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Unitarising measures for the representations of affine group and associated invariant operators

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Abstract

Let G denote the group of affine transformations on $\mathbb{C}^n$, or one of its subgroups such as the translation group or the Euclidean motion group. In this paper, we study unitarisability of holomorphic representations T of G into the space of square integrable $\mathbb{C}$-valued functions with respect to a positive real measure μ with support in a domain D in $\mathbb{C}^n$. Moreover, we investigate the infinitesimal representation approach of a unitary representation (T_g, μ) on the Lie algebra $\mathcal{G}$ of G in order to determine a class of second-order differential operators (called of Ornstein–Uhlenbeck type) invariant with respect to the unitarising measures μ.

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1. Introduction

Unitary representation theory of Lie groups plays a significant role in pure and applied mathematics, particularly in harmonic analysis [9–11,15,17–19], as well as in relativistic physics, quantum mechanics [20] and coherent states theory [8,13]. Among modern problems in harmonic

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analysis related to group representation theory, we study unitarity of holomorphic representations of a given Lie group G on $L^2(D,\mu)$, where D is a domain in $\mathbb{C}^n$ ($n \geqslant 1$), having the form

$$(T_g f)(z) = h_g(z) f\big(k_g(z)\big), \tag{1.1}$$

where $h_g : D \to \mathbb{C}$ and $k_g : D \to D$ are holomorphic functions. Our central goal is to establish a correspondence between the existence of unitarising measures μ and the corresponding second-order differential operators of Ornstein–Uhlenbeck type, denoted Δ^{OU}, invariant with respect to μ, i.e.,

$$\int_D \Delta^{OU} \Phi(z,\bar{z})\, \mathrm{d}\mu(z) = 0 \tag{1.2}$$

for all Φ in a suitable class of test functions. This problem was suggested by P. Malliavin in [16] and [2,4] for infinite dimensional representations of the group of diffeomorphisms of the circle $\mathrm{Diff}(S^1)$ acting on the manifold of univalent functions in the unit disc. Motivated by Malliavin ideas and the approach of infinitesimal representation, H. Airault and H. Ouerdiane studied in [6] the problem of unitarising measure corresponding to holomorphic representation like (1.1) for finite dimensional Lie groups as $(\mathbb{R},+)$, $(\mathbb{R}^\star,\times)$, the linear group $GL(2,\mathbb{R})$ and the 3-dimensional Heisenberg group. Recently, H. Airault constructed in [3] the mentioned correspondence for the representation of the symplectic group $Sp(2n)$ acting on the Siegel disc of complex symmetric matrices. In this paper, we study holomorphic representations of the semidirect product group $G = G_0 \ltimes V$, where G_0 is a subgroup of $GL(n,\mathbb{C})$ and V is the Euclidean vector space $\mathbb{R}^n$ or $\mathbb{C}^n$ with the non-commutative group law

$$(A,h) \star \big(A',h'\big) = \big(AA', Ah' + h\big), \tag{1.3}$$

for $A, A' \in G_0$ and $h, h' \in V$, and acting on $\mathbb{C}^n$ via the affine action

$$\psi\big((A,h),z\big) = Az + h. \tag{1.4}$$

Let $\mathcal{G}_0$ be the Lie algebra of G_0 with Lie bracket $[\,,\,]_0$. The Lie algebra of G is the vector space $\mathcal{G} = \mathcal{G}_0 \times V$ with Lie bracket $[(K,h),(K',h')] = ([K,K']_0, Kh' - K'h)$ for $K, K' \in \mathcal{G}_0$ and $h, h' \in V$.

The paper is organized as follows. In Section 2, we consider unitarising measures, see [5,6], for holomorphic representations of a Lie group G acting on a domain D in $\mathbb{C}^n$. In Theorem 2.1, we give a sufficient condition for the existence of unitarising measure when h_g in (1.1) is given by

$$h_g(z) = \big(\theta(g)\big)^\beta \big[\det\big(k_g'(z)\big)\big]^\alpha e^{\varphi(k_g(z))-\varphi(z)}, \tag{1.5}$$

where $\theta : G \mapsto \mathbb{C}^\star$ is a group homomorphism and $k_g'(z)$ is the holomorphic Jacobian of k_g. Let $\rho(v) = H(v) + \beta(v)I$ be the infinitesimal representation of a unitary holomorphic representation (T_g,μ), let v and w belong to the Lie algebra $\mathcal{G}$ and assume that $\rho(v)$, $\mathrm{i}\rho(v)$, $\rho(w)$ and $\mathrm{i}\rho(w)$ are vectors fields coming from the infinitesimal representation, we show in Theorem 2.2 that the operator $\Delta_{v,w} = \rho(v)\overline{\rho(w)} + \rho(w)\overline{\rho(v)}$ is invariant with respect to μ. In Section 3, with the affine action (1.4) of the complex affine group $\mathbf{A}_c(n) = GL(n,\mathbb{C}) \ltimes \mathbb{C}^n$ on $\mathbb{C}^n$, we obtain the holomorphic representation

$$\big(\Xi_g^{p,\varphi} F\big)(z) = \big(\det(A)\big)^p \exp\big(\varphi\big(A^{-1}(z-h)\big) - \varphi(z)\big) F\big(A^{-1}(z-h)\big), \tag{1.6}$$

where p is an integer and φ is a $\mathbb{C}$-valued holomorphic function on $\mathbb{C}^n$. If the unitarising measure μ is absolutely continuous w.r.t. the Lebesgue measure λ on $\mathbb{C}^n$, i.e., $d\mu(z) = \Theta(z,\bar{z})\, d\lambda(z)$, then $\Lambda(z) := \Theta(z)e^{-\varphi(z)-\overline{\varphi(z)}}$ satisfies the functional equation

$$\Lambda(z) = \left|\det(A)\right|^{2p+2}\Lambda(Az+h) \tag{1.7}$$

for all $z \in \mathbb{C}^n$ and all $(A,h) \in GL(n,\mathbb{C}) \times \mathbb{C}^n$. See Theorem 3.1 and Theorem 3.4. Non-zero constants are solutions of (1.7) when $p = -1$. If we suppose in addition that the density $\Theta(z,\bar{z})$ is a continuous function on $\mathbb{C}^n$ such that $\sup_{z\in\mathbb{C}^n}(\Theta(z,\bar{z})e^{-\varphi(z)-\overline{\varphi(z)}}) < +\infty$, they are the only solutions of (1.7). This leads us to determine a class of non-Gaussian unitarising measures for $\Xi_g^{-1,\varphi}$, given by

$$d\mu_\varphi(z) = e^{\varphi(z)+\overline{\varphi(z)}}\, d\lambda(z). \tag{1.8}$$

Since $(\Xi_g^{-1,\varphi}, d\mu_\varphi)$ is also a unitary holomorphic representation of the real affine group $A_r(n) = GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$, by direct calculation we show that the unitarising measure $d\mu_\varphi$ in (1.8) is invariant for the second-order differential operators

$$\Delta_1 := \sum_{k=1}^{n}\left\{\left(\rho(A_{kk}) + \overline{\rho(A_{kk})}\right)\overline{H_k} - \left(\rho(V_k) + \overline{\rho(V_k)}\right)\overline{H_{kk}}\right\} \quad \text{for } n \geqslant 1$$

and $\Delta_2 := \sum_{k,l=1,\,k\neq l}^{n} \rho(A_{k,l})\overline{H_{k,l}}$ where $n \geqslant 2$, $(A_{k,l}, V_k)_{1\leqslant k,l\leqslant n}$ is a basis of the Lie algebra of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$, $H_k = H(V_k)$, $H_{kk} = H(A_{kk})$.

In Sections 4 and 5, we obtain non-constant solutions of (1.7) by considering actions of the translation and the Euclidean subgroups on $\mathbb{C}^n$. This provides a large class of unitarising measures (see Theorem 4.1 and Theorem 5.1). Hence, with parametric actions, we find a family of complex OU operators invariant w.r.t. the Gaussian measure.

2. Preliminaries and theoretical results

2.1. The problem of unitarising measures

Given a Lie group G and a non-negative real measure μ on a domain D in the Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$, let $\mathcal{H}L^2(D,\mu) := \mathcal{H}(D) \cap L^2(\mu)$ be the vector space of holomorphic functions $f : D \to \mathbb{C}$ such that $|f|^2$ is μ-integrable. The space $\mathcal{H}L^2(D,\mu)$ is a Hilbert space with the inner product

$$\langle f_1, f_2\rangle = \int_D f_1(z)\overline{f_2(z)}\, d\mu(z), \qquad f_1, f_2 \in \mathcal{H}L^2(D,\mu).$$

Consider the operators $T_g : \mathcal{H}L^2(D,\mu) \to \mathcal{H}L^2(D,\mu)$, $g \in G$, with the following properties:

1. T_g is given by $(T_g f)(z) = h_g(z) f(k_g(z))$ where $h_g : D \to \mathbb{C}$ and $k_g : D \to D$ are holomorphic functions.
2. The map $g \mapsto T_g$ is group homomorphism from G into the group of automorphisms on $\mathcal{H}(D)$, i.e., for any $g_1, g_2 \in G$ and for any holomorphic function f on D, we have $T_{g_1 g_2} f = T_{g_1}(T_{g_2} f)$.

3. T_g is unitary, that is for $f \in \mathcal{H}L^2(D,\mu)$, it holds

$$\int_D \left|T_g(f)(z)\right|^2 \mathrm{d}\mu(z) = \int_D \left|f(z)\right|^2 \mathrm{d}\mu(z).$$

If e denotes the neutral element of G then $(T_e f)(z) = f(z)$. Therefore, for any $g \in G$, the operator T_g is invertible with $(T_g)^{-1} = T_{g^{-1}}$.

Conditions 1 and 2 together give the following system on h_g and k_g,

$$h_{g_1}(z)h_{g_2}\big(k_{g_1}(z)\big) = h_{g_1g_2}(z), \tag{2.1}$$

$$k_{g_2}\big(k_{g_1}(z)\big) = k_{g_1g_2}(z). \tag{2.2}$$

When conditions 1, 2 and 3 are satisfied, we say that the measure μ is *unitarising* for T_g and we denote the unitary representation of G by (T_g,μ).

The following remarks (see [6]) are useful to obtain functions h_g satisfying (2.1).

Remarks 2.1. Assuming that the map k_g is determined, then:

1. For any $\mathbb{C}$-valued holomorphic function φ on D,

$$h_g(z) = \exp\big(\varphi\big(k_g(z)\big) - \varphi(z)\big) \tag{2.3}$$

satisfies (2.1).
2. If $\hat{h}_g$ and $\tilde{h}_g$ are both solutions of (2.1), then the product $h_g = \hat{h}_g\tilde{h}_g$ is also solution of (2.1).
3. Let $\det(k'_g(z)) =: \det(\frac{\partial k^j_g(z)}{\partial z_p})$ be the Jacobian determinant of the map k_g. Then $h_g(z) = \det(k'_g(z))$ is solution of (2.1).
4. Let θ be a group homomorphism from G into the complex multiplicative group, then h_g given by $h_g(z) = \theta(g)$ for $(z,g) \in D \times G$, is solution of (2.1).

Notation 2.1. Let $n \geqslant 1$, for $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $z_k = x_k + \mathrm{i}y_k$, we denote the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$ by $\mathrm{d}\lambda(z) = \prod_{k=1}^n \mathrm{d}x_k\,\mathrm{d}y_k$.

Theorem 2.1. *Let k_g be a holomorphic solution of (2.2) on the domain $D \subset \mathbb{C}^n$, θ be a group homomorphism from G to the complex multiplicative group and α, β be two real numbers. Then*

$$T_g^{\alpha,\varphi}(f)(z) = \big(\theta(g)\big)^\beta\big[\det\big(k'_g(z)\big)\big]^\alpha e^{\varphi(k_g(z))-\varphi(z)} f\big(k_g(z)\big) \tag{2.4}$$

defines a holomorphic representation of the group G. Assume that there is a real-valued map $\Lambda(z,\bar{z})$ on D satisfying

$$\left|\theta(g)\right|^{2\beta}\left|\det\big(k'_g(z)\big)\right|^{2\alpha-2}\Lambda(z,\bar{z}) = \Lambda\big(k_g(z),\overline{k_g(z)}\big) \quad \forall g \in G,\ \forall z \in D. \tag{2.5}$$

Then the measure

$$\mathrm{d}\mu(z) = \exp\big(\varphi(z) + \overline{\varphi(z)}\big)\Lambda(z,\bar{z})\,\mathrm{d}\lambda(z) \tag{2.6}$$

is unitarising for $T_g^{\alpha,\varphi}$.

Proof. Eq. (2.2) and Remarks 2.1 imply that the $T_g^{\alpha,\varphi}$ given in (2.4) satisfy $T_{g_1}^{\alpha,\varphi}T_{g_2}^{\alpha,\varphi} = T_{g_1g_2}^{\alpha,\varphi}$. The change of variables $u = k_g(z)$ shows the unitarity of $T_g^{\alpha,\varphi}$. □

If $\alpha = 1$ and $\beta = 0$, then $\Lambda = 1$ is always solution of (2.5). If $\alpha \neq 1$ or $\beta \neq 0$, there exist groups G where (2.5) has non-constant solutions Λ. We give below some simple examples. See (4.3)–(4.4) and (5.3)–(5.4).

2.2. Invariant operators of Ornstein–Uhlenbeck type

We introduce a class of second-order differential operators, called of Ornstein–Uhlenbeck type, invariant with respect to the unitarising measure of an holomorphic representation T_g of a Lie group G with Lie algebra $\mathcal{G}$.

Definition 2.1. A real measure μ on the domain D is called invariant for the operator Δ, or equivalently we say that Δ is invariant with respect to μ if

$$\int_D (\Delta\Psi)(z,\bar{z})\,\mathrm{d}\mu(z) = 0, \tag{2.7}$$

for all differentiable functions Ψ with compact support strictly contained in D.

Definition 2.2. Let T_g be an unitary representation of a Lie group into $\mathcal{H}L^2(D,\mu)$. We say that a second-order differential operator Δ^{OU} is of Ornstein–Uhlenbeck type if it satisfies the following conditions:

1. Δ^{OU} has the form
$$\Delta^{OU} = \sum_{j,k=1}^{n} a_{j,k}(z,\bar{z})\frac{\partial^2}{\partial z_j \partial \bar{z}_k} + \sum_{k=1}^{n} u_k(z,\bar{z})\frac{\partial}{\partial z_k} + \sum_{k=1}^{n} v_k(z,\bar{z})\frac{\partial}{\partial \bar{z}_k}, \tag{2.8}$$
where $a_{j,k}(z,\bar{z})$, $u_k(z,\bar{z})$ and $v_k(z,\bar{z})$ are continuous functions on D.
2. Δ^{OU} is invariant w.r.t. the unitarising measure μ of the representation T_g.

Lemma 2.1. *Let $\mathrm{d}\mu(z) = \Theta(z,\bar{z})\,\mathrm{d}\lambda(z)$, where $\Theta(z,\bar{z})$ is a positive differentiable function defined on $D \subset \mathbb{C}^n$. Let $u_{j,k}(z,\bar{z})$ and $Q_{j,k}(z,\bar{z})$ be two differentiable functions on D satisfying for all $j,k = 1,\dots,n$*

$$\frac{\partial Q_{j,k}}{\partial z_j} - \frac{\partial \log(\Theta u_{j,k})}{\partial z_j} = 0. \tag{2.9}$$

Then

$$\Delta := \sum_{j,k=1}^{n} u_{j,k}(z,\bar{z})\left(\frac{\partial^2}{\partial z_j \partial \bar{z}_k} + \frac{\partial Q_{j,k}}{\partial z_j}(z,\bar{z})\frac{\partial}{\partial \bar{z}_k}\right) \tag{2.10}$$

is invariant with respect to μ.

Proof. Integrating by parts with respect to $\frac{\partial}{\partial z_j}$, we prove with condition (2.9) that $\int_{\mathbb{C}^n} \Delta\Psi(z,\bar{z})\,\mathrm{d}\mu(z) = 0$ when Ψ is continuously twice-differentiable with compact support. □

Of course, there exist several operators of the form (2.10) which are invariant with respect to the *same* measure μ. For example, see (4.10) for such operators. In our work, we consider the cases when Δ^{OU} is associated to a Kählerian metric on D, thus the matrix $u = (u_{j,k})$ satisfies ${}^t\bar{u} = u$, see [7], and furthermore we would like Δ^{OU} to be written with vector fields of the infinitesimal representation of the Lie algebra $\mathcal{G}$.

2.3. Infinitesimal representation of the Lie algebra $\mathcal{G}$ and OU operators on $\mathbb{C}^n$

In [1,3,2,5,6] and [16], the unitarising measure and its relation to the OU operator has been studied for the representation of the Lie algebra $\mathcal{G}$ and not directly for the group G. Let $\epsilon \mapsto g_\epsilon$ be a differentiable curve on G such that $g_0 = e$, we put

$$v = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g_\epsilon. \tag{2.11}$$

For $f \in \mathcal{H}L^2(D,\mu)$, the infinitesimal representation $\rho(v)$ of T_g is given by

$$\big(\rho(v)f\big)(z) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} T_{g_\epsilon} f(z). \tag{2.12}$$

Let $z = (z_1,\dots,z_n) \in D$. Since $k_g(z) = ((k_g)_1(z), (k_g)_2(z), \dots, (k_g)_n(z)) \in D$ and $h_g(z) \in \mathbb{C}$, we put

$$\alpha_j(v)(z) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (k_{g_\epsilon})_j(z), \quad \text{and} \quad \beta(v)(z) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} h_{g_\epsilon}(z),$$

then

$$\rho(v) = H(v) + \beta(v)I, \tag{2.13}$$

where $H(v)$ is the holomorphic vector field given by

$$\big(H(v)f\big)(z) = \sum_{j=1}^n \alpha_j(v)(z)\frac{\partial f}{\partial z_j}(z) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} f\big(k_{g_\epsilon}(z)\big). \tag{2.14}$$

Let f and ϕ be two holomorphic functions. Since the representation T_g is unitary, we have

$$\int \big(\rho(v)f\big)(z)\overline{\phi(z)}\,d\mu(z) + \int f(z)\overline{\big(\rho(v)\phi\big)(z)}\,d\mu(z) = 0. \tag{2.15}$$

Remark 2.1. The divergence of a vector field V with respect to a measure μ is the function, denoted div_μ, satisfying $\int (div_\mu V)(z)\Phi(z)\,d\mu(z) = \int (V\Phi)(z)\,d\mu(z)$, for any differentiable function Φ with compact support in D and vanishing out of the support of μ. Then Eq. (2.15) implies that $div_\mu(H(v) + \overline{H(v)}) = -(\beta(v) + \overline{\beta(v)})$.

Lemma 2.2. *Let $F(z,\bar{z}) = f(z)\overline{\phi(z)}$ where $f, \phi \in \mathcal{H}(D)$. Then,*

$$\int \big(\rho(v) + \overline{\rho(v)}\big)F(z,\bar{z})\,d\mu(z) = 0. \tag{2.16}$$

This stays true for functions $F(z,\bar{z})$ which can be approximated by polynomials in $(z,\bar{z})$.

Proof. With (2.15), since we have $\int \rho(v)F(z)\,d\mu(z) = \int (\rho(v)f(z))\overline{\phi(z)}\,d\mu(z)$ and $\int \overline{\rho(v)}F(z)\,d\mu(z) = \int f(z)\overline{\rho(v)\phi(z)}\,d\mu(z)$. □

Theorem 2.2. *Let G be a complex group. Let v and w be two independent elements of its Lie algebra $\mathcal{G}$. We consider the vector fields $\rho(v)$, $\rho(w)$. Assume that $\mathrm{i}\rho(v)$ is also a vector field in the infinitesimal representation, then*

$$\Delta_{v,w} := \rho(v)\overline{\rho(w)} + \rho(w)\overline{\rho(v)} \tag{2.17}$$

is a real operator invariant with respect the unitarising measure, more precisely for all F such that (2.16) is true, it holds: $\int_D \Delta_{v,w} F(z,\bar{z})\,\mathrm{d}\mu(z) = 0$.

Proof. The commutation $\overline{\rho(v)}\rho(w) = \rho(w)\overline{\rho(v)}$ implies $\overline{\Delta_{v,w}} = \Delta_{v,w}$. Then, by Lemma 2.2, $\int[\rho(v) + \overline{\rho(v)}]\overline{\rho(w)}F\,\mathrm{d}\mu = 0$ and $\int[\mathrm{i}\rho(v) + \overline{\mathrm{i}\rho(v)}]\overline{\rho(w)}F\,\mathrm{d}\mu = 0$. This implies $\int \rho(v)\overline{\rho(w)}F\,\mathrm{d}\mu = 0$. □

Corollary 2.1. *Let G be a complex Lie group and $(e_k)_{1\leqslant k\leqslant N}$, $(\mathrm{i}e_k)_{1\leqslant k\leqslant N}$ be independent elements in its Lie algebra. Then, for any sequence $\alpha = (\alpha_{k,j})_{1\leqslant k,j\leqslant N}$ of complex numbers, the operators*

$$\Delta_\alpha = \sum_{k,j=1}^{N} \alpha_{k,j}\Delta_{e_k,e_j} \tag{2.18}$$

are invariant with respect to the unitarising measure.

3. Unitary representations of the group of affine transformations on $\mathbb{C}^n$

The group $\mathbf{A}_c(n)$ of affine transformations of $\mathbb{C}^n$ is the semi-product $GL_n(\mathbb{C}) \ltimes \mathbb{C}^n$. Elements of $\mathbf{A}_c(n)$ are written as pairs $g = (A,h)$, where $A \in GL(n,\mathbb{C})$ and $h \in \mathbb{C}^n$. The product in $\mathbf{A}_c(n)$ is given by

$$(A,h) \circ (A',h') = (AA', Ah' + h). \tag{3.1}$$

The inverse of $g = (A,h)$ is $g^{-1} = (A^{-1}, -A^{-1}h)$. The group $\mathbf{A}_c(n)$ is isomorphic to a subgroup of $GL(n+1,\mathbb{C})$, via the assignment to each (A,h) the matrix $\begin{pmatrix} A & h \\ 0_n & 1 \end{pmatrix}$, where $h = {}^t(h_1,\ldots,h_n) \in \mathbb{C}^n$, $0_n = (0,\ldots,0) \in \mathbb{C}^n$.

Throughout this section, we consider the affine group action on $\mathbb{C}^n$ given by

$$\psi(g,z) = Az + h, \quad \text{for } g = (A,h) \in \mathbf{A}_c(n),\ z \in \mathbb{C}^n. \tag{3.2}$$

Theorem 3.1. *Let φ be a complex-valued holomorphic function on $\mathbb{C}^n$ and $p \in \mathbb{Z}$, then for $g = (A,h)$, the transformation $\Xi_g^{p,\varphi}$ given by (1.6) defines a holomorphic representation of G into $\mathcal{H}(\mathbb{C}^n)$. For $g^{-1} = (A^{-1}, -A^{-1}h)$,*

$$\left(\Xi_{g^{-1}}^{p,\varphi}F\right)(z) = \left(\det(A)\right)^{-p} e^{\varphi(Az+h)-\varphi(z)} F(Az+h). \tag{3.3}$$

Moreover, the measure $\mathrm{d}\mu(z) = \Theta(z,\bar{z})\,\mathrm{d}\lambda(z)$ is a unitarising for $\Xi_g^{p,\varphi}$ if the function

$$\Lambda(z) = \Theta(z,\bar{z})e^{-\varphi(z)-\overline{\varphi(z)}} \tag{3.4}$$

satisfies (1.7) for all $z \in \mathbb{C}^n$ and all $(A,h) \in GL(n,\mathbb{C}) \times \mathbb{C}^n$.

Proof. Remarks 2.1 imply that the transformation $\Xi_g^{p,\varphi}$ in (1.6) is a holomorphic representation. We verify by change of variables that (3.4) satisfies (1.7). □

Corollary 3.1. *If $p=-1$, the holomorphic representation $\Xi_g^{-1,\varphi}$ is unitary into the Hilbert space $\mathcal{H}L^2(\mathbb{C}^n, \mathrm{d}\mu_\varphi)$ where $\mathrm{d}\mu_\varphi$ is the non-Gaussian measure*

$$\mathrm{d}\mu_\varphi(z) = e^{\varphi(z)+\overline{\varphi(z)}}\,\mathrm{d}\lambda(z). \tag{3.5}$$

Proof. For $p=-1$, (1.7) has constants as only solutions. The measure $\mathrm{d}\mu_\varphi$ is non-Gaussian since no holomorphic function φ satisfies $\varphi(z)+\overline{\varphi(z)} = \sum_{k=1}^n z_k\bar{z}_k$, $\forall z=(z_1,\dots,z_n)\in\mathbb{C}^n$. □

Remark 3.1. If we assume the condition $\sup_{z\in\mathbb{C}^n}(\Theta(z,\bar{z})e^{-\varphi(z)-\overline{\varphi(z)}}) < +\infty$, for the continuous function Θ, we can pass to the limit in (1.7) when $\|A\|\to 0$ to find that the unique solution of (1.7) is the null map if $p>-1$. Similarly, for $p<-1$, just pass to the limit in (1.7) when $\|A\|\to+\infty$, to see that Λ is the null map, where $\|.\|$ is the classical maximum norm on $M(n,\mathbb{C})$.

Theorem 3.1 and corollary 3.1 remain valid for affine action on $\mathbb{C}^n$ of the real Lie group $\mathbf{A}_r(n) = GL(n,\mathbb{R})\ltimes\mathbb{R}^n$ with the group law (3.1). In that case, by considering subgroups of $\mathbf{A}_r(n)$, we shall find more non-trivial solutions for (1.7). See Theorems 4.1 and 5.1.

3.1. The Lie algebra $\mathcal{A}_r(n)$ and its infinitesimal representation

The Lie algebra $\mathcal{A}_r(n)$ of the real affine group $\mathbf{A}_r(n)$ is the vector space of $(n+1)\times(n+1)$ real matrices spanned by

$$\left\{ A_{k,l} = \begin{pmatrix} & & & 0\\ & E_{k,l} & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & 0\end{pmatrix},\ V_k = \begin{pmatrix} 0 & \cdots & 0 & \\ \vdots & & \vdots & \delta_k\\ \vdots & \ddots & & \\ 0 & \cdots & 0 & 0\end{pmatrix},\ 1\leqslant k,l\leqslant n \right\}, \tag{3.6}$$

where the matrix $E_{k,l}$ has exactly one non-zero entry which is 1 in the (k,l) position, δ_k is the vector of $\mathbb{R}^n$ whose components are zero except the k-th equal to 1.

Proposition 3.1. *Let $n\geqslant 1$, $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ and $\varphi\in\mathcal{H}(\mathbb{C}^n)$. Consider $(\Xi_g^{-1,\varphi}, \mathrm{d}\mu_\varphi)$ as in Corollary 3.1, then the associated infinitesimal representation on the Lie algebra $\mathcal{A}_r(n)$ is given by*

$$\rho_{k,k} = \rho(A_{k,k})F(z) = -z_k\frac{\partial F}{\partial z_k}(z) - \left(1 + z_k\frac{\partial\varphi}{\partial z_k}(z)\right)F(z), \tag{3.7}$$

$$\rho_{k,l} = \rho(A_{k,l})F(z) = -z_l\frac{\partial F}{\partial z_k}(z) - z_l\frac{\partial\varphi}{\partial z_k}(z)F(z), \quad \text{if } k\neq l, \tag{3.8}$$

$$\rho_q = \rho(V_q)F(z) = -\frac{\partial F}{\partial z_q}(z) - \frac{\partial\varphi}{\partial z_q}(z)F(z). \tag{3.9}$$

Proof. As in (2.12), for the curves $g_\epsilon^{(k,l)} := I_{n+1} + \epsilon A_{k,l}$ and $\tilde{g}_\epsilon^k := I_{n+1} + \epsilon V_k$. □

Theorem 3.2. *We denote $\overline{H_k} = -\frac{\partial}{\partial\bar{z}_k}$ and $\overline{H_{k,k}} = -\bar{z}_k\frac{\partial}{\partial\bar{z}_k}$. The unitarising measure $\mathrm{d}\mu_\varphi$ in (3.5) is invariant for*

$$\Delta_1 := \sum_{k=1}^{n} \big\{ (\rho_{kk} + \overline{\rho_{kk}}) \overline{H_k} - \big(\rho(V_k) + \overline{\rho(V_k)}\big) \overline{H_{kk}} \big\}$$
$$= \sum_{k=1}^{n} \left\{ (z_k - \bar{z}_k) \frac{\partial^2}{\partial z_k \partial \bar{z}_k} + \left(1 + (z_k - \bar{z}_k) \frac{\partial \varphi}{\partial z_k}(z)\right) \frac{\partial}{\partial \bar{z}_k} \right\}. \tag{3.10}$$

Hence, the OU operator is given by $\Delta^{OU} = \mathrm{i}(\Delta_1 - \overline{\Delta_1})$*,*

$$\Delta^{OU} = -\sum_{k=1}^{n} \left\{ \frac{1}{2} y_k \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) + 2 y_k \frac{\partial \varphi_1}{\partial x_k} \frac{\partial}{\partial x_k} + \left(1 - 2 y_k \frac{\partial \varphi_2}{\partial x_k}\right) \frac{\partial}{\partial y_k} \right\}, \tag{3.11}$$

where φ_1 *and* φ_2 *are respectively the real and the imaginary part of* φ*.*

On the domain $D = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \forall k = 1, \dots, n;\ z_k \neq \bar{z}_k\}$*, the metric associated to the differential operator* Δ_1 *on* D *is Kählerian with potential* K *given by*

$$K(z) = \sum_{k=1}^{n} (\bar{z}_k - z_k) \log(z_k - \bar{z}_k) + (\bar{z}_k - z_k). \tag{3.12}$$

Proof. To prove the invariance of the operator Δ_1 w.r.t. the unitarising measure $\mathrm{d}\mu_\varphi$, we can use Lemma 2.2 or just integrate by parts. Moreover, according to [7, Theorem 4.17], the metric associated to the potential K in (3.12) is Kählerian. □

Remark 3.2. Let $n \geqslant 2$. Integrating by parts, we can verify that the unitarising measure $\mathrm{d}\mu_\varphi$ (3.5) is also invariant for

$$\Delta_2 := \sum_{k,l=1,\, k\neq l}^{n} \rho_{k,l} \overline{H_{k,l}}$$
$$= \sum_{k,l=1,\, k\neq l}^{n} z_l \bar{z}_l \left\{ \frac{\partial^2}{\partial z_k \partial \bar{z}_k} + \frac{\partial \varphi}{\partial z_k}(z) \frac{\partial}{\partial \bar{z}_k} \right\}. \tag{3.13}$$

The real part of Δ_2 is given by

$$\Delta_2 + \overline{\Delta_2} = \sum_{k,l=1,\, k\neq l}^{n} \frac{1}{2}\big(x_l^2 + y_l^2\big) \left\{ \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + 2 \frac{\partial \varphi_1}{\partial x_k} \frac{\partial}{\partial x_k} - 2 \frac{\partial \varphi_2}{\partial x_k} \frac{\partial}{\partial y_k} \right\}. \tag{3.14}$$

We note that there is no Kählerian structure associated to Δ_2 and we observe that the unitarising measure $\mathrm{d}\mu_\varphi$ is not invariant for $\sum_{k=1}^{n} \rho_{kk} \overline{H_{kk}}$.

3.2. Infinitesimal representation of the Lie algebra $\mathcal{A}_c(n)$ and Ornstein–Uhlenbeck operators

Let $(A_{k,l}, V_j)$ as in (3.6). The Lie algebra $\mathcal{A}_c(n)$ of the complex affine group $\mathbf{A}_c(n)$ is the real vector space spanned by $(A_{k,l}, V_j)$ and $(\mathrm{i}A_{k,l}, \mathrm{i}V_j)$. Consider the representation $(\Xi_g^{-1}, \mathrm{d}\mu_\varphi)$ obtained in Corollary 3.1. The infinitesimal representation of $\mathcal{A}_c(n)$ associated to $(\Xi_g^{-1}, \mathrm{d}\mu_\varphi)$ is given by the vector fields $(\rho_{k,k}, \rho_{k,l}, \rho_q)$ obtained in (3.7)–(3.8)–(3.9) and in addition the vector fields $(\mathrm{i}\rho_{k,k}, \mathrm{i}\rho_{k,l}, \mathrm{i}\rho_q)$. We can apply Theorem 2.2 and its corollary. We give an example when $n = 2$.

Example 3.1. The measure $d\mu_\varphi$ is invariant for

$$\rho_{1,2}\overline{\rho_{1,2}} + \rho_{2,1}\overline{\rho_{2,1}} + \sum_{j=1,2} \rho_{j,j}\overline{\rho_{j,j}} + \rho_j\overline{\rho_j}$$
$$= p(z_1, z_2)\left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}\right) + \text{first-order differential operator} \tag{3.15}$$

where $p(z_1, z_2) = 1 + |z_1|^2 + |z_2|^2$; No Kählerian structure is associated to (3.15).

3.3. Non-constant solutions of the functional equation (1.7)

Let G_0 be a subgroup of $GL(n, \mathbb{R})$. For an affine action of the semidirect product $G_0 \ltimes \mathbb{R}^n$ on $\mathbb{C}^n$, we may have non-constant solutions of the functional equation (1.7). To obtain unitarising measures for representations of such groups, we proceed as follows:

Lemma 3.1. *Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $t \in \mathbb{C}$, the map $Y_t : \mathbb{C}^n \to \mathbb{R}^n$, defined by*

$$Y_t(z) = \mathrm{i}(\bar{t}z - t\bar{z}) \tag{3.16}$$

satisfies the functional equation

$$Y_t(Az + th) = AY_t(z), \quad \forall A \in G_0,\ \forall h \in \mathbb{R}^n. \tag{3.17}$$

Theorem 3.3. *Let $L : \mathbb{R}^n \to \mathbb{R}^+$ such that $(\det(A))^{2p+2}L(AY) = L(Y)$ for all $A \in G_0$ and $Y \in \mathbb{R}^n$. Then, the functional Λ defined by*

$$\Lambda(z) = L\big(Y_t(z)\big) \tag{3.18}$$

is solution of (1.7). Furthermore, the measures $(\mu_t)_{t\in\mathbb{C}}$ defined by

$$\mathrm{d}\mu_t(z) = L\big(Y_t(z)\big)e^{\varphi(z)+\overline{\varphi(z)}}\,\mathrm{d}\lambda(z) \tag{3.19}$$

are unitarising for the holomorphic representation $\Xi_g^{p,\varphi}$ given by (1.6).

In the next two sections, we give applications of Theorem 3.3. On $\mathbb{R}^n$, consider the usual scalar product $\langle x, y\rangle = x_1y_1 + \dots + x_ny_n$, where $x = (x_j)_{j=1,\dots,n}$ and $y = (y_j)_{j=1,\dots,n}$. We extend the bilinear form $\langle x, y\rangle$ on $\mathbb{R}^n$ to $\mathbb{C}^n$ by putting

$$\langle z, w\rangle = z_1w_1 + \dots + z_nw_n \tag{3.20}$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. In particular $\langle z, z\rangle = \sum_{k=1}^n z_k^2$.

4. Parametric holomorphic actions of the translation group on $\mathcal{H}(\mathbb{C}^n)$

In this section we study the problem of unitarising measure for the holomorphic representation of the group of translations $G_1 = \{(I_n, h),\ h \in \mathbb{R}^n\}$ with the parametric group action ψ_U ($U \in GL(n, \mathbb{C})$), defined by $\psi_U(h, z) = z + Uh$.

Theorem 4.1. *Let $U \in GL(n, \mathbb{C})$ be an invertible complex matrix such that $\overline{U}U = U\overline{U}$, ϕ_U be a holomorphic function on $\mathbb{C}^n$ and L be a real-valued mapping defined on $\mathbb{R}$. Then the holomorphic parametric representation T_h^{U,ϕ_U} given by*

$$\left(T_h^{U,\phi_U} F\right)(z) = \exp\left(\phi_U(z - Uh) - \phi_U(z)\right) F(z - Uh), \quad h \in \mathbb{R}^n \tag{4.1}$$

is unitary in $\mathcal{H}L^2(\mathbb{C}^n, \mu_{L,U,\phi_U})$*, where* μ_{L,U,ϕ_U} *is given by*

$$\mathrm{d}\mu_{L,U,\phi_U}(z) = L\left(\langle \overline{U} z - U\bar{z}, \overline{U} z - U\bar{z}\rangle\right) \exp\left(\phi_U(z) + \overline{\phi_U(z)}\right) \mathrm{d}\lambda(z), \tag{4.2}$$

and $\langle\, , \rangle$ *is the symmetric bilinear form on* $\mathbb{C}^n$ *defined in Eq.* (3.20).

Proof. $k_g^U(z) = z - Uh$ is solution of (2.2) and $H_h^U(z) = \exp(\phi_U(z - Uh) - \phi_U(z))$ is solution of (2.1). Hence, the transformations T_h^{U,ϕ_U} in Eq. (4.1) define a holomorphic representation of the translation group, $T_{h_1}^{U,\phi_U} T_{h_2}^{U,\phi_U} = T_{h_1+h_2}^{U,\phi_U}$ for every $h_1, h_2 \in \mathbb{R}^n$. Let

$$\Lambda_U(z) = \Theta_U(z, \bar{z}) \exp\left(-\phi_U(z) - \overline{\phi_U(z)}\right)$$

where Θ_U is the density of the unitarising measure with respect to the Lebesgue measure. Unitarity is obtained when Λ_U satisfies

$$\Lambda_U(z) = \Lambda_U(z + Uh). \tag{4.3}$$

Let $M_U : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $M_U(z) = \overline{U} z - U\bar{z}$. Since $h \in \mathbb{R}^n$ and $\overline{U}U = U\overline{U}$, we have $M_U(z) = M_U(z + Uh)$. This implies that

$$\Lambda_U(z) = L\left(\langle \overline{U} z - U\bar{z}, \overline{U} z - U\bar{z}\rangle\right) \tag{4.4}$$

is a solution of (4.3). □

Corollary 4.1. *Let* σ *be a positive number, assuming that the complex matrix* U *in the previous theorem is unitary, i.e., it satisfies* ${}^tU\overline{U} = I_n$*. Then for*

$$L(x) = e^{\sigma x} \quad \text{and} \quad \phi_U(z) := -\sigma \langle \overline{U} z, \overline{U} z\rangle,$$

the measure in (4.2) *is Gaussian*

$$\mathrm{d}\mu_\sigma(z) = e^{-2\sigma\langle z, \bar{z}\rangle} \mathrm{d}\lambda(z) \tag{4.5}$$

and also unitarising for the Fock representation

$$\left(T_h^{U,\sigma} F\right)(z) = e^{2\sigma\langle z, \overline{U} h\rangle - \sigma\langle h, h\rangle} F(z - Uh), \quad h \in \mathbb{R}^n. \tag{4.6}$$

Proof. $\phi_U(z - Uh) - \phi_U(z) = -\sigma\langle \overline{U} U h, \overline{U} U h\rangle + 2\sigma\langle \overline{U} z, \overline{U} U h\rangle$. Since $\overline{U}U = U\overline{U}$, we deduce that $\langle \overline{U} U h, \overline{U} U h\rangle = \langle h, h\rangle$. □

Lemma 4.1. *The infinitesimal representation associated to the unitary representation* $T_h^{U,\sigma}$ *in* (4.6) *is given by*

$$\begin{aligned}(\rho_{j,U} F)(z) &:= \frac{\mathrm{d}}{\mathrm{d}h_j}\bigg|_{h=0_{\mathbb{R}^n}} \left(T_h^{U,\sigma} F\right)(z) \\ &= -\sum_{k=1}^n U_{k,j} \frac{\partial F}{\partial z_k}(z) + 2\sigma \sum_{k=1}^n \overline{U}_{k,j} z_k F(z).\end{aligned} \tag{4.7}$$

We put $H_{j,U} := -\sum_{k=1}^n U_{k,j} \frac{\partial}{\partial z_k}$.

Remark 4.1. Let U and V be two $n \times n$ symmetric unitary matrices, i.e., ${}^t\overline{U}U = I_n$, ${}^tU = U$ and ${}^t\overline{V}V = I_n$, ${}^tV = V$. If $UV = VU$, then we have $U\overline{V} = \overline{V}U$. But in general $U\overline{V} \neq V\overline{U}$. For example, let t_1, t_2 be two distinct real numbers. For $t \in \mathbb{R}$, we put

$$U(t) = \begin{pmatrix} \cos(t) & \mathrm{i}\sin(t) \\ \mathrm{i}\sin(t) & \cos(t) \end{pmatrix}, \qquad V(t) = \begin{pmatrix} \mathrm{i}\sin(t) & \cos(t) \\ \cos(t) & \mathrm{i}\sin(t) \end{pmatrix}. \tag{4.8}$$

Then $U(t_1)V(t_2) = V(t_2)U(t_1)$ and $U(t_1)\overline{V}(t_2) - V(t_2)\overline{U}(t_1) = 2\mathrm{i}\sin(t_1 - t_2)I$, where I is the 2×2 identity matrix.

Proposition 4.1. *Let U and V be two complex matrices which commute and each of them satisfy: ${}^t\overline{U}U = I_n$ and ${}^tU = U$. We put*

$$\Delta = \sum_{j=1}^{n} \big((\rho_{j,U} + \overline{\rho_{j,U}}) \overline{H_{j,V}} - (\rho_{j,V} + \overline{\rho_{j,V}}) \overline{H_{j,U}} \big). \tag{4.9}$$

Then

$$\Delta = \sum_{k,q=1}^{n} (U\overline{V} - V\overline{U})_{k,q} \left[\frac{\partial^2}{\partial z_k \partial \bar{z}_q} - 2\sigma \bar{z}_k \frac{\partial}{\partial \bar{z}_q} \right], \tag{4.10}$$

where $(U\overline{V} - V\overline{U})_{k,q}$ is the entry at line k and column q of $U\overline{V} - V\overline{U}$. The metric associated to $\mathrm{i}\Delta$ is Kählerian.

Furthermore, we have

$$\int_{\mathbb{C}^n} \Delta\Phi(z) e^{-2\sigma\langle z,\bar{z}\rangle} \,\mathrm{d}\lambda(z) = 0, \tag{4.11}$$

for all continuously twice-differentiable function Φ with compact support.

Proof. For all $j \in \{1, \ldots, n\}$,

$$\begin{aligned} \Delta_j &:= (\rho_{j,U} + \overline{\rho_{j,U}}) \overline{H_{j,V}} - (\rho_{j,V} + \overline{\rho_{j,V}}) \overline{H_{j,U}} \\ &= \sum_{k,q=1}^{n} (U_{k,j} \overline{V_{q,j}} - V_{k,j} \overline{U_{q,j}}) \frac{\partial^2}{\partial z_k \partial \bar{z}_q} \\ &\quad - 2\sigma \sum_{k,q=1}^{n} (U_{k,j} \overline{V_{q,j}} - V_{k,j} \overline{U_{q,j}}) \bar{z}_k \frac{\partial}{\partial \bar{z}_q} \\ &\quad - 2\sigma \sum_{k,q=1}^{n} (\overline{U_{k,j}}\, \overline{V_{q,j}} - \overline{V_{k,j}}\, \overline{U_{q,j}}) z_k \frac{\partial}{\partial \bar{z}_q}. \end{aligned} \tag{4.12}$$

The coefficient of $z_k \frac{\partial}{\partial \bar{z}_q}$ in $\Delta = \sum_{j=1}^{n} \Delta_j$ is equal to 0, in fact

$$\sum_{j=1}^{n} (\overline{U_{k,j}}\, \overline{V_{q,j}} - \overline{V_{k,j}}\, \overline{U_{q,j}}) = (\overline{U}\,\overline{V})_{k,q} - (\overline{V}\,\overline{U})_{k,q} = 0, \tag{4.13}$$

since U and V are symmetric and commute. On the other hand, the matrix $A = U\overline{V} - V\overline{U}$ satisfies $\overline{A} + A = 0$ and ${}^tA = -\overline{A}$. Thus $P = \mathrm{i}A$ satisfies ${}^t\overline{P} = P$. This proves that the metric associated to $\mathrm{i}\Delta$ is Kählerian. □

Remark 4.2. Let $\beta \in \mathbb{C}$. Assume that $U\overline{V} - V\overline{U} = \beta I$ as it is the case for the example in Remark 4.1. We obtain

$$\Delta = \beta \sum_{k=1}^{n} \left[\frac{\partial^2}{\partial z_k \partial \bar{z}_k} - 2\sigma \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right]. \tag{4.14}$$

However, we cannot obtain (4.14) as an expression $\sum_{j=1}^{n} (\rho_{j,U} + \overline{\rho_{j,U}}) \overline{H_{j,U}}$. We need to use two different parametric actions to obtain Δ. with the vector fields of the infinitesimal representation. The operator Δ given in Eq. (4.14) is a complex OU operator corresponding to the Kählerian metric $\mathrm{d}s^2 = \sum_{k=1}^{n} \mathrm{d}z_k \, \mathrm{d}\bar{z}_k$.

5. The Euclidean Motion group $E(n) = SO(n) \ltimes \mathbb{R}^n$

The Euclidean motion group $E(n)$ is the group of one-to-one maps preserving the Euclidean distance of $\mathbb{R}^n$. Elements of $E(n)$ can be written as $X \mapsto RX + h$, with $R \in SO(n)$, where $SO(n)$ is the special orthogonal group and $h \in \mathbb{R}^n$. The group $E(n)$ is isomorphic to the subgroup of $GL(n+1, \mathbb{R})$ consisting of matrices $\begin{pmatrix} R & h \\ 0_n & 1 \end{pmatrix}$ and is isomorphic to the semi-product group $SO(n) \ltimes \mathbb{R}^n$ equipped with the (non-commutative) law, $(R, h) \circ (R', h') = (RR', Rh' + h)$, see [12].

5.1. Euclidean motion group actions and unitarising measures

In order to apply Theorem 3.3, we consider the action $\psi_t(g, z) = Rz + th$, with $g = (R, h) \in SO(n) \ltimes \mathbb{R}^n$ and where $t \in \mathbb{C}$ is a parameter.

Theorem 5.1. *Let $t \in \mathbb{C}$, $\varphi_t \in \mathcal{H}(\mathbb{C}^n)$, L be a real continuous mapping defined on $\mathbb{R}$, then the measure*

$$\mathrm{d}\mu_{L,\varphi_t}(z) = L\big(\big\langle \mathrm{i}(\bar{t}z - t\bar{z}), \mathrm{i}(\bar{t}z - t\bar{z})\big\rangle\big) e^{\varphi_t(z) + \overline{\varphi_t(z)}} \, \mathrm{d}\lambda(z) \tag{5.1}$$

is unitarising for the holomorphic representation $\Xi_g^{\varphi_t}$ defined by

$$\big(\Xi_g^{\varphi_t} F\big)(z) = e^{\varphi_t(R^{-1}(z - th)) - \varphi_t(z)} F\big(R^{-1}(z - th)\big), \quad g = (R, h) \in \mathbf{E}(n). \tag{5.2}$$

Proof. Let $\varphi_t \in \mathcal{H}(\mathbb{C}^n)$, we assume that the unitarising measure is absolutely continuous w.r.t. the Lebesgue measure, let Θ be its density. We put $\Lambda_t(z) = \Theta(z) e^{-\varphi_t(z) - \overline{\varphi_t(z)}}$. The unitarising condition implies

$$\Lambda_t(z) = \Lambda_t(Rz + th), \tag{5.3}$$

for all $z \in \mathbb{C}^n$ and all $(R, h) \in \mathbf{E}(n)$. According to Theorem 3.3, the function

$$\Lambda_t(z) = L\big(\big\langle \mathrm{i}(\bar{t}z - t\bar{z}), \mathrm{i}(\bar{t}z - t\bar{z})\big\rangle\big) \tag{5.4}$$

is a solution of the functional equation (5.3). □

Corollary 5.1. *Let $\sigma \in \mathbb{R}^+$ and $t \in \mathbb{C}^*$, assume that $\varphi_t(z) = -\sigma \bar{t}^2 \langle z, z \rangle$ and $L(y) = e^{-\sigma y}$. Then, the Gaussian measure*

$$\mathrm{d}\mu_{t,\sigma}(z) = e^{-2\sigma t\bar{t}\langle z, \bar{z}\rangle} \, \mathrm{d}\lambda(z) \tag{5.5}$$

is unitarising for the holomorphic representation $\Xi_g^{t,\sigma}$ *given by*

$$\big(\Xi_g^{t,\sigma} F\big)(z) = e^{2\sigma t(\bar{t})^2\langle z,h\rangle - \sigma(\bar{t}t)^2\langle h,h\rangle} F\big(R^{-1}(z - thf)\big). \tag{5.6}$$

Remark 5.1. The unitary Euclidean representation $\Xi_g^{t,\sigma}$ on $\mathcal{H}L^2(\mathbb{C}^n, \mu_{t,\sigma})$ and defined in Eq. (5.6), generalizes the Gaussian holomorphic representation $V_{(R,h)}$,

$$V_{(R,h)} = \Xi_{(R,h)}^{\frac{1}{2},2}, \quad (R,h) \in E(n),$$

studied by Driver and Hall in [8].

5.2. The Lie algebra $\mathcal{E}(n)$ and its infinitesimal representation

The Lie algebra $\mathcal{E}(n)$ is the real vector space of dimension $\frac{n(n+1)}{2}$ generated by the $(n+1) \times (n+1)$ matrices $A_{k,l} - A_{l,k}$ and V_j, $1 \leqslant k < l \leqslant n$, $1 \leqslant j \leqslant n$ where $A_{k,l}$ and V_j are defined in (3.6). Let $g_\epsilon^{(k,l)} := \exp(\epsilon A_{k,l})$ and $\tilde{g}_\epsilon^j := I_{n+1} + \epsilon V_j$, we calculate the infinitesimal representation as in (2.12).

Lemma 5.1. *Let $n \geqslant 2$. Let $t = 1$ and put $\varphi := \varphi_1$ in (5.2). The infinitesimal representation for $(\Xi_g^{\varphi_1}, \mathrm{d}\mu_{L,\varphi_1})$ defined in (5.2) is given by*

$$\begin{aligned} \rho_{k,l} &= \rho(A_{k,l}) = z_l \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_l} + \left(z_l \frac{\partial \varphi}{\partial z_k}(z) - z_k \frac{\partial \varphi}{\partial z_l}(z) \right) I, \quad 1 \leqslant k < l \leqslant n, \\ \rho_j &= \rho(V_j) = -\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j}(z) I, \quad j \in \{1, \ldots, n\}. \end{aligned} \tag{5.7}$$

Let $n \geqslant 3$, we put

$$H_{k,l} = z_l \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_l} \quad \text{and} \quad H_j = -\frac{\partial}{\partial z_j}. \tag{5.8}$$

If (k, l, j) are three distinct integers in $\{1, \ldots, n\}$, we have the commutation relations $[H_{k,l}, H_{l,j}] = H_{j,k} = -H_{k,j}$ and $[H_{k,l}, H_k] = H_l$. For $1 \leqslant k < j < l \leqslant n$, we have

$$H_{k,j} H_l - H_{k,l} H_j + H_{j,l} H_k = 0 \tag{5.9}$$

and

$$\begin{aligned} D_{k,j,l} &:= \rho_{k,j} H_l - \rho_{l,k} H_j + \rho_{j,l} H_k \\ &= (H_{k,j}\varphi)(z) \frac{\partial}{\partial z_l} - (H_{k,l}\varphi)(z) \frac{\partial}{\partial z_j} + (H_{j,l}\varphi)(z) \frac{\partial}{\partial z_k}. \end{aligned} \tag{5.10}$$

As consequence of (5.9)–(5.10) and Lemma 2.2, we obtain

Theorem 5.2. *For any $1 \leqslant k < j < l \leqslant n$, the operators*

$$\Delta_{k,j,l} = \rho_{k,j} \overline{H_l} - \rho_{k,l} \overline{H_j} + \rho_{j,l} \overline{H_k} + \overline{D_{k,j,l}} \tag{5.11}$$

are invariant with respect to the unitarising measure in (5.1).

5.3. The case $n = 3$

Let $(k, j, l) = (1, 2, 3)$, then $\Delta_{1,2,3} = \Delta^S_{1,2,3} + \Delta^F_{1,2,3}$ where

$$\Delta^S_{1,2,3} = \left(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}\right)\frac{\partial}{\partial \bar{z}_3} + \left(z_3 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_3}\right)\frac{\partial}{\partial \bar{z}_1} + \left(z_1 \frac{\partial}{\partial z_3} - z_3 \frac{\partial}{\partial z_1}\right)\frac{\partial}{\partial \bar{z}_2},$$

$$\Delta^F_{1,2,3} = 2\Re\big[(H_{2,3}\varphi)(z)\big]\frac{\partial}{\partial \bar{z}_1} + 2\Re\big[(H_{3,2}\varphi)(z)\big]\frac{\partial}{\partial \bar{z}_2} + 2\Re\big[(H_{1,2}\varphi)(z)\big]\frac{\partial}{\partial \bar{z}_3}. \tag{5.12}$$

Lemma 5.2. *In real coordinates, $z_j = x_j + \mathrm{i}y_j$, the second-order part $\Delta_{(2)}$ of the operator $\Delta := -(\Delta_{1,2,3} + \overline{\Delta_{1,2,3}})$ is equal to*

$$\Delta_{(2)} = \left(y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2}\right)\frac{\partial}{\partial x_1} + \left(y_3 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_3}\right)\frac{\partial}{\partial x_2} + \left(y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1}\right)\frac{\partial}{\partial x_3}. \tag{5.13}$$

In the space $\mathbb{R}^3 \times \mathbb{R}^3 = \{(X, Y) = (x_1, x_2, x_3, y_1, y_2, y_3)\}$, we define

$$W_1 = \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_2}, \qquad W_2 = \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_3},$$
$$W_3 = \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_1}.$$

Proposition 5.1. *The operator $\Delta_{(2)}$ in (5.13) can be expressed as*

$$\Delta_{(2)} = \frac{1}{2}\left(W_1^2 + W_2^2 + W_3^2 - \Delta^Y_{S^2} - \Delta^X_{\mathbb{R}^3}\right), \tag{5.14}$$

where $\Delta^X_{\mathbb{R}^3} := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and $\Delta^Y_{S^2}$ is the usual Laplacian on the 2-dimensional sphere $S^2 = \{Y = (y_1, y_2, y_3) \in \mathbb{R}^3\colon y_1^2 + y_2^2 + y_3^2 = 1\}$ in $\mathbb{R}^3$.

Proof. We deduce (5.14) with the expression of $\Delta^Y_{S^2}$ in [14, p. 99]. □

Remark 5.2. We fix $1 \leqslant j < k < l \leqslant n$. If $\varphi(z) = z_j^2 + z_k^2 + z_l^2$, then by Eqs. (5.7), (5.10) and (5.11), we have $\Delta^F_{1,2,3} = 0$ in (5.12).

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