = \{e\}$, $\Gamma \subset G$ a subgroup with a symmetric system of generators $S = \{g_1, \ldots, g_d, g_1^{-1}, \ldots, g_d^{-1}\}$, and μ the spectral measure of (Γ, S). Then $\mu_\alpha \to \mu$ in the weak-*-topology as $\alpha \to \infty$.*

Proof. One has

$$R_\alpha(q^s) = \sum_{g \in W_s} R_\alpha(g),$$

hence

$$\operatorname{tr} R_\alpha(q^s) = \sum_{g \in W_s} \chi_\alpha(g).$$

By Lemma 1, it follows that

$$\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} t^s d\mu_\alpha = \lim_{\alpha \to \infty} \frac{\operatorname{tr} R_\alpha(q^s)}{d_\alpha} = \lim_{\alpha \to \infty} \sum_{g \in W_s} \frac{\chi_\alpha(g)}{d_\alpha} = m_s = \int_{-\infty}^{\infty} t^s d\mu.$$

Since the supports of μ_α and μ are contained in $[-2d, 2d]$, we get the weak-*-convergence $\mu_\alpha \to \mu$. □

The previous lemma for $G = \mathrm{SO}(3)$ is Theorem 1.1 in [4]. We need another generalization. In the next lemma, $G = G_1 \times \ldots \times G_m$ where each G_i is a centerless, connected, simple, compact Lie group. Let $p_i : G \to G_i,\ i = 1, \dots, m$, denote the projection maps. For each $\alpha \in \hat{G}$ there exist uniquely defined $\beta_i \in \hat{G}_i,\ i = 1, \dots, m$, such that R_α is equivalent to $R_{\beta_1} \otimes \dots \otimes R_{\beta_m}$. Writing $\beta_i = \pi_i(\alpha)$ we obtain the mappings $\pi_i : \hat{G} \to \hat{G}_i$.

Lemma 3. *Let $\mathscr{C} \subset \hat{G}$ be an infinite subset such that all maps $\pi_i|\mathscr{C}$ are proper. Then $\mu_\alpha \to \mu$ in the weak-*-topology as $\alpha \to \infty, \alpha \in \mathscr{C}$.*

Proof. Let $s \geqq 0$ be fixed. By Lemma 1, for a given $\epsilon > 0$ there exist finite sets $K_i \subset \hat{G}_i$ such that

$$\frac{|\chi_\beta(p_i(g))|}{d_\beta} < \frac{\epsilon}{(2d)^s}$$

if $g \in W_s,\ p_i(g) \neq e$ and $\pi_i(\alpha) \notin F_i$. The set $F = (\pi_1^{-1}(F_1) \cap \mathscr{C}) \cup \dots \cup (\pi_m^{-1}(F_m) \cap \mathscr{C})$ is finite by the assumption of lemma. Now,

$$\frac{\operatorname{tr} R_\alpha(q^s)}{d_\alpha} = \sum_{g \in W_s} \prod_{i=1}^m \frac{\chi_{\pi_i(\alpha)}(p_i(g))}{d_{\pi_i(\alpha)}}.$$

For $\alpha \in \mathscr{C} - F$ and $g \in W_s$, it can happen that $p_i(g) = e$ for all i so that $g = e$ in G. Otherwise, there is at least one factor in the above product with absolute value smaller than $\epsilon/(2d)^s$. In this case

$$\left| \prod_{i=1}^m \frac{\chi_{\pi_i(\alpha)}(p_i(g))}{d_{\pi_i(\alpha)}} \right| < \frac{\epsilon}{(2d)^s},$$

since the absolute value of each factor is smaller than or equal to 1. Therefore

$$\left| \sum_{g \in W_s} \prod_{i=1}^m \frac{\chi_{\pi_i(\alpha)}(p_i(g))}{d_{\pi_i(\alpha)}} - m_s \right| \leqq \epsilon.$$

The end of the proof is the same as in Lemma 2. □

4. Proof of theorem. Recall that we have a symmetric sequence in G. Let q be the sum of its elements in the group ring, i.e., $q = g_1 + \dots + g_d + g_1^{-1} + \dots + g_d^{-1} \in \mathbb{R}[G]$. We need the following proposition.

Proposition. *Let $G = G_1 \times \dots \times G_m$ be the product of centerless, connected, simple, compact Lie groups, R a continuous representation of G in a Hilbert space $\mathscr{H}$, and $\mathscr{C} \subset \hat{G}$ an infinite subset such that all maps $\pi_i|\mathscr{C}$ are proper and*

$$\operatorname{Hom}_G(\mathscr{H}_\alpha, \mathscr{H}) \neq \{0\}$$

for $\alpha \in \mathscr{C}$. Then one has

(4) $$\|R(q)\| \geqq \nu_{\Gamma,S}.$$

Proof. In view of (1), $\nu_{\Gamma,S}$ or $-\nu_{\Gamma,S}$ is in $\operatorname{supp}(\mu)$. By Lemma 3, there exist $\lambda_\alpha \in \operatorname{supp}(\mu_\alpha)$ such that

(5) $$\overline{\lim} |\lambda_\alpha| \geqq \nu_{\Gamma,S} \quad \text{as} \quad \alpha \to \infty, \alpha \in \mathscr{C}.$$

But λ_α is an eigenvalue of $R_\alpha(q)$. By our assumption, for $\alpha \in \mathscr{C}$ there exists $v_\alpha \in \mathscr{H}$, $v_\alpha \neq 0$, such that $R(q)v_\alpha = \lambda_\alpha \cdot v_\alpha$. Therefore $\|R(q)\| \geqq |\lambda_\alpha|$ for all $\alpha \in \mathscr{C}$, and so (4) follows from (5). □

Proof of theorem. We have to prove the inequality

$$\delta_S \geqq \frac{1}{2d}\nu_{\Gamma,S}.$$

If this is done then (i), (ii) and (iii) follow from the corresponding assertions of the theorem of H. Kesten. Let R be the unitary representation of G in $\mathscr{H} = L_0^2(M)$. Then

$$\delta_S = \frac{1}{2d}\|R(q)\|,$$

and so we have to prove (4). We first make several reductions.

(a) If (4) is proved for $M_1 = G/H_1$, where $H \subset H_1 \subset G$, $H_1 \neq G$, then (4) also holds for $M = G/H$. Indeed, $L_0^2(M_1)$ embeds in $L_0^2(M)$ as a closed G-invariant subspace. Let R_1 be the representation of G in $L_0^2(M_1)$. Then $\|R_1(q)\| \leqq |R(q)\|$, which implies (4) for $M = G/H$.

(b) Suppose that H contains a closed normal subgroup N of G, put $G^* = G/N$, $H^* = H/N$ and write M in the Klein form $M = G^*/H^*$. We claim that it suffices to prove the theorem for this Klein form. Let $\varphi : G \to G^*$ be the canonical epimorphism, $S^* = \varphi(S)$, $\Gamma^* = \varphi(G)$, and $q^* = \varphi(g_1) + \cdots + \varphi(g_d) + \varphi(g_1)^{-1} + \cdots + \varphi(g_d)^{-1}$. Further, let R^* denote the representation of G^* in $L_0^2(M)$. Then, clearly, $R^*(q^*) = R(q)$, and so our claim follows from (3).

(c) If (4) is proved for G (semisimple) centerless then (4) is also proved for the general case. Indeed, let $H_1 = H \cdot Z_G$, where Z_G is the center of G. Then the following two cases are possible.

   (c1) $H_1 = G$. Then H contains the commutator subgroup G^s of G. By (b), it suffices to prove the theorem for $M = G/G^s/H/G^s$. But then M is a torus, in which case $\delta_S = 1$, see [1]. This implies (4).

   (c2) $H_1 \neq G$. By (a), our assertion is reduced to $M_1 = G/H_1$. By (b), we may consider the Klein form $M_1 = G/Z_G/H_1/Z_G$, in which the transitive group is centerless.

We reduced the proof to the case of $G = G_1 \times \ldots \times G_m$, where all G_i are as in the proposition above. By (b), we assume without loss of generality that H does not contain G_i for each i. Consider the complexified group $G^{\mathbb{C}} = G_1^{\mathbb{C}} \times \ldots \times G_m^{\mathbb{C}}$ and fix a Borel subgroup $B_i \subset G_i^{\mathbb{C}}$. Then $B = B_1 \times \ldots \times B_m$ is a Borel subgroup in $G^{\mathbb{C}}$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is of the form $\mathfrak{h} = \mathfrak{h}_1 + \cdots + \mathfrak{h}_m$, where $\mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}_i^{\mathbb{C}}$. We take $\mathfrak{h}_i$ in $\mathfrak{b}_i$ for all i. We will use B and $\mathfrak{h}$ (resp. B_i and $\mathfrak{h}_i$) to determine the irreducible representations of G (resp. G_i) by their highest weights.

Since G_i acts non-trivially on M, there is at least one $\alpha = \alpha(i) \in \hat{G}$ which occurs in $\mathscr{H}$, such that $\pi_i(\alpha)$ is the class of a non-trivial representation. Let f_i be a highest weight vector of an irreducible G-submodule in $\mathscr{H}$ of class α. Its highest weight Λ_i has the property that

$$\Lambda_i|_{\mathfrak{h}_i} \neq 0.$$

Note that f_i is a continuous function on M so that $f = f_1 \cdots f_m$ is in $L^2(M)$. Also, f is again a highest weight vector whose highest weight Λ equals $\Lambda_1 + \cdots + \Lambda_m$. Thus

$$\Lambda|_{\mathfrak{h}_i} \neq 0 \quad \text{forall} \quad i,\ i = 1, \ldots, m.$$

Each power f^k is a highest weight vector of an irreducible G-submodule $V_k \subset L^2(M)$ with highest weight $k\Lambda$. For $k \geqq 1$ the representation in V_k is non-trivial. Thus, the G-invariant linear functional given by integration over M vanishes on V_k identically, i.e., $V_k \subset \mathscr{H}$. Let $\mathscr{C} \subset \hat{G}$ consist of representation classes determined by the highest weights $k\Lambda$, $k \geqq 1$. By construction, $\mathscr{C}$ satisfies all requirements of the proposition above, and so we obtain (4). □

R e m a r k. For each prime $p \equiv 1(\text{mod}4)$, A. Lubotzky, R. Phillips and P. Sarnak constructed sequences S of $2d = p + 1$ elements in $G = \mathrm{SO}(3)$ for which one has equality in (i) of the theorem, see [5]. To the best of the author's knowledge, these are the only examples of this type.

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