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Identities for vector fields in the infinitesimal representation of the symplectic group into the Siegel disk of complex symmetric matrices

Helene Airault [a,b,*]

[a] *Insset, 48, rue Raspail, 02100, Saint-Quentin (Aisne), France*
[b] *LAMFA, UMR 7352 CNRS, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039, Amiens, France*

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Abstract

We discuss the notion of Ornstein–Uhlenbeck operator on a complex manifold endowed with a Kählerian metric. We give the example of the Siegel disk. We consider the infinitesimal holomorphic representation of $Sp(2n)$, the symplectic group of order n, into the Siegel disk $\mathcal{D}_n$ of symmetric complex $n \times n$ matrices. Let $\rho(v) = L(v) + \beta(v)I$, the first order differential operator on $\mathcal{D}_n$ associated to the element v in the Lie algebra $\mathcal{G}$ of $Sp(2n)$. We denote $L(v)$ a vector field, $\beta(v)$ a function on $\mathcal{D}_n$ and $\beta(v)I$ is the operator of multiplication by $\beta(v)$. We show the existence of a basis (e_k) in the Lie algebra $\mathcal{G}$ and of constants (a_k) such that the operator $\sum_k a_k \rho(e_k)^2$ is equal to the multiplication by a constant. The constants (a_k) can be taken equal to 1 for $n^2 + n$ of them and to -1 for the others. Varying the coefficients in the modular factor of the representation, we obtain Ornstein–Uhlenbeck type operators on $\mathcal{D}_n$ of the form $\sum_k a_k \rho(e_k)\overline{L(e_k)}$ where $\overline{L(e_k)}$ is the complex conjugate of $L(e_k)$. In particular the Kählerian Laplacian on $\mathcal{D}_n$ is expressed as $\sum_k a_k L(e_k)\overline{L(e_k)}$. The imaginary part of the vector field $\sum_k a_k \beta(e_k)\overline{L(e_k)}$ is divergence free for the measure of the holomorphic representation. This extends some of the identities obtained for the Poincaré disk in H. Airault and H. Ouerdiane (2011, 2009) [4,3].

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* Correspondence to: Insset, 48, rue Raspail, 02100, Saint-Quentin (Aisne), France.
E-mail address: helene.airault@u-picardie.fr.

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Part 1: Kähler geometry. Unitary holomorphic representation

1. Introduction

Consider the holomorphic representation T_g of the symplectic group $G = Sp(2n)$ of order n into the Siegel disk $\mathcal{D}_n$ of $n \times n$ complex symmetric matrices $\mathcal{Z}$ such that $I - \mathcal{Z}\overline{\mathcal{Z}} > 0$. This representation is defined as follows. Let $g = \left(\begin{smallmatrix} A & B \\ \overline{B} & \overline{A} \end{smallmatrix}\right) \in Sp(2n)$ where A, B are $n \times n$ complex matrices and where ${}^t g J g = J$, we denote J the $2n \times 2n$ matrix $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and I is the $n \times n$ identity matrix. The action of G on $\mathcal{D}_n$ is given by

$$\mathcal{W} = k_g(\mathcal{Z}) = (A\mathcal{Z} + B)(\overline{B}\mathcal{Z} + \overline{A})^{-1}. \tag{1.1}$$

We define the holomorphic representation of $Sp(2n)$ into $\mathcal{D}_n$ with the classical formula

$$(T_g\Phi)(\mathcal{Z}) = \det(\overline{B}\mathcal{Z} + \overline{A})^{\gamma}\Phi\big(k_g(\mathcal{Z})\big) \tag{1.2}$$

where Φ is a holomorphic function on $\mathcal{D}_n$, γ is a constant. We call $\det(\overline{B}\mathcal{Z} + \overline{A})^{\gamma}$ the modular factor.

Let $\mathcal{D}$ be a complex domain and let G be a complex group operating on $\mathcal{D}$ by a holomorphic map $\mathcal{Z} \to k_g(\mathcal{Z})$, if $\mathcal{Z} \in \mathcal{D}$, then $k_g(\mathcal{Z}) \in \mathcal{D}$. Following [5,11], for a holomorphic function Φ on $\mathcal{D}$, we define

$$(T_g\Phi)(\mathcal{Z}) = h_g(\mathcal{Z})\Phi\big(k_g(\mathcal{Z})\big) \tag{1.3}$$

where $h_g(\mathcal{Z})$ is a holomorphic function of $\mathcal{Z}$ with values in the set of complex numbers. We assume $k_e(\mathcal{Z}) = \mathcal{Z}$ and $h_e(\mathcal{Z}) = 1$ when e is the neutral element of G. The condition

$$T_{g_1g_2}\Phi(\mathcal{Z}) = T_{g_2}(T_{g_1}\Phi)(\mathcal{Z}) \tag{1.4}$$

must be satisfied. This implies that

$$h_{g_1g_2}(\mathcal{Z}) = h_{g_2}(\mathcal{Z})h_{g_1}k_{g_2}(\mathcal{Z}) \quad \text{and} \quad k_{g_1g_2}(\mathcal{Z}) = (k_{g_1}ok_{g_2})(\mathcal{Z}). \tag{1.5}$$

A particular solution of (1.5) is given by

$$h_g(\mathcal{Z}) = \big(\det\big(k'_g(\mathcal{Z})\big)\big)^{\alpha} \tag{1.6}$$

where $k'_g(\mathcal{Z})$ is the complex holomorphic Jacobian matrix $(\frac{\partial k_m}{\partial z_j})$ of the map $\mathcal{Z} \to k_g(\mathcal{Z})$. For the representation (1.2), $h_g(\mathcal{Z})$ is obtained with (1.6). However for the representation of the 3-dimensional Heisenberg group given in [4], the factor $h_g(\mathcal{Z})$ is not as in (1.6). It would be interesting to know the relation between the factor $h_g(\mathcal{Z})$ and the action $k_g(\mathcal{Z})$ in the case of a holomorphic representation of $Diff_+(S^1)$, the set of orientation preserving diffeomorphisms of the unit circle, into the space of functions univalent in the unit disk. Such representations have been suggested by Kirillov, Neretin and Yureev, see [9,10]. In [2], it has been shown how to embed $Diff_+(S^1)$ in the infinite dimensional Grassmannian. In the present work, from the Kählerian potential on $\mathcal{D}_n$, we recall how to obtain the Laplace–Beltrami operator. We define the complex Ornstein–Uhlenbeck operator (complex O–U operator) on $\mathcal{D}_n$. We calculate the infinitesimal representation of $Sp(2n)$, generalizing [6] where $n = 1$. For a vector v in the Lie algebra of $Sp(2n)$, let $\rho(v) = L(v) + \beta(v)I$ be the operator on $\mathcal{D}_n$ associated to v in the infinitesimal representation. $L(v)$ is a first order differential operator and $\beta(v)I$ is the multiplication by $\beta(v)$. We show the existence of a basis (v_k) in the Lie algebra and constants (a_k) such that $\sum_k a_k\rho(v_k)^2$ is the multiplication by a constant. By normalizing the vectors (v_k), we can take a_k either 1 or -1. Let K be the Kähler potential on $\mathcal{D}_n$ and

$$\Delta = \sum_{j,k} m_{jk} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \tag{1.7}$$

be the associated Riemannian Laplacian on $\mathcal{D}_n$ for the Kähler metric. We show that Δ can be written as

$$\Delta = \sum_k a_k L(v_k) \overline{L(v_k)}. \tag{1.8}$$

We calculate Δ^c, the complex O–U operator on $\mathcal{D}_n$, in the form

$$\Delta^c = \Delta - c \sum_{j,k} m_{jk} \left[\frac{\partial}{\partial z_j} \log K \right] \frac{\partial}{\partial \bar{z}_k} \tag{1.9}$$

by requiring that Δ^c has μ^c as invariant measure. The representation T_g is unitary in $L^2_{Hol}(\mu^c)$, the set of square integrable holomorphic functions. We prove (see (9.6)–(9.7)) that

$$\Delta^c = \sum_k a_k \rho(v_k) \overline{L(v_k)} = \Delta - cV. \tag{1.10}$$

Since μ^c is a real measure, then for any real-valued differentiable function Ψ on $\mathcal{D}_n$, we deduce $\int_{\mathcal{D}_n} (\overline{V} - V)\Psi \, d\mu^c = 0$. The case $n = 1$ is discussed in [3,4]. Our main theorems are in Sections 8, 9 for $n = 2$ and in Section 11 for arbitrary n. They extend (4.20)–(4.21)–(4.23) concerning the Poincaré disk.

2. Laplacian and O–U operators on a Kähler domain, their expressions in terms of the infinitesimal representation

In this section, we explain our motivation. A real measure μ on the domain $\mathcal{D}$ is invariant for the operator D if

$$\int_{\mathcal{D}} (D\Psi)(\mathcal{Z}, \overline{\mathcal{Z}}) \, d\mu = 0 \tag{2.1}$$

for all differentiable Ψ such that the integral in (2.1) is well defined. The representation (1.3) is unitary in the space of square integrable holomorphic functions $L^2_{Hol}(\mu)$ if

$$\int \left| (T_g \Phi)(\mathcal{Z}) \right|^2 d\mu(\mathcal{Z}) = \int \left| \Phi(\mathcal{Z}) \right|^2 d\mu(\mathcal{Z}). \tag{2.2}$$

We say that μ is unitarizing for T_g. In the following, we relate invariant measures and unitarizing measures.

We say that a vector field V on $\mathcal{D}$ is a free divergence vector field for the real measure μ if

$$\int_{\mathcal{D}} (V\Psi)(\mathcal{Z}, \overline{\mathcal{Z}}) \, d\mu = 0 \tag{2.3}$$

for any real-valued function Ψ.

2.1. Unitary holomorphic representations on a Kähler domain

Assume that $\mathcal{D}$ is a Kähler manifold. See [14]. We write an element of $\mathcal{D}$ as

$$\mathcal{Z} = (z_1, z_2, \ldots, z_p, \ldots). \tag{2.4}$$

We assume the existence of a globally defined Kähler potential,

$$U = \log K(\mathcal{Z}, \overline{\mathcal{Z}}). \tag{2.5}$$

This means:

(1) $K(\mathcal{Z}, \overline{\mathcal{Z}})$ is a positive *real* valued function.
(2) The metric on $\mathcal{D}$ is given by

$$ds^2 = -\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}})\, dz_j\, d\overline{z_k}. \tag{2.6}$$

We put

$$\omega = i \sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}})\, dz_j \wedge d\overline{z_k}. \tag{2.7}$$

If $\mathcal{D}$ is of complex dimension p, then

$$dv = (\omega)^{\wedge p} \tag{2.8}$$

defines the volume element on $\mathcal{D}$. We denote by Δ the Riemannian Laplace operator on $\mathcal{D}$ with the Kählerian metric (2.6). We define the matrix

$$P = (p_{jk}) \quad \text{with } p_{jk} = \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}}). \tag{2.9}$$

Since K is a real function, we have

$${}^t\overline{P} = P \tag{2.10}$$

and we put

$$M = (m_{jk}) = \text{constant } \overline{P}^{-1}. \tag{2.11}$$

Let

$$\Delta = \sum_{jk} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \tag{2.12}$$

then:

(1) Δ is a real operator. Up to a multiplicative constant, Δ is the Laplace–Beltrami operator associated to the Kählerian metric.
(2) Δ is an invariant operator with respect to the volume measure dv,

$$\int \Delta\Psi\, dv = 0. \tag{2.13}$$

(3) Let

$$\Delta^c = \Delta - c\sum_{j,k} m_{jk}\left[\frac{\partial}{\partial z_j}\log K\right]\frac{\partial}{\partial \overline{z_k}} \tag{2.14}$$

where c is a constant. Then Δ^c has

$$\mu^c = \text{constant } \exp(-c\log K)\,dv \tag{2.15}$$

for invariant measure ($\int \Delta^c \Psi\, d\mu^c = 0$). The imaginary part $\Im V$ of the vector field

$$V = \sum_{j,k} m_{jk}\left[\frac{\partial}{\partial z_j}\log K\right]\frac{\partial}{\partial \overline{z_k}} \tag{2.16}$$

is divergence free ($\int \Im V \Psi\, d\mu^c = 0$) for μ^c.

Following [11], we define a Berezinian measure as a probability measure μ^c of the form (2.15) where K is the Kähler potential, dv is the volume on $\mathcal{D}$ and where the constant is a normalizing constant in order to have a probability measure.

Definition 2.1. Let

$$\Delta^c = \Delta - cV \quad \text{with } V = \sum_{j,k} m_{jk}\left[\frac{\partial}{\partial z_j}\log K\right]\frac{\partial}{\partial \overline{z_k}} \tag{2.17}$$

then Δ^c is called complex Ornstein–Uhlenbeck (O–U operator) and

$$\Delta^c + \overline{\Delta^c} \tag{2.18}$$

is called real O–U operator. If μ^c defined in (2.15) is a unitarizing measure for the representation T_g, then the O–U operator is said to be associated to T_g.

The Berezinian measure μ^γ in (3.10) is associated to the representation (1.2). It is an invariant measure for the complex Ornstein–Uhlenbeck operator on $\mathcal{D}_n$.

The aim of this note is to show that it is possible to express Laplacian and O–U operator in terms of the infinitesimal representation. On $\mathcal{D}_n$, we establish the formulas giving such expressions of Δ and Δ^c. The interest of such expressions comes from the infinite dimensional setting, see [12]: The difficulty in infinite dimension is to define the volume measure dv and the unitarizing measure μ^c of the representation. Thus it is convenient to first define Δ and Δ^c and then obtain dv and μ^c as invariant measures associated respectively to the elliptic operators Δ and Δ^c.

2.2. *Infinitesimal representation and differential operators*

The infinitesimal representation is defined as follows: To $v \in \mathcal{G}$, the Lie algebra of G, we associate the differential operator

$$\rho(v)\Phi(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} T_{g_\epsilon}\Phi(\mathcal{Z}) \tag{2.19}$$

where

$$v = \frac{d}{d\epsilon}_{|\epsilon=0} g_\epsilon \quad \text{with } g_0 = e \text{ and } g_\epsilon \in G,\ \epsilon \text{ is a real parameter.} \tag{2.20}$$

The Lie algebra $\mathcal{G}$ is a real vector space. Let $[v_1, v_2]$ be the Lie bracket on $\mathcal{G}$, then

$$\rho\bigl([v_1, v_2]\bigr) = \rho(v_1)\rho(v_2) - \rho(v_2)\rho(v_1). \tag{2.21}$$

For a holomorphic function Φ, we have

$$\begin{aligned}\rho(v)\Phi(\mathcal{Z}) &= \frac{d}{d\epsilon}_{|\epsilon=0} k_{g_\epsilon}(\mathcal{Z})\Phi'(\mathcal{Z}) + \left[\frac{d}{d\epsilon}_{|\epsilon=0} h_{g_\epsilon}(\mathcal{Z})\right]\Phi(\mathcal{Z}) \\ &= \alpha(v)(\mathcal{Z})\Phi'(\mathcal{Z}) + \beta(v)\Phi(\mathcal{Z})\end{aligned} \tag{2.22}$$

where

$$\alpha(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} k_{g_\epsilon}(\mathcal{Z}), \qquad \beta(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} h_{g_\epsilon}(\mathcal{Z}) \tag{2.23}$$

and Φ' is the complex derivative of Φ. In other words, if $\mathcal{Z} = (z_1, z_2, \ldots)$ and $\mathcal{W} = k_g(\mathcal{Z})$ is given by

$$k_g(\mathcal{Z}) = \bigl((k_g)_1(\mathcal{Z}), (k_g)_2(\mathcal{Z}), \ldots\bigr) \tag{2.24}$$

then

$$\bigl(\rho(v)\Phi\bigr)(\mathcal{Z}) = \sum_j \alpha_j(v)(\mathcal{Z}) \frac{\partial}{\partial z_j}\Phi + \beta(v)\Phi(\mathcal{Z}) \tag{2.25}$$

with

$$\alpha_j(v)(\mathcal{Z}) = \frac{d}{d\epsilon}_{|\epsilon=0} (k_{g_\epsilon})_j(\mathcal{Z}), \quad j = 1, 2, \ldots. \tag{2.26}$$

We denote by $L(v)$ the vector field

$$\begin{aligned}&\bigl(L(v)\Phi\bigr)(\mathcal{Z}) = \sum_j \alpha_j(v)(\mathcal{Z}) \frac{\partial}{\partial z_j}\Phi = \frac{d}{d\epsilon}_{|\epsilon=0} \Phi\bigl(k_{g_\epsilon}(\mathcal{Z})\bigr) \\ &\text{then} \quad \rho(v) = L(v) + \beta(v)I.\end{aligned} \tag{2.27}$$

2.3. Adjoint for the first order differential operators related to the representation and unitarity

Definition 2.2. A first order differential operator L on $\mathcal{D}$ is said to be holomorphic if it is of the form

$$L = \sum_j a_j(\mathcal{Z}) \frac{\partial}{\partial z_j} \tag{2.28}$$

where a_j are holomorphic functions. We say also that L is a holomorphic vector field.

Lemma 2.3. *Assume that T_g is unitary as in (2.2) where μ is a positive real measure, then for any holomorphic functions Φ and Ψ defined on $\mathcal{D}$, we have*

$$\int \bigl(\rho(v)\Phi\bigr)(\mathcal{Z})\overline{\Psi(\mathcal{Z})}\, d\mu(\mathcal{Z}) + \int \Phi(\mathcal{Z})\overline{\bigl(\rho(v)\Psi\bigr)(\mathcal{Z})}\, d\mu(\mathcal{Z}) = 0. \tag{2.29}$$

Proof. Let g_ϵ be as in (2.2), we take the derivative with respect to ϵ. It gives the result since $L(v)$ is a holomorphic vector field and we have

$$L(v)(\Phi\overline{\Psi}) = \overline{\Psi}L(v)(\Phi). \qquad \square \tag{2.30}$$

Notation 2.4. Let V be a vector field, we denote $div_\mu V$ the function such that for any differentiable function F null outside a compact set in $\mathcal{D}$ and vanishing out of the support of μ, we have

$$\int (div_\mu V)(\mathcal{Z})F(\mathcal{Z})\,d\mu(\mathcal{Z}) = \int (VF)(\mathcal{Z})\,d\mu(\mathcal{Z}). \tag{2.31}$$

Then (2.29) is expressed as $div_\mu(L(v) + \overline{L(v)}) = -(\beta(v) + \overline{\beta(v)})$.

2.4. Second order differential operators related to the representation

In this section, we raise several questions in a more general setting. Let $\rho(v)$ and $L(v)$ be as in (2.22)–(2.27). Let $(e_1, e_2, \ldots)$ be a basis of the Lie algebra $\mathcal{G}$ of G and real constants A_{jk}. We consider second order differential operators of the form

$$\sum_{j,k} A_{jk}\rho(e_j)\overline{L(e_k)}. \tag{2.32}$$

Note that the second order derivatives in (2.32) are all of the form $\frac{\partial^2}{\partial z_j \partial \overline{z_k}}$ There are no terms like $\frac{\partial^2}{\partial z_j \partial z_k}$ or $\frac{\partial^2}{\partial \overline{z_j}\partial \overline{z_k}}$. We ask the following questions:

(1) Do there exist constants A_{jk} and an appropriate basis $(e_1, e_2, \ldots)$ such that the Laplacian Δ is given by $\Delta = \sum_{j,k} A_{jk}L(e_j)\overline{L(e_k)}$ and such that the operator $\sum_{j,k} A_{jk}\rho(e_j)L(e_k)$ is the multiplication by a constant?

(2) Assume that μ_0 is a real measure such that

$$\int \left|(T_g\Phi)(\mathcal{Z})\right|^2 d\mu_0(\mathcal{Z}) = \int \left|\Phi(\mathcal{Z})\right|^2 d\mu_0(\mathcal{Z}) \quad \text{for } T_g\Phi(\mathcal{Z}) = \Phi\big(k_g(\mathcal{Z})\big). \tag{2.33}$$

Is the measure μ_0 an invariant measure for the Laplacian Δ?

(3) Consider the measure μ of the representation T_g such that (2.2) holds. With the same constants (A_{jk}) and basis $(e_1, e_2, \ldots)$ as in (2.32), does the operator (2.32) have the measure μ for invariant measure?

(4) If the answer to (3) is positive, how is the first order part in (2.32) expressed in terms of the derivatives of the Kähler potential on $\mathcal{D}$?

(5) Assume that μ is the measure of the unitary representation; this means: μ is a real positive measure and we have (2.2). Does there exist a second order differential operator like (2.32) and admitting μ as invariant measure?

3. The group $Sp(2n)$. Action on $\mathcal{D}_n$

Let J be the $2n \times 2n$ matrix, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is the unit matrix of order n. The complex group $Sp(2n)$ is the set of $2n \times 2n$ complex matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which satisfy

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \bar{g} \tag{3.1}$$

and

$$^t g J g = J \tag{3.2}$$

$^t g$ is the transposed matrix of g. The condition (3.1) is equivalent to the fact that g is of the form $g = \left(\begin{smallmatrix} A & B \\ \overline{B} & \overline{A} \end{smallmatrix}\right)$. The condition (3.2) is equivalent to

$$^t A\overline{B} = {}^t\overline{B}A \quad \text{and} \quad {}^t\overline{A}A - {}^tB\overline{B} = I. \tag{3.3}$$

If $\mathcal{Z}$ is an $n \times n$ matrix, then $(\mathcal{Z}^t A + {}^t B)(\overline{B}\mathcal{Z} + \overline{A}) = (\mathcal{Z}^t \overline{B} + {}^t\overline{A})(A\mathcal{Z} + B)$. Conversely, if this identity holds for any symmetric $\mathcal{Z}$, then $^t A\overline{B} = {}^t\overline{B}A$ and $^t\overline{A}A - {}^tB\overline{B} = \text{constant } I$.

Remark 3.1. If (A, B) is a solution of (3.3) then (A, iB) is also a solution of (3.3).

The group $G = Sp(2n)$ acts on the domain $\mathcal{D}_n$ with (1.1). Since $\mathcal{Z}$ is symmetric, the conditions on A and B imply that $k_g(\mathcal{Z})$ is also symmetric. We have

$$\begin{aligned} I - k_g(\mathcal{Z})\overline{k_g(\mathcal{Z})} &= \left({}^t\overline{A} + \mathcal{Z}^t\overline{B}\right)^{-1}(I - \mathcal{Z}\overline{\mathcal{Z}})(B\overline{\mathcal{Z}} + A)^{-1}, \\ \det\left(I - k_g(\mathcal{Z})\overline{k_g(\mathcal{Z})}\right) &= \det(I - \mathcal{Z}\overline{\mathcal{Z}}) \times \frac{1}{|\det(\overline{B}\mathcal{Z} + \overline{A})|^2}. \end{aligned} \tag{3.4}$$

Compare with the "cocycle" identity in [9, p. 744]. From (3.4), it results that the Kählerian metric

$$ds^2 = -\partial\bar{\partial} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) \tag{3.5}$$

is invariant under the action of the group G. Using that $\mathcal{Z}$ is *symmetric*, differentiating $(\overline{B}\mathcal{Z} + \overline{A})^{-1}(\overline{B}\mathcal{Z} + \overline{A})$, we obtain

$$d(\overline{B}\mathcal{Z} + \overline{A})^{-1} = -(\overline{B}\mathcal{Z} + \overline{A})^{-1}\overline{B}[d\mathcal{Z}](\overline{B}\mathcal{Z} + \overline{A})^{-1} \tag{3.6}$$

and $d\mathcal{W} = (\mathcal{Z}^t\overline{B} + {}^t\overline{A})^{-1}[d\mathcal{Z}](\overline{B}\mathcal{Z} + \overline{A})^{-1}$ or equivalently

$$d\mathcal{W} = {}^tN[d\mathcal{Z}]N \quad \text{where } N = (\overline{B}\mathcal{Z} + \overline{A})^{-1}. \tag{3.7}$$

Lemma 3.2. *Let* $\mathcal{Z} = (z_1, z_2, \ldots) \to \mathcal{W} = k_g(\mathcal{Z})$ *with* $k_g(\mathcal{Z}) = (k_g(\mathcal{Z})_1, k_g(\mathcal{Z})_2, \ldots) = (w_1, w_2, \ldots)$. *The Jacobian matrix of the map* $\mathcal{Z} \to k_g(\mathcal{Z})$ *has determinant equal to*

$$Jac = \det\left[\left(\frac{\partial}{\partial z_j} k_g(\mathcal{Z})_p\right)_{1 \leqslant j, p \leqslant \frac{n(n+1)}{2}}\right] = \left(\det\left((\overline{B}\mathcal{Z} + \overline{A})^{-1}\right)\right)^{n+1}. \tag{3.8}$$

Proof. The matrix $\mathcal{Z}$ has $\frac{n(n+1)}{2}$ independent coefficients. If N were a diagonal matrix, we would find that $Jac = [\det(N)]^{n+1}$ and

$$dw_1 \wedge dw_2 \wedge \cdots \wedge dw_{\frac{n(n+1)}{2}} = (\det N)^{n+1} dz_1 \wedge \cdots \wedge dz_{\frac{n(n+1)}{2}}. \tag{3.9}$$

The proof for general N is more delicate and we may use [8, p. 53] where the volume element for the domain $\mathcal{D}$ is calculated. □

The volume is invariant under the maps $\mathcal{Z} \to k_g(\mathcal{Z})$. If $\mathcal{W} = k_g(\mathcal{Z})$, then

$$\frac{dw_1 \wedge d\overline{w}_1 \wedge dw_2\, d\overline{w}_2 \wedge \cdots \wedge dw_{n(n+1)/2} \wedge d\overline{w}_{n(n+1)/2}}{|\det(I - \mathcal{W}\overline{\mathcal{W}})|^{n+1}}$$
$$= \frac{dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_{n(n+1)/2} \wedge dz_{n(n+1)/2}}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^{n+1}}.$$

The representation (1.2) is unitary in the space of square integrable holomorphic functions $L^2_{Hol}(\mu^\gamma)$ with

$$d\mu^\gamma = \frac{dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_{n(n+1)/2} \wedge d\overline{z_{n(n+1)/2}}}{[\det(I - \mathcal{Z}\overline{\mathcal{Z}})]^{n+1+\gamma}} = \frac{dv}{[\det(I - \mathcal{Z}\overline{\mathcal{Z}})]^\gamma}$$

where dv is the volume measure on $\mathcal{D}_n$.

$$\mu^\gamma = \exp\bigl[-\gamma \log\det(I - \mathcal{Z}\overline{\mathcal{Z}})\bigr]\, dv = \exp\bigl[-\gamma \log K(\mathcal{Z}, \overline{\mathcal{Z}})\bigr]\, dv. \tag{3.10}$$

$\mathcal{D}_n$ is a Kähler manifold with Kähler potential $\log\det(I - \mathcal{Z}\overline{\mathcal{Z}})$. We shall explicit the Laplacian and O–U operator Δ^γ on $\mathcal{D}_n$ and verify that $\int \Delta^\gamma \Psi\, d\mu^\gamma = 0$.

We obtain the infinitesimal representation on $\mathcal{D}_n$ as follows: As in Section 2, from (1.2), we have

$$\rho(v)\Phi(\mathcal{Z}) = \sum_{j=1}^{n(n+1)/2} \frac{d}{d\epsilon}_{|\epsilon=0} \bigl(k_{g_\epsilon}(\mathcal{Z})\bigr)_j \frac{\partial}{\partial z_j} \Phi(\mathcal{Z}) + \gamma \frac{d}{d\epsilon}_{|\epsilon=0} \det(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon})\Phi(\mathcal{Z}) \tag{3.11}$$

with

$$k_{g_\epsilon}(\mathcal{Z}) = (A_\epsilon \mathcal{Z} + B_\epsilon)(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon})^{-1}$$

and

$$g_\epsilon = \begin{pmatrix} A_\epsilon & B\epsilon \\ \overline{B}_\epsilon & \overline{A}_\epsilon \end{pmatrix}, \qquad \frac{d}{d\epsilon}_{|\epsilon=0} g_\epsilon = v. \tag{3.12}$$

As in (2.27), define $L(v)$ corresponding to the infinitesimal representation $\rho(v)$ when $\gamma = 0$,

$$L(v)\Phi(\mathcal{Z}) = \sum_{j=1}^{n(n+1)/2} \frac{d}{d\epsilon}_{|\epsilon=0} \bigl(k_{g_\epsilon}(\mathcal{Z})\bigr)_j \frac{\partial}{\partial z_j} \Phi(\mathcal{Z})$$

and

$$\rho(v) = L(v) + \gamma \beta(v) I. \tag{3.13}$$

Lemma 3.3.

$$\frac{d}{d\epsilon}_{|\epsilon=0} \bigl(k_{g_\epsilon}(\mathcal{Z})\bigr) = \frac{d}{d\epsilon}_{|\epsilon=0} A_\epsilon \mathcal{Z} - \mathcal{Z}\frac{d}{d\epsilon}_{|\epsilon=0} \overline{A}_\epsilon + \frac{d}{d\epsilon}_{|\epsilon=0} B_\epsilon - \mathcal{Z}\frac{d}{d\epsilon}_{|\epsilon=0} \overline{B}_\epsilon \mathcal{Z}, \tag{3.14}$$

$$\frac{d}{d\epsilon}_{|\epsilon=0} \det(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon}) = \operatorname{trace}\left[\frac{d}{d\epsilon}_{|\epsilon=0} \overline{B}_\epsilon \mathcal{Z} + \frac{d}{d\epsilon}_{|\epsilon=0} \overline{A}_\epsilon\right]. \tag{3.15}$$

4. The Poincaré disk

Let $G = Sp(2)$ be the group of matrices

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{where } |a|^2 - |b|^2 = 1. \tag{4.1}$$

Taking the differential of $a\bar{a} - b\bar{b} = 1$ at the identity, it is immediate that the Lie algebra $\mathcal{G}$ of G is the set of matrices

$$v = \begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix} \quad \text{where } \alpha \text{ is real.} \tag{4.2}$$

This is a real vector space of dimension 3. We take for basis of $\mathcal{G}$,

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{4.3}$$

Let $g_t^j = \exp(te_j)$,

$$g_1 = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} ch(t/2) & sh(t/2) \\ sh(t/2) & ch(t/2) \end{pmatrix},$$
$$g_3 = \begin{pmatrix} ch(t/2) & i\,sh(t/2) \\ -i\,sh(t/2) & ch(t/2) \end{pmatrix}. \tag{4.4}$$

The Poincaré disk is the unit disk $\mathcal{D} = \{z\bar{z} < 1\}$ with the metric

$$ds^2 = -\frac{\partial^2}{\partial z \partial \bar{z}} \log(1 - z\bar{z}) = \frac{dz\, d\bar{z}}{(1 - z\bar{z})^2}, \tag{4.5}$$

$$ds^2 = -\frac{\partial^2}{\partial z \partial \bar{z}} U(z, \bar{z}) \quad \text{with the Kähler potential } U(z, \bar{z}) = \log(1 - z\bar{z}). \tag{4.6}$$

The group G acts on the unit disk,

$$\text{if } g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \text{we put } u = k_g(z) = \frac{az + b}{\bar{b}z + \bar{a}}. \tag{4.7}$$

Then

$$k_{g_1 g_2}(z) = k_{g_1}\big(k_{g_2}(z)\big) \quad \text{and} \quad k'_g(z) = \frac{1}{(\bar{b}z + \bar{a})^2}. \tag{4.8}$$

We have the two fundamental identities

$$\frac{du \wedge d\bar{u}}{(1 - u\bar{u})^2} = \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2} \quad \text{and} \quad 1 - u\bar{u} = \frac{1 - z\bar{z}}{(\bar{b}z + \bar{a})(b\bar{z} + a)}. \tag{4.9}$$

The second relation in (4.9) can be written as

$$U\big(k_g(z)\big) = U(z) - 2\Re \log(\bar{b}z + \bar{a}). \tag{4.10}$$

The holomorphic representation is given by

$$[T_g \Phi](\mathcal{Z}) = \big(k'_g(\mathcal{Z})\big)^\alpha \Phi\big(k_g(\mathcal{Z})\big). \tag{4.11}$$

From (4.9), we deduce that the operator T_g is unitary in $L^2_{Hol}(\mu)$ with

$$d\mu = (1-z\bar{z})^{2\alpha}\frac{dz\,d\bar{z}}{(1-z\bar{z})^2}. \tag{4.12}$$

By direct calculation, we prove

Theorem 4.1. *The measure μ is an invariant measure for*

$$\Delta^{OU} = (1-z\bar{z})^2\left[\frac{\partial^2}{\partial z\partial\bar{z}} + \alpha\frac{\partial}{\partial z}\log(1-z\bar{z})\frac{\partial}{\partial\bar{z}} + \alpha\frac{\partial}{\partial\bar{z}}\log(1-z\bar{z})\frac{\partial}{\partial z}\right]. \tag{4.13}$$

We have

$$\Delta^{OU} = \Delta - \alpha(1-z\bar{z})\left(\bar{z}\frac{\partial}{\partial\bar{z}} + z\frac{\partial}{\partial z}\right) \tag{4.14}$$

where Δ is the Laplacian on the unit disk,

$$\Delta = (1-z\bar{z})^2\frac{\partial^2}{\partial z\partial\bar{z}}. \tag{4.15}$$

Moreover the vector field

$$W = (1-z\bar{z})\left(\bar{z}\frac{\partial}{\partial\bar{z}} - z\frac{\partial}{\partial z}\right) \tag{4.16}$$

is a free-divergence vector field, this means $div_\mu(W) = 0$ or equivalently $\int (W\psi)\,d\mu = 0$.

For (4.16), passing in polar coordinates as in [3],

$$z = re^{i\theta}, \qquad \frac{\partial}{\partial z} = \frac{1}{2}e^{-i\theta}\left[\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right], \qquad \left(\bar{z}\frac{\partial}{\partial\bar{z}} - z\frac{\partial}{\partial z}\right) = i\frac{\partial}{\partial\theta}$$

it is immediate that $\int (W\psi)\,d\mu = 0$. See [3] for the commutation relations between Δ, Δ^{OU} and W.

Remark 4.2. We put $u = 1 - z\bar{z}$. Let Φ be a differentiable function on the unit disk which depends only on u, let $\Phi(u)$, then

$$-(1-z\bar{z})\bar{z}\frac{\partial}{\partial\bar{z}} = u(1-u)\frac{\partial\Phi}{\partial u}. \tag{4.17}$$

Our problem is whether it is possible to find Δ^{OU} in terms of the infinitesimal representation. As in (2.19), we define

$$\rho(e_j)\Phi(z) = \frac{d}{d\epsilon}_{|\epsilon=0} T^{\gamma}_{g^j_\epsilon}\Phi(z) = \alpha_j(z)\frac{d}{dz}f(z) + \beta_j(z)f(z). \tag{4.18}$$

We put $L_j\Phi(z) = \alpha_j(z)\frac{d}{dz}\Phi(z)$, then

$$\rho(e_1)\Phi(z) = i\left[z\Phi'(z) + \alpha\Phi(z)\right], \qquad \rho(e_2)\Phi(z) = \frac{1}{2}\left(1-z^2\right)\Phi'(z) - \alpha z\Phi(z),$$
$$\rho(e_3)\Phi(z) = \frac{i}{2}\left(1+z^2\right)\Phi'(z) + i\alpha z\Phi(z). \tag{4.19}$$

Theorem 4.3. *Let*

$$A=\rho(e_2)\overline{L_2}+\rho(e_3)\overline{L_3}-\rho(e_1)\overline{L_1}. \tag{4.20}$$

We have

$$A=\frac{1}{2}(1-z\bar{z})^2\left[\frac{\partial^2}{\partial z\partial\bar{z}}-\frac{2\alpha\bar{z}}{(1-z\bar{z})}\frac{\partial}{\partial\bar{z}}\right]$$

or equivalently

$$A=\frac{1}{2}(1-z\bar{z})^2\left[\frac{\partial^2}{\partial z\partial\bar{z}}+2\alpha\frac{\partial}{\partial z}\log(1-z\bar{z})\frac{\partial}{\partial\bar{z}}\right]. \tag{4.21}$$

The measure μ^γ of the representation is an invariant measure for A. We have

$$\Delta^{OU}=A+\overline{A} \tag{4.22}$$

where Δ^{OU} is given by (4.13). Moreover,

$$\left[\rho(e_2)^2+\rho(e_3)^2-\rho(e_1)^2\right]\Phi=\left(\alpha^2-\alpha\right)\Phi. \tag{4.23}$$

Proof. By direct calculation,

$$\begin{aligned}
\rho(e_1)^2\Phi&=-z^2\Phi''(z)-(1+2\alpha)z\Phi'(z)-\alpha^2\Phi(z),\\
\rho(e_2)^2\Phi&=\frac{1}{4}\left(1-z^2\right)^2\Phi''(z)-\frac{1+2\alpha}{2}z\left(1-z^2\right)\Phi'(z)-\frac{\alpha}{2}\left(1-(1+2\alpha)z^2\right)\Phi(z),\\
\rho(e_3)^2\Phi&=-\frac{1}{4}\left(1+z^2\right)^2\Phi''(z)-\frac{1+2\alpha}{2}z\left(1+z^2\right)\Phi'(z)-\frac{\alpha}{2}\left(1+(1+2\alpha)z^2\right)\Phi(z).
\end{aligned}$$

We deduce (4.23). Note that the signs of the coefficients in (4.23) are related to the signs of the metric tensor on $Sp(2)$. The geometry of the group $Sp(2)$ has been studied in detail in [7]. The O–U operators on the Poincaré disk can be lifted to operators on $Sp(2)$ as explained in [6,3]. □

Part 2: Kähler geometry on $\mathcal{D}_2$

5. Metric and Laplacian on $\mathcal{D}_2$. Action of $Sp(2\times 2)$

We specify Sections 1 and 2 to the case of $\mathcal{D}_2$. In this particular case, we give elementary proofs, compare with Section 11. Let

$$\mathcal{Z}=\begin{pmatrix} z_1 & z_2\\ z_2 & z_3\end{pmatrix}\quad\text{and}\quad I-\mathcal{Z}\overline{\mathcal{Z}}=\begin{pmatrix} u & -w\\ -\overline{w} & v\end{pmatrix} \tag{5.1}$$

with

$$u=1-z_1\overline{z_1}-z_2\overline{z_2},\qquad v=1-z_3\overline{z_3}-z_2\overline{z_2},\qquad w=z_1\overline{z_2}+z_2\overline{z_3}. \tag{5.2}$$

The condition $I-\mathcal{Z}\overline{\mathcal{Z}}>0$ means that the eigenvalues of the hermitian matrix $I-\mathcal{Z}\overline{\mathcal{Z}}$ are strictly positive. For u, v, w, it implies that $u>0$, $v>0$, $uv-w\overline{w}>0$. Since $u\leqslant 1$ and $v\leqslant 1$, $\mathcal{D}_2$ is a bounded domain in C^3.

In the case of the Poincaré disk, the Laplacian, see (4.15), is given by

$$\Delta = u_0^2 \frac{\partial^2}{\partial z \partial \bar{z}} \quad \text{where } u_0 = 1 - z\bar{z}.$$

Similarly, in Section 5.1, we obtain the coefficients of $\frac{\partial^2}{\partial z_j \partial \bar{z}_k}$ in the $\mathcal{D}_2$-Laplacian as functions of the coefficients of the matrix $I - \mathcal{Z}\overline{\mathcal{Z}}$, in that case in terms of u, v, w. This is also a consequence of (10.5). In Section 5.2, we obtain the Jacobian of the map $\mathcal{Z} \to k_g(\mathcal{Z})$ and we determine the measure μ^γ which makes T_g a unitary operator.

5.1. Metric and Laplacian on $\mathcal{D}_2$

Let $K(\mathcal{Z}, \overline{\mathcal{Z}}) = \det(I - \mathcal{Z}\overline{\mathcal{Z}})$. Consider the matrix

$$P = (p_{ik}) = \left(\frac{\partial^2}{\partial z_i \overline{\partial z_k}} \log K(\mathcal{Z}, \overline{\mathcal{Z}})\right) = \left(\frac{\partial^2}{\partial z_i \overline{\partial z_k}} \log(uv - w\overline{w})\right). \tag{5.3}$$

Lemma 5.1. *We have* $\overline{P} = {}^t P$,

$$P = \frac{1}{(uv - w\overline{w})^2} \begin{pmatrix} -v^2 & -2v\overline{w} & -\overline{w}^2 \\ -2vw & -2(uv + w\overline{w}) & -2u\overline{w} \\ -w^2 & -2uw & -u^2 \end{pmatrix} \tag{5.4}$$

and

$$\det P = -\frac{2}{(uv - w\overline{w})^3} = -2\big[\det(I - \mathcal{Z}\overline{\mathcal{Z}})\big]^{-3}. \tag{5.5}$$

Proof.

$$\begin{aligned}
&\frac{\partial}{\partial z_1}(uv - w\overline{w}) = -\overline{z_1}v - \overline{z_2}\overline{w} = -\overline{z_1}(1 - z_3\overline{z_3}) - z_3\overline{z_2}^2, \\
&\frac{\partial^2}{\partial z_1 \partial \overline{z_1}}(uv - w\overline{w}) = z_3\overline{z_3} - 1, \qquad \frac{\partial^2}{\partial z_1 \partial \overline{z_2}}(uv - w\overline{w}) = -2\overline{z_2}z_3, \\
&\frac{\partial^2}{\partial z_1 \partial \overline{z_3}}(uv - w\overline{w}) = \overline{z_1}z_3, \\
&\frac{\partial}{\partial z_2}(uv - w\overline{w}) = -\overline{z_2}(u + v) - \overline{z_1}w - \overline{z_3}\overline{w} = -2\overline{z_2}(1 - z_2\overline{z_2}) - 2z_2\overline{z_1}\overline{z_3}, \\
&\frac{\partial^2}{\partial z_2 \partial \overline{z_2}}(uv - w\overline{w}) = 4z_2\overline{z_2} - 2, \qquad \frac{\partial^2}{\partial z_2 \partial \overline{z_3}}(uv - w\overline{w}) = -2z_2\overline{z_1}, \\
&\frac{\partial}{\partial z_3}(uv - w\overline{w}) = -u\overline{z_3} - w\overline{z_2} = -\overline{z_3}(1 - z_1\overline{z_1}) - z_1\overline{z_2}^2, \\
&\frac{\partial^2}{\partial z_3 \partial \overline{z_3}}(uv - w\overline{w}) = -(1 - z_1\overline{z_1}).
\end{aligned}$$

We have

$$\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \log\phi = \frac{1}{\phi^2}\left[\phi \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} - \left(\frac{\partial \phi}{\partial z_i}\right)\left(\frac{\partial \phi}{\partial \overline{z_j}}\right)\right].$$

For $\phi = uv - w\overline{w}$, we put

$$a_{ij} = \phi \frac{\partial^2 \phi}{\partial z_i \partial \overline{z_j}} - \left(\frac{\partial \phi}{\partial z_i}\right)\left(\frac{\partial \phi}{\partial \overline{z_j}}\right).$$

We calculate

$$a_{11} = -v^2, \qquad a_{12} = -2v\overline{w}, \qquad a_{13} = -\overline{w}^2, \qquad a_{22} = -2(uv + w\overline{w}).$$

We deduce $\det P$ with Sarrus rule

$$\det \begin{pmatrix} -v^2 & -2v\overline{w} & -\overline{w}^2 \\ -2vw & -2(uv + w\overline{w}) & -2u\overline{w} \\ -w^2 & -2uw & -u^2 \end{pmatrix} = -2(uv - w\overline{w})^3. \qquad \square$$

Lemma 5.2. *Let $M = -4\overline{P}^{-1} = (m_{jk})$ then*

$$M = (m_{ij}) = \begin{pmatrix} 4u^2 & -4uw & 4w^2 \\ -4u\overline{w} & 2(uv + w\overline{w}) & -4vw \\ 4\overline{w}^2 & -4v\overline{w} & 4v^2 \end{pmatrix}. \tag{5.6}$$

Theorem 5.3 *(Laplacian on $\mathcal{D}_2$). Let*

$$\Delta = \sum_{j,k} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}}. \tag{5.7}$$

We have

$$\begin{aligned} \Delta = {} & 4u^2 \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + 2(uv + w\overline{w}) \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + 4v^2 \frac{\partial^2}{\partial z_3 \partial \overline{z_3}} - 4u\left(w \frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + \overline{w} \frac{\partial^2}{\partial z_2 \partial \overline{z_1}}\right) \\ & - 4v\left(\overline{w} \frac{\partial^2}{\partial z_3 \partial \overline{z_2}} + w \frac{\partial^2}{\partial z_2 \partial \overline{z_3}}\right) + 4\left(w^2 \frac{\partial^2}{\partial z_1 \partial \overline{z_3}} + \overline{w}^2 \frac{\partial^2}{\partial z_3 \partial \overline{z_1}}\right). \end{aligned} \tag{5.8}$$

Consider the metric

$$H = -\sum_{j,k} \frac{\partial}{\partial z_j \partial \overline{z_k}} \big[\log \det(I - \mathcal{Z}\overline{\mathcal{Z}})\big] \, dz_j \, d\overline{z_k}. \tag{5.9}$$

Then H defines a Kählerian metric on $\mathcal{D}_2$ and Δ is the associated Laplacian.

5.2. Integration by parts and O–U operator on $\mathcal{D}_2$

Since the matrix $\overline{\mathcal{Z}}(I - \mathcal{Z}\overline{\mathcal{Z}})$ is symmetric, we have

$$\overline{z_2}u - \overline{z_3}\overline{w} = \overline{z_2}v - \overline{z_1}w. \tag{5.10}$$

We put

$$d\mu^\gamma = \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{(uv - w\overline{w})^{\gamma+3}}. \tag{5.11}$$

Theorem 5.4. *Let $M = (m_{ij})$ and Δ as in (5.6)–(5.8). Let μ^γ as in (5.11). We have*

$$\int \Delta\Phi \, d\mu^\gamma = 4\gamma \int U\Phi \, d\mu^\gamma \tag{5.12}$$

with

$$U = (\overline{w}\overline{z_2} - \overline{z_1}u)\frac{\partial}{\partial \overline{z_1}} + (w\overline{z_2} - \overline{z_3}v)\frac{\partial}{\partial \overline{z_3}} + (\overline{w}\overline{z_3} - \overline{z_2}u)\frac{\partial}{\partial \overline{z_2}}. \tag{5.13}$$

In particular

$$U = \sum_{j,k} m_{jk} \frac{\partial}{\partial z_j} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial}{\partial \overline{z_k}}. \tag{5.14}$$

Proof. Integrating by parts (5.12) (in $\frac{\partial}{\partial z}$), this gives $\int \Delta f \, d\mu^\gamma = I_1 + I_2 + I_3$, with

$$\begin{aligned}
I_1 = {} & -\int \biggl[\frac{\partial}{\partial z_1}\biggl(\frac{4u^2}{(uv - w\overline{w})^{\gamma+3}}\biggr) + \frac{\partial}{\partial z_2}\biggl(-\frac{4u\overline{w}}{(uv - w\overline{w})^{\gamma+3}}\biggr) \\
& + \frac{\partial}{\partial z_3}\biggl(\frac{4\overline{w}^2}{(uv - w\overline{w})^{\gamma+3}}\biggr)\biggr]\frac{\partial}{\partial \overline{z_1}}\Phi, \\
I_2 = {} & -\int \biggl[\frac{\partial}{\partial z_1}\biggl(-\frac{4uw}{(uv - w\overline{w})^{\gamma+3}}\biggr) + \frac{\partial}{\partial z_2}\biggl(2\frac{uv + w\overline{w}}{(uv - w\overline{w})^{\gamma+3}}\biggr) \\
& + \frac{\partial}{\partial z_3}\biggl(-\frac{4v\overline{w}}{(uv - w\overline{w})^{\gamma+3}}\biggr)\biggr]\frac{\partial}{\partial \overline{z_2}}\Phi, \\
I_3 = {} & -\int \biggl[\frac{\partial}{\partial z_1}\biggl(\frac{4w^2}{(uv - w\overline{w})^{\gamma+3}}\biggr) + \frac{\partial}{\partial z_2}\biggl(-\frac{4vw}{(uv - w\overline{w})^{\gamma+3}}\biggr) \\
& + \frac{\partial}{\partial z_3}\biggl(\frac{4v^2}{(uv - w\overline{w})^{\gamma+3}}\biggr)\biggr]\frac{\partial}{\partial \overline{z_3}}\Phi.
\end{aligned}$$

To calculate I_1, let

$$A_1 = \frac{\partial}{\partial z_1}(4u^2) - \frac{\partial}{\partial z_2}(4u\overline{w}) + \frac{\partial}{\partial z_3}(4\overline{w}^2) = 12(\overline{w}\overline{z_2} - u\overline{z_1})$$

and

$$\begin{aligned}
B_1 &= -(\gamma + 3)\biggl[4u^2\frac{\partial}{\partial z_1} - 4u\overline{w}\frac{\partial}{\partial z_2} + 4\overline{w}^2\frac{\partial}{\partial z_3}\biggr](uv - w\overline{w}) \\
&= -4(\gamma + 3)(uv - w\overline{w})(\overline{w}\overline{z_2} - \overline{z_1}u).
\end{aligned} \tag{5.15}$$

This gives the first term in (5.13). In the same way, to calculate I_2, let

$$\begin{aligned}
A_2 &= \frac{\partial}{\partial z_2}\bigl(2(uv + w\overline{w})\bigr) - \frac{\partial}{\partial z_1}(4uw) - \frac{\partial}{\partial z_3}(4v\overline{w}) = -6(u + v)\overline{z_2} + 6(w\overline{z_1} + \overline{w}\overline{z_3}), \\
B_2 &= -(\gamma + 3)\biggl[2(uv + w\overline{w})\frac{\partial}{\partial z_2} - 4uw\frac{\partial}{\partial z_1} - 4v\overline{w}\frac{\partial}{\partial z_3}\biggr](uv - w\overline{w}) \\
&= 2(\gamma + 3)(uv - w\overline{w})\bigl(\overline{z_2}(u + v) - \overline{z_3}\overline{w} - \overline{z_1}w\bigr).
\end{aligned} \tag{5.16}$$

We deduce

$$A_2 + \frac{B_2}{(uv - w\overline{w})} = 2\gamma\big[(u+v)\overline{z_2} - w\overline{z_1} - \overline{w}\overline{z_3}\big].$$

From the important identity (5.10), we obtain the second term in (5.13). For I_3, we put

$$A_3 = \frac{\partial}{\partial z_1}\big(4w^2\big) + \frac{\partial}{\partial z_2}(-4vw) + \frac{\partial}{\partial z_3}\big(4v^2\big) = 12(w\overline{z_2} - v\overline{z_3})$$

and

$$\begin{aligned} B_3 &= -(\gamma+3)\left[4w^2\frac{\partial}{\partial z_1} - 4vw\frac{\partial}{\partial z_2} + 4v^2\frac{\partial}{\partial z_3}\right](uv - w\overline{w}) \\ &= -4(\gamma+3)(uv - w\overline{w})(-v\overline{z_3} + w\overline{z_2}), \\ A_3 &+ \frac{B_3}{(uv - w\overline{w})} = 4\gamma(v\overline{z_3} - w\overline{z_2}). \end{aligned} \tag{5.17}$$

This gives (5.13). From (5.15)–(5.16)–(5.17), we deduce that U is given by (5.14). □

Corollary 5.5. *Let Φ be a function on $\mathcal{D}_2$ which depends only on u, v, w, let $\Phi(u,v,w)$. Consider the vector field U in* (5.13)*, then*

$$\begin{aligned} U\Phi &= \big(u(1-u) - w\overline{w}\big)\frac{\partial\Phi}{\partial u} + \big(v(1-v) - w\overline{w}\big)\frac{\partial\Phi}{\partial v} \\ &\quad + (1-u-v)\left(w\frac{\partial\Phi}{\partial w} + \overline{w}\frac{\partial\Phi}{\partial\overline{w}}\right). \end{aligned} \tag{5.18}$$

Compare (5.18) with (4.17) and (10.20).

5.3. Radial coordinates on $\mathcal{D}_2$

We take $b_1 = \psi = u + v$ and $b_2 = \phi = uv - w\overline{w}$ as radial variables. These are the coefficients of the characteristic polynomial of $I - \mathcal{Z}\overline{\mathcal{Z}}$. The domain $\mathcal{D}_2$ is characterized by $0 \leqslant b_1 \leqslant 2$ and $0 \leqslant b_2 \leqslant 1$. Let F be a function depending only on b_1, b_2. Then F is spectral in the sense of [13]. Let Δ be as in (5.6)–(5.8), we wish to calculate $(\Delta F)(b_1, b_2)$.

Lemma 5.6. *Let $b_1 = \psi = u + v$ and $b_2 = \phi = uv - w\overline{w}$. We have*

$$\Delta b_1 = 4\big(b_2 - b_1^2\big), \qquad \Delta b_2 = -4b_2(1 + b_1). \tag{5.19}$$

Proof. From the proof of Lemma 5.1, the Hessian matrix $H = (H_{jk}) = \frac{\partial^2\phi}{\partial z_j \partial \overline{z_k}}$ is given by

$$H = \begin{pmatrix} z_3\overline{z_3} - 1 & -2z_3\overline{z_2} & z_3\overline{z_1} \\ -2z_2\overline{z_3} & 4z_2\overline{z_2} - 2 & -2z_2\overline{z_1} \\ z_1\overline{z_3} & -2z_1\overline{z_2} & z_1\overline{z_1} - 1 \end{pmatrix} \tag{5.20}$$

and $\Delta b_2 = \operatorname{trace}[(M^t)H]$. Thus

$$\begin{aligned} \Delta b_2 &= -4u^2(v + z_2\overline{z_2}) + 8u\overline{w}z_2\overline{z_3} + 4z_1\overline{z_3}\overline{w}^2 \\ &\quad + 8uw\overline{z_2}z_3 + 8(uv + w\overline{w})z_2\overline{z_2} - 4(uv + w\overline{w}) + 8v\overline{w}\overline{z_2}z_1 \\ &\quad + 4w^2\overline{z_1}z_3 + 8z_2\overline{z_1}vw - 4v^2(u + z_2\overline{z_2}). \end{aligned} \tag{i}$$

Using (5.10), $\overline{z_2}u + \overline{z_1}w = \overline{z_2}v + \overline{z_3}\overline{w}$, we take the sum of the first line and the third line in (i),

$$-4uv(u+v) - 4\left(u^2+v^2\right)z_2\overline{z_2} + 4u\overline{w}z_2\overline{z_3} + 4\overline{w}\overline{z_3}(vz_2 + wz_3) + 4vwz_2\overline{z_1} + 4w\overline{z_1}(uz_2 + \overline{w}z_1). \tag{ii}$$

The expression (ii) is equal to

$$-4uv(u+v) - 4\left(u^2+v^2\right)z_2\overline{z_2} + 4w\overline{w}(z_1\overline{z_1} + z_3\overline{z_3}) + 4(u+v)(\overline{w}z_2\overline{z_3} + wz_2\overline{z_1}). \tag{iii}$$

The second line in (i) is equal to

$$8(u+v)w\overline{w} - 8uwz_1\overline{z_2} - 8v\overline{w}z_2\overline{z_3} + 8(uv + w\overline{w})z_2\overline{z_2} - 4(uv + w\overline{w}). \tag{iv}$$

Assuming (iii) and (iv), we obtain

$$-4uv(u+v) - 4\left(u^2+v^2\right)z_2\overline{z_2} + 4w\overline{w}(z_1\overline{z_1} + z_3\overline{z_3}) + 4(u-v)z_2(\overline{w}\overline{z_3} - w\overline{z_1}) + 8(u+v)w\overline{w} + 8(uv + w\overline{w})z_2\overline{z_2} - 4(uv + w\overline{w}).$$

Since $\overline{w}\overline{z_3} - w\overline{z_1} = \overline{z_2}(u-v)$, we deduce the result $-4b_2(1+b_1)$. On the other hand the relation

$$m_{11} + 2m_{22} + m_{33} = -4\left(b_2 - b_1^2\right) \tag{5.21}$$

permits to calculate Δb_1. □

To calculate the radial Laplacian, we use

$$\frac{\partial F}{\partial \overline{z_k}} = \frac{\partial F}{\partial b_1}\frac{\partial b_1}{\partial \overline{z_k}} + \frac{\partial F}{\partial b_2}\frac{\partial b_2}{\partial \overline{z_k}},$$

$$\frac{\partial^2 F}{\partial z_j \partial \overline{z_k}} = \frac{\partial^2 F}{\partial b_1^2}\frac{\partial b_1}{\partial z_j}\frac{\partial b_1}{\partial \overline{z_k}} + \frac{\partial^2 F}{\partial b_1 \partial b_2}\left[\frac{\partial b_2}{\partial z_j}\frac{\partial b_1}{\partial \overline{z_k}} + \frac{\partial b_1}{\partial z_j}\frac{\partial b_2}{\partial \overline{z_k}}\right] + \frac{\partial^2 F}{\partial b_2^2}\frac{\partial b_2}{\partial z_j}\frac{\partial b_2}{\partial \overline{z_k}}.$$

To obtain $\sum_{j,k} m_{jk}\frac{\partial^2 F}{\partial z_j \partial \overline{z_k}}$, we express

$$\sum_{j,k} m_{jk}\frac{\partial b_1}{\partial z_j}\frac{\partial b_1}{\partial \overline{z_k}}, \qquad \sum_{j,k} m_{jk}\left[\frac{\partial b_2}{\partial z_j}\frac{\partial b_1}{\partial \overline{z_k}} + \frac{\partial b_1}{\partial z_j}\frac{\partial b_2}{\partial \overline{z_k}}\right], \qquad \sum_{j,k} m_{jk}\frac{\partial b_2}{\partial z_j}\frac{\partial b_2}{\partial \overline{z_k}}$$

as functions of (b_1, b_2). For that purpose, the following identities relative to the matrix M will be useful. We define the matrices Q and T with

$$\begin{pmatrix} \frac{\partial \phi}{\partial \overline{z_1}} \\ \frac{\partial \phi}{\partial \overline{z_2}} \\ \frac{\partial \phi}{\partial \overline{z_3}} \end{pmatrix} = \begin{pmatrix} -v & -w & 0 \\ -\overline{w} & -(u+v) & -w \\ 0 & -\overline{w} & -u \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

$$\begin{pmatrix} \frac{\partial \psi}{\partial \overline{z_1}} \\ \frac{\partial \psi}{\partial \overline{z_2}} \\ \frac{\partial \psi}{\partial \overline{z_3}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = T\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \tag{5.22}$$

Lemma 5.7. *Let M be as in (5.6) and $Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, we have*

$$QMQ = 2(uv - w\overline{w}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + Q_1$$

$$\text{with } Q_1 = \begin{pmatrix} w\overline{w} & w(u-v) & -w^2 \\ (u-v)\overline{w} & (u-v)^2 & -w(u-v) \\ -\overline{w}^2 & -\overline{w}(u-v) & w\overline{w} \end{pmatrix}.$$

There holds

$$(\overline{Z}^t)Q_1 Z = 0 \quad \text{and} \quad (\overline{Z}^t)QMQZ = 2(uv - w\overline{w})(2 - u - v),$$

$$TMT = \begin{pmatrix} 4u^2 & -8uw & 4w^2 \\ -8u\overline{w} & 8(uv + w\overline{w}) & -8vw \\ 4\overline{w}^2 & -8v\overline{w} & 4v^2 \end{pmatrix}$$

$$= -4(uv - w\overline{w})^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} P \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$(\overline{Z}^t)TMTZ = 4(u+v)^2 - 4(u+v)^3 + 4(uv - w\overline{w})\big(3(u+v) - 2\big),$$

$$J = TMQ = QMT = 4(uv - w\overline{w}) \begin{pmatrix} u & -w & 0 \\ -\overline{w} & u+v & -w \\ 0 & -\overline{w} & v \end{pmatrix}$$

$$= -4(uv - w\overline{w}) \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} (Q^t) \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2(\overline{Z}^t)JZ = 8(uv - w\overline{w})\big[(u+v) - (u+v)^2 + 2(uv - w\overline{w})\big].$$

Proof. To calculate $(\overline{Z}^t)TMTZ$, we can obtain first

$$(\overline{Z}^t) \begin{pmatrix} 0 & 0 & 4w^2 \\ -8u\overline{w} & 8w\overline{w} & -8vw \\ 4\overline{w}^2 & 0 & 0 \end{pmatrix} Z$$

$$= 4w\overline{w}\big(2 - (u+v)\big) - 4(u+v)\overline{z_2}(wz_3 + \overline{w}z_1). \qquad \square$$

Theorem 5.8 *(Radial Laplacian and O–U operator). Let $b_1 = u + v$, $b_2 = uv - w\overline{w}$. Let F be a function depending only on b_1, b_2 and let Δ be as in (5.8), we have*

$$(\Delta F)(b_1, b_2) = 4\big[3b_1b_2 - b_1^3 + b_1^2 - 2b_2\big]\frac{\partial^2 F}{\partial b_1^2} + 8b_2\big(b_1 - b_1^2 + 2b_2\big)\frac{\partial^2 F}{\partial b_1 \partial b_2}$$
$$+ 2b_2(2 - b_1)\frac{\partial^2 F}{\partial b_2^2} + 4\big(b_2 - b_1^2\big)\frac{\partial F}{\partial b_1} - 4b_2(1 + b_1)\frac{\partial F}{\partial b_2} \tag{5.23}$$

and

$$\big(\Delta^{OU} F\big)(b_1, b_2) = (\Delta F)(b_1, b_2) + (UF)(b_1, b_2),$$
$$(UF)(b_1, b_2) = \big(2b_2 + b_1 - b_1^2\big)\frac{\partial F}{\partial b_1} + b_2(2 - b_1)\frac{\partial F}{\partial b_2}. \tag{5.24}$$

We have obtained an O–U operator in the rectangle $\{(b_1, b_2) \mid 0 \leqslant b_1 \leqslant 2,\ 0 \leqslant b_2 \leqslant 1\}$*. Note that the coefficients of the second order derivatives in* Δ *as well as* U *vanish on the border* $b_1 = 2$*,* $b_2 = 1$.

Proof. This is a consequence of Lemma 5.7. We obtain U with (5.18). □

5.4. *Jacobian of the action of* $Sp(2 \times 2)$ *on* $\mathcal{D}_2$

Let

$$\mathcal{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \quad \text{and} \quad \mathcal{W} = k_g(\mathcal{Z}) = (A\mathcal{Z} + B)(\overline{B}\mathcal{Z} + \overline{A})^{-1} = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_3 \end{pmatrix}. \tag{5.25}$$

For the Jacobian of the map $\mathcal{Z} \to k_g(\mathcal{Z})$, we write $k_g : (z_1, z_2, z_3) \to (w_1, w_2, w_3)$ and we express $dw_1 \wedge dw_2 \wedge dw_3$ in terms of $dz_1 \wedge dz_2 \wedge dz_3$.

Lemma 5.9.

$$dw_1 \wedge dw_2 \wedge dw_3 = \left(\det\left[(\overline{B}\mathcal{Z} + \overline{A})^{-1}\right]\right)^3 dz_1 \wedge dz_2 \wedge dz_3. \tag{5.26}$$

Proof. Denote $N = (\overline{B}\mathcal{Z} + \overline{A})^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = (C_1 \quad C_2)$ where C_1, C_2 are the columns of the matrix N. According to (3.7),

$$\begin{pmatrix} dw_1 & dw_2 \\ dw_2 & dw_3 \end{pmatrix} = {}^tN \begin{pmatrix} dz_1 & dz_2 \\ dz_2 & dz_3 \end{pmatrix} N.$$

Making the product of the matrices, it gives

$$\begin{aligned} dw_1 &= \alpha_1^2\, dz_1 + 2\alpha_1\alpha_3\, dz_2 + \alpha_3^2\, dz_3 = {}^tC_1 d\mathcal{Z} C_1, \\ dw_2 &= \alpha_1\alpha_2\, dz_1 + (\alpha_1\alpha_4 + \alpha_2\alpha_3)\, dz_2 + \alpha_3\alpha_4\, dz_3 = {}^tC_1 d\mathcal{Z} C_2, \\ dw_3 &= \alpha_2^2\, dz_1 + 2\alpha_2\alpha_4\, dz_2 + \alpha_4^2\, dz_3 = {}^tC_2 d\mathcal{Z} C_2. \end{aligned}$$

We find

$$\begin{aligned} &dw_1 \wedge dw_3 \\ &\quad = (\alpha_1\alpha_4 - \alpha_2\alpha_3)\left[2\alpha_1\alpha_2\, dz_1 \wedge dz_2 + (\alpha_1\alpha_4 + \alpha_2\alpha_3)\, dz_1 \wedge dz_3 + 2\alpha_3\alpha_4\, dz_2 \wedge dz_3\right] \end{aligned}$$

and $dw_1 \wedge dw_2 \wedge dw_3 = (\alpha_1\alpha_4 - \alpha_2\alpha_3)^3\, dz_1 \wedge dz_2 \wedge dz_3$. □

Lemma 5.10. *We have*

$$\frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^3} = \frac{dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \wedge dw_3 \wedge d\overline{w_3}}{|\det(I - \mathcal{W}\overline{\mathcal{W}})|^3}. \tag{5.27}$$

Proof.

$$\begin{aligned} d\mathcal{W} \wedge d\overline{\mathcal{W}} &= dw_1 \wedge d\overline{w_1} \wedge dw_2 \wedge d\overline{w_2} \wedge dw_3 \wedge d\overline{w_3} \\ &= \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det[\overline{B}\mathcal{Z} + \overline{A}]|^6} \end{aligned}$$

and $(d\mathcal{W} \wedge d\overline{\mathcal{W}})/|\det(I - \mathcal{W}\overline{\mathcal{W}})|^p$ is invariant if $p = 3$. □

Theorem 5.11. *The representation* (1.2) *is unitary for* μ^γ *where*

$$d\mu^\gamma(z) = \frac{dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge dz_3 \wedge d\overline{z_3}}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^3 |\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^\gamma}. \tag{5.28}$$

Proof. Let $w = k_g(z)$, using (5.27), we have

$$I = \int T_g f(z) \overline{T_g f(z)}\, d\mu(z) = \int f(w) \overline{f(w)}\, d\mu(w). \qquad \square$$

Part 3: The Lie algebra $\mathcal{SP}_{2n}$. Infinitesimal representation

6. The Lie algebra $\mathcal{SP}_{2n}$, $n = 2$

We need the classical lemma,

Lemma 6.1. *The Lie algebra* $\mathcal{SP}_{2n}$ *of* $Sp(2n)$ *has* real *dimension equal to* $2n^2 + n$.

Proof. We take the differentials of (3.1) and (3.2) at $g = Id$ and we denote dg the differential of g, this gives

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} dg \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = d\bar{g} \tag{6.1}$$

and

$$d({}^t g) J + J dg = 0. \tag{6.2}$$

From (6.1) and (6.2), we deduce that dg is of the form

$$e = \begin{pmatrix} a_1 & a_2 \\ \overline{a_2} & \overline{a_1} \end{pmatrix} \tag{6.3}$$

where the $n \times n$ matrices a_1, a_2 satisfy

$$a_1 + {}^t\overline{a_1} = 0 \quad \text{and} \quad a_2 \text{ is complex symmetric.} \tag{6.4}$$

The set of matrices $a_2 = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma \end{smallmatrix}\right)$ has *real* dimension $2\frac{n(n+1)}{2} = n(n+1)$. The set of matrices $a_1 = \left(\begin{smallmatrix} i\alpha & \beta \\ -\bar{\beta} & i\delta \end{smallmatrix}\right)$ where α, δ are real matrices has *real* dimension $2\frac{n(n-1)}{2} + n = n^2$. $\square$

Notation 6.2. We denote (δ_{pk}) the $n \times n$ matrix where the only non-zero coefficient is equal to one and is located at the intersection of the pth line and kth column.

$$(\delta_{pp})(\delta_{ps}) = (\delta_{ps}),\ (\delta_{pp})(\delta_{rs}) = 0 \text{ if } r \neq p \text{ and } (\delta_{rp})(\delta_{pp}) = (\delta_{rp}),\ (\delta_{rs})(\delta_{pp}) = 0 \text{ if } s \neq p.$$

Definition 6.3. The Lie algebra $\mathcal{SP}_{2n}$ is a direct sum of seven subspaces. They are defined with the following basis: In the set matrices $\left(\begin{smallmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{smallmatrix}\right)$, the set $\mathcal{E}_1$ consists of matrices where $a_2 = (\delta_{pp})$. For $\mathcal{E}_2$, we take $a_2 = i(\delta_{pp})$, then $\mathcal{E}_3$ corresponds to $a_2 = (\delta_{pq}) + (\delta_{qp})$, $p \neq q$ and $\mathcal{E}_4$ corresponds to $a_2 = i[(\delta_{pq}) + (\delta_{qp})]$, $p \neq q$.

In the set matrices $\left(\begin{smallmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{smallmatrix}\right)$, the set $\mathcal{F}_5$ consists of matrices where $a_1 = i(\delta_{pp})$. For $\mathcal{F}_6$, we take $a_1 = (\delta_{pq}) - (\delta_{qp})$, $p < q$ and $\mathcal{F}_7$ corresponds to $a_2 = i[(\delta_{pq}) + (\delta_{qp})]$, $p \neq q$.

Lemma 6.4. *For $n = 2$, we have for basis of the Lie algebra $\mathcal{SP}_{2n}$, $dg = \begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix}$ where a_2 is one of the matrices*

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \qquad e_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_6 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{6.5}$$

and $dg = \begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix}$ where a_1 is one of the matrices

$$e_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_8 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_9 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{10} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}. \tag{6.6}$$

As in Definition 6.3, we consider matrices $\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix}$ where $a_2 \in E_j$, $j = 1, \dots, 4$,

$$E_1 = \{\tau_1 e_1 + \tau_2 e_2\}, \qquad E_2 = \{\tau_1 e_3 + \tau_2 e_4\}, \qquad E_3 = \{\tau e_5\}, \qquad E_4 = \{\tau e_6\}, \tag{6.7}$$

and the matrices $\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix}$ where $a_1 \in F_j$, $j = 5, 6, 7$,

$$F_5 = \{\alpha e_9 + \delta e_{10}\}, \qquad F_6 = \{\tau e_7\}, \qquad F_7 = \{\tau e_8\} \tag{6.8}$$

and all parameters $\tau_1, \dots$ are real. Note that for $e = \begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix}$ and $u = \begin{pmatrix} 0 & u_2 \\ \overline{u_2} & 0 \end{pmatrix}$, we have $[e, u] = \begin{pmatrix} a_2\overline{u_2} - u_2\overline{a_2} & 0 \\ 0 & \overline{a_2}u_2 - \overline{u_2}a_2 \end{pmatrix}$.

6.1. Exponentiating the matrices of the basis (6.5)–(6.6)

In order to calculate the infinitesimal representation as in (2.19)–(2.22), we exponentiate the matrices as follows.

Notation 6.5. For $v = \begin{pmatrix} a_1 & a_2 \\ \bar{a}_2 & \bar{a}_1 \end{pmatrix}$ in the basis of $\mathcal{SP}_{2n}$, we consider a real parameter ϵ and the matrix ϵv. We denote $g_\epsilon = \exp(\epsilon v)$ a matrix in the group $Sp(2n)$ such that $\frac{d}{d\epsilon}_{|\epsilon=0} g_\epsilon = v$. There may be several solutions for g_ϵ in the same way that on the 1-dimensional real space, the two curves $y = sh(x)$ and $y = \sin(x)$ have the same tangent $y = x$ at $x = 0$. Since in the infinitesimal representation, we consider the tangent space to the Lie group G only at the neutral element of G, our choice for $g_\epsilon = \exp(\epsilon v)$ is justified. Moreover in our calculation, from the case E_1 to the case E_2 for example, we use Remark 3.1.

We consider the case $n = 2$, but we keep in mind the number of parameters in the subspaces (E_j), (F_j), for arbitrary n. This is summarized in (6.9)–(6.15):

(E_1) (n parameters):

$$a_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \quad \text{with } \tau_1, \tau_2 \text{ real},$$
$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau_1) & 0 & sh(\tau_1) & 0 \\ 0 & ch(\tau_2) & 0 & sh(\tau_2) \\ sh(\tau_1) & 0 & ch(\tau_1) & 0 \\ 0 & sh(\tau_2) & 0 & ch(\tau_2) \end{pmatrix}. \tag{6.9}$$

(E_2) (n parameters):

$$a_2 = \begin{pmatrix} i\tau_1 & 0 \\ 0 & i\tau_2 \end{pmatrix} \quad \text{with } \tau_1, \tau_2 \text{ real},$$

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau_1) & 0 & i\,sh(\tau_1) & 0 \\ 0 & ch(\tau_2) & 0 & i\,sh(\tau_2) \\ -i\,sh(\tau_1) & 0 & ch(\tau_1) & 0 \\ 0 & -i\,sh(\tau_2) & 0 & ch(\tau_2) \end{pmatrix}. \tag{6.10}$$

(E_3) ($\frac{n(n-1)}{2}$ parameters):

$$a_2 = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \quad \text{with } \tau \text{ real},$$

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau) & 0 & 0 & sh(\tau) \\ 0 & ch(\tau) & sh(\tau) & 0 \\ 0 & sh(\tau) & ch(\tau) & 0 \\ sh(\tau) & 0 & 0 & ch(\tau) \end{pmatrix}. \tag{6.11}$$

(E_4) ($\frac{n(n-1)}{2}$ parameters):

$$a_2 = \begin{pmatrix} 0 & i\tau \\ i\tau & 0 \end{pmatrix} \quad \text{with } \tau \text{ real},$$

$$\exp\begin{pmatrix} 0 & a_2 \\ \overline{a_2} & 0 \end{pmatrix} = \begin{pmatrix} ch(\tau) & 0 & 0 & i\,sh(\tau) \\ 0 & ch(\tau) & i\,sh(\tau) & 0 \\ 0 & -i\,sh(\tau) & ch(\tau) & 0 \\ -i\,sh(\tau) & 0 & 0 & ch(\tau) \end{pmatrix}. \tag{6.12}$$

Now consider $a_1 = \left(\begin{smallmatrix} i\alpha & \beta \\ -\bar\beta & i\delta \end{smallmatrix}\right)$ where α and δ are real.

(E_5) (n parameters):

$$a_1 = \begin{pmatrix} i\alpha & 0 \\ 0 & i\delta \end{pmatrix} \quad \text{with } \alpha, \delta \text{ real},$$

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 & 0 & 0 \\ 0 & e^{i\delta} & 0 & 0 \\ 0 & 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 0 & e^{-i\delta} \end{pmatrix}. \tag{6.13}$$

(E_6) ($\frac{n(n-1)}{2}$ parameters):

$$a_1 = \begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix} \quad \text{with } \tau \text{ real},$$

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{pmatrix}. \tag{6.14}$$

(E_7) ($\frac{n(n-1)}{2}$ parameters):

$$a_1 = \begin{pmatrix} 0 & i\tau \\ i\tau & 0 \end{pmatrix} \quad \text{with } \tau \text{ real},$$

$$\exp\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1} \end{pmatrix} = \begin{pmatrix} \cos(\tau) & i\sin(\tau) & 0 & 0 \\ i\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -i\sin(\tau) \\ 0 & 0 & -i\sin(\tau) & \cos(\tau) \end{pmatrix}. \tag{6.15}$$

6.2. The infinitesimal representation

In the following, we calculate $\rho(v)$ for each vector v of the basis given in Lemma 6.4. We consider separately each case $(E_1), \ldots, (E_7)$ as given in (6.7)–(6.8). We consider a_1, a_2 as in (6.3)–(6.4). In the cases E_1, E_2, E_3, E_4, we have $ch(\tau)$ on the diagonal of A_τ, then $\frac{d}{d\tau}_{|\tau=0} A_\tau = 0$. We deduce from Lemma 3.14 that for E_1, E_2, E_3, E_4, there holds

$$\frac{d}{d\tau}_{|\tau=0}\big(k_{g_\tau}(\mathcal{Z})\big) = \frac{d}{d\tau}_{|\tau=0} B_\tau - \mathcal{Z}\frac{d}{d\tau}_{|\tau=0}\overline{B}_\tau\mathcal{Z},$$
$$\frac{d}{d\tau}_{|\tau=0}\det(\overline{B_\tau}\mathcal{Z} + \overline{A_\tau}) = \operatorname{trace}\left[\frac{d}{d\tau}_{|\tau=0}\overline{B}_\tau\mathcal{Z}\right]. \tag{6.16}$$

For E_5, E_6, E_7, there holds $B = 0$ and

$$\frac{d}{d\epsilon}_{|\epsilon=0}\big(k_{g_\epsilon}(\mathcal{Z})\big) = \frac{d}{d\epsilon}_{|\epsilon=0} A_\epsilon\mathcal{Z} - \mathcal{Z}\frac{d}{d\epsilon}_{|\epsilon=0}\overline{A}_\epsilon,$$
$$\frac{d}{d\epsilon}_{|\epsilon=0}\det(\overline{B_\epsilon}\mathcal{Z} + \overline{A_\epsilon}) = \operatorname{trace}\left[\frac{d}{d\epsilon}_{|\epsilon=0}\overline{A}_\epsilon\right]. \tag{6.17}$$

7. Infinitesimal representation on $\mathcal{D}_2$

Notation 7.1. $\rho(\tau_1)$, $\rho(\tau_2)$ correspond to $a_1 = 0$, $a_2 = \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$ and $\widetilde{\rho}(\tau_1)$, $\widetilde{\rho}(\tau_2)$ correspond to $a_1 = 0$, $a_2 = \left(\begin{smallmatrix} i\tau_1 & 0 \\ 0 & i\tau_2 \end{smallmatrix}\right)$, τ_1, τ_2 are real.

$\rho(\tau)$ corresponds to $a_1 = 0$, $a_2 = \left(\begin{smallmatrix} 0 & \tau \\ \tau & 0 \end{smallmatrix}\right)$ and $\widetilde{\rho}(\tau)$ corresponds to $a_1 = 0$, $a_2 = \left(\begin{smallmatrix} 0 & i\tau \\ i\tau & 0 \end{smallmatrix}\right)$ where τ is real.

On the other hand, $\rho(\alpha)$, $\rho(\delta)$ correspond to $a_2 = 0$, $a_1 = \left(\begin{smallmatrix} i\alpha & 0 \\ 0 & i\delta \end{smallmatrix}\right)$ where α, δ are real.

$\rho_m(\tau)$ corresponds to $a_2 = 0$, $a_1 = \left(\begin{smallmatrix} 0 & \tau \\ -\tau & 0 \end{smallmatrix}\right)$ and $\widetilde{\rho}_m(\tau)$ corresponds to $a_2 = 0$, $a_1 = \left(\begin{smallmatrix} 0 & i\tau \\ i\tau & 0 \end{smallmatrix}\right)$ where τ is real.

Theorem 7.2. *We identify $\mathcal{Z} = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right)$ with $\mathcal{Z} = (z_1, z_2, z_3) \in C^3$. The infinitesimal representation is given by*

(1) $$\rho(\tau_1)\Phi = \big(1 - z_1^2\big)\frac{\partial\Phi}{\partial z_1} - z_1 z_2\frac{\partial\Phi}{\partial z_2} - z_2^2\frac{\partial\Phi}{\partial z_3} + \gamma z_1\Phi,$$

$$\rho(\tau_2)\Phi = -z_2^2\frac{\partial\Phi}{\partial z_1} - z_3 z_2\frac{\partial\Phi}{\partial z_2} + \big(1 - z_3^2\big)\frac{\partial\Phi}{\partial z_3} + \gamma z_3\Phi.$$

(2) $$\widetilde{\rho}(\tau_1)\Phi = i\left(1+z_1^2\right)\frac{\partial \Phi}{\partial z_1} + iz_1z_2\frac{\partial \Phi}{\partial z_2} + iz_2^2\frac{\partial \Phi}{\partial z_3} - i\gamma z_1\Phi,$$
$$\widetilde{\rho}(\tau_2)\Phi = iz_2^2\frac{\partial \Phi}{\partial z_1} + iz_3z_2\frac{\partial \Phi}{\partial z_2} + i\left(1+z_3^2\right)\frac{\partial \Phi}{\partial z_3} - i\gamma z_3\Phi.$$

(3) $$\rho(\tau)\Phi = -2z_1z_2\frac{\partial \Phi}{\partial z_1} + \left(1-\left(z_1z_3+z_2^2\right)\right)\frac{\partial \Phi}{\partial z_2} - 2z_3z_2\frac{\partial \Phi}{\partial z_3} + 2\gamma z_2\Phi.$$

(4) $$\widetilde{\rho}(\tau)\Phi = 2iz_1z_2\frac{\partial \Phi}{\partial z_1} + i\left(1+\left(z_1z_3+z_2^2\right)\right)\frac{\partial \Phi}{\partial z_2} + 2iz_3z_2\frac{\partial \Phi}{\partial z_3} - 2i\gamma z_2\Phi.$$

(5) $$\rho(\alpha)\Phi = 2iz_1\frac{\partial \Phi}{\partial z_1} + iz_2\frac{\partial \Phi}{\partial z_2} - i\gamma\Phi, \qquad \rho(\delta)\Phi = iz_2\frac{\partial \Phi}{\partial z_2} + 2iz_3\frac{\partial \Phi}{\partial z_3} - i\gamma\Phi.$$

(6) $$\rho_m(\tau)\Phi = 2z_2\frac{\partial \Phi}{\partial z_1} + (z_3-z_1)\frac{\partial \Phi}{\partial z_2} - 2z_2\frac{\partial \Phi}{\partial z_3}.$$

(7) $$\widetilde{\rho}_m(\tau)\Phi = 2iz_2\frac{\partial \Phi}{\partial z_1} + i(z_3+z_1)\frac{\partial \Phi}{\partial z_2} + 2iz_2\frac{\partial \Phi}{\partial z_3}.$$

Proof. We calculate (3.11)–(3.12) in each case where A and B are as in Section 6. For
$$\mathcal{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$$
by (6.16), we have

(1) $$A = \begin{pmatrix} ch(\tau_1) & 0 \\ 0 & ch(\tau_2) \end{pmatrix}, \qquad B = \begin{pmatrix} sh(\tau_1) & 0 \\ 0 & sh(\tau_2) \end{pmatrix},$$
$$\frac{d}{d\tau_1}_{|\tau_1=0,\,\tau_2=0} k_g(\mathcal{Z}) = \begin{pmatrix} (1-z_1^2) & -z_1z_2 \\ -z_1z_2 & -z_2^2 \end{pmatrix} \quad \text{and} \quad \frac{d}{d\tau_1}_{|\tau_1=0,\,\tau_2=0} \det(B\mathcal{Z}+A) = z_1$$

and
$$\frac{d}{d\tau_2}_{|\tau_1=0,\,\tau_2=0} k_g(\mathcal{Z}) = \begin{pmatrix} -z_2^2 & -z_2z_3 \\ -z_2z_3 & 1-z_3^2 \end{pmatrix} \quad \text{and} \quad \frac{d}{d\tau_2}_{|\tau_1=0,\,\tau_2=0} \det(B\mathcal{Z}+A) = z_3.$$
We deduce $\rho(\tau_1)$ and $\rho(\tau_2)$. For $\widetilde{\rho}(\tau_1)$ and $\widetilde{\rho}(\tau_2)$,

(2) $$A = \begin{pmatrix} ch(\tau_1) & 0 \\ 0 & ch(\tau_2) \end{pmatrix}, \qquad B = \begin{pmatrix} i\,sh(\tau_1) & 0 \\ 0 & i\,sh(\tau_2) \end{pmatrix},$$
$$\frac{d}{d\tau_1}_{|\tau_1=0,\,\tau_2=0} k_g(\mathcal{Z}) = \begin{pmatrix} i(1+z_1^2) & iz_1z_2 \\ iz_1z_2 & iz_2^2 \end{pmatrix} \quad \text{and}$$
$$\frac{d}{d\tau_1}_{|\tau_1=0,\,\tau_2=0} \det(\overline{B}\mathcal{Z}+\overline{A}) = -iz_1$$

and
$$\frac{d}{d\tau_2}_{|\tau_1=0,\,\tau_2=0} k_g(\mathcal{Z}) = \begin{pmatrix} iz_2^2 & iz_3z_2 \\ iz_3z_2 & i(1+z_3^2) \end{pmatrix}, \quad \frac{d}{d\tau_2}_{|\tau_1=0,\,\tau_2=0} \det(\overline{B}\mathcal{Z}+\overline{A}) = -iz_3.$$
We deduce $\widetilde{\rho}(\tau_1)$ and $\widetilde{\rho}(\tau_2)$. Now we calculate $\rho(\tau)$.

(3) $$A=\begin{pmatrix} ch(\tau) & 0 \\ 0 & ch(\tau)\end{pmatrix},\qquad B=\begin{pmatrix} 0 & sh(\tau) \\ sh(\tau) & 0\end{pmatrix},$$
$$\frac{d}{d\tau}_{|\tau=0}k_g(\mathcal{Z})=\begin{pmatrix} -2z_1z_2 & 1-(z_1z_3+z_2^2) \\ 1-(z_1z_3+z_2^2) & -2z_3z_2\end{pmatrix}$$

and

$$\frac{d}{d\tau}_{|\tau=0}\det(B\mathcal{Z}+A)=2z_2.$$

This gives $\rho(\tau)$. For $\widetilde{\rho}(\tau)$, we have

(4) $$A=\begin{pmatrix} ch(\tau) & 0 \\ 0 & ch(\tau)\end{pmatrix},\qquad B=\begin{pmatrix} 0 & i\,sh(\tau) \\ i\,sh(\tau) & 0\end{pmatrix},$$
$$\frac{d}{d\tau}_{|\tau=0}k_g(\mathcal{Z})=\begin{pmatrix} 2iz_1z_2 & i(1+z_1z_3+z_2^2) \\ i(1+z_1z_3+z_2^2) & 2iz_3z_2\end{pmatrix}$$

and

$$\frac{d}{d\tau}_{|\tau=0}\det(B\mathcal{Z}+A)=-2iz_2.$$

Now we consider the cases (E_5)–(E_6)–(E_7) where $B=0$. We have $k_g(\mathcal{Z})=A\mathcal{Z}\overline{A}^{-1}$ and $h_g(\mathcal{Z})=[\det(A)]^\gamma$. With (6.17), we obtain

(5) $$A=\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\delta}\end{pmatrix}\quad\text{then } k_g(\mathcal{Z})=\begin{pmatrix} e^{2i\alpha}z_1 & e^{i(\alpha+\delta)}z_2 \\ e^{i(\alpha+\delta)}z_2 & e^{2i\delta}z_3\end{pmatrix},$$
$$\frac{d}{d\alpha}_{|\alpha=0,\,\beta=0}k_g(\mathcal{Z})=\begin{pmatrix} 2iz_1 & iz_2 \\ iz_2 & 0\end{pmatrix},\qquad \frac{d}{d\beta}_{|\alpha=0,\,\beta=0}k_g(\mathcal{Z})=\begin{pmatrix} 0 & iz_2 \\ iz_2 & 2iz_3\end{pmatrix}.$$

On the other hand

$$h_g(\mathcal{Z})=e^{-i\gamma(\alpha+\delta)},$$
$$\frac{d}{d\alpha}_{|\alpha=0,\,\beta=0}h_g(\mathcal{Z})=-i\gamma\quad\text{and}\quad \frac{d}{d\beta}_{|\alpha=0,\,\beta=0}h_g(\mathcal{Z})=-i\gamma.$$

This gives $\rho(\alpha)$ and $\rho(\beta)$.

We calculate $\rho_m(\tau)$,

(6) $$A=\begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau)\end{pmatrix},\qquad \frac{d}{d\tau}_{|\tau=0}k_g(\mathcal{Z})=\begin{pmatrix} 2z_2 & z_3-z_1 \\ z_3-z_1 & -2z_2\end{pmatrix}.$$

For $\widetilde{\rho}_m(\tau)$, we have

(7) $$A=\begin{pmatrix} \cos(\tau) & i\sin(\tau) \\ i\sin(\tau) & \cos(\tau)\end{pmatrix},\qquad \frac{d}{d\tau}_{|\tau=0}k_g(\mathcal{Z})=\begin{pmatrix} 2iz_2 & i(z_3+z_1) \\ i(z_3+z_1) & 2iz_2\end{pmatrix}.$$

We identify $\mathcal{Z}=\left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3\end{smallmatrix}\right)$ with $\mathcal{Z}=(z_1,z_2,z_3)\in C^3$, let $k_g(\mathcal{Z})=(w_1,w_2,w_3)$. We have obtained the first order operators for the infinitesimal representation $\frac{d}{d\epsilon}_{|\epsilon=0}(T_{g_\epsilon}f)(z)$. $\square$

8. Identities for square of vector fields on $\mathcal{D}_2$

In Theorem 7.1, the operators are of the form $\rho_j = L_j + \gamma\phi_j I$ where L_j is a first order operator with holomorphic coefficients and $\phi_j(z_1, z_2, z_3)$ is a holomorphic function.

Theorem 8.1. *Taking $\gamma = 0$, we have*

$$2\big(\rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 + \rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 - \big(\rho(\alpha)^2 + \rho(\delta)^2\big)\big) + \rho(\tau)^2 + \widetilde{\rho}(\tau)^2 - \big(\rho_m(\tau)^2 + \widetilde{\rho}_m(\tau)^2\big) = 0. \tag{8.1}$$

In other words, there exist constants A_j such that

$$\mathcal{C} = \sum_j A_j L_j^2 = 0. \tag{8.2}$$

Proof.

$$\rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 = -4z_1^2\frac{\partial^2}{\partial z_1^2} - 4z_1z_2\frac{\partial^2}{\partial z_1\partial z_2} - 4z_2^2\frac{\partial^2}{\partial z_1\partial z_3} - 4z_1\frac{\partial}{\partial z_1} - 2z_2\frac{\partial}{\partial z_2},$$

$$\rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 = -4z_3^2\frac{\partial^2}{\partial z_3^2} - 4z_3z_2\frac{\partial^2}{\partial z_3\partial z_2} - 4z_2^2\frac{\partial^2}{\partial z_1\partial z_3} - 4z_3\frac{\partial}{\partial z_3} - 2z_2\frac{\partial}{\partial z_2},$$

$$\rho(\alpha)^2 + \rho(\delta)^2 = -2z_2^2\frac{\partial^2}{\partial z_2^2} - 4z_1^2\frac{\partial^2}{\partial z_1^2} - 4z_1z_2\frac{\partial^2}{\partial z_1\partial z_2} - 4z_3^2\frac{\partial^2}{\partial z_3^2} - 4z_3z_2\frac{\partial^2}{\partial z_3\partial z_2} - \left(4z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + 4z_3\frac{\partial}{\partial z_3}\right)$$

thus

$$\begin{aligned} D_1 &= \rho(\tau_1)^2 + \widetilde{\rho}(\tau_1)^2 + \rho(\tau_2)^2 + \widetilde{\rho}(\tau_2)^2 - \big(\rho(\alpha)^2 + \rho(\delta)^2\big) \\ &= 2z_2^2\frac{\partial^2}{\partial z_2^2} - 8z_2^2\frac{\partial^2}{\partial z_3\partial z_1} - 4z_1\frac{\partial}{\partial z_1} - 4z_2\frac{\partial}{\partial z_2} - 4z_3\frac{\partial}{\partial z_3} \\ &\quad + \left(4z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} + 4z_3\frac{\partial}{\partial z_3}\right). \end{aligned} \tag{8.3}$$

On the other hand, with (E_3)–(E_4), we have

$$\begin{aligned} D_2 &= \rho(\tau)^2 + \widetilde{\rho}(\tau)^2 \\ &= -4\big(z_1z_3 + z_2^2\big)\frac{\partial^2}{\partial z_2^2} - 8z_1z_2\frac{\partial^2}{\partial z_1\partial z_2} - 8z_3z_2\frac{\partial^2}{\partial z_3\partial z_2} \\ &\quad - 4z_1\frac{\partial}{\partial z_1} - 4z_2\frac{\partial}{\partial z_2} - 4z_3\frac{\partial}{\partial z_3} \end{aligned} \tag{8.4}$$

and from (E_6)–(E_7),

$$\begin{aligned} D_3 &= \rho_m(\tau)^2 + \widetilde{\rho}_m(\tau)^2 \\ &= -4z_1z_3\frac{\partial^2}{\partial z_2^2} - 8z_1z_2\frac{\partial^2}{\partial z_1\partial z_2} - 8z_3z_2\frac{\partial^2}{\partial z_3\partial z_2} - 16z_2^2\frac{\partial^2}{\partial z_3\partial z_1} \\ &\quad - 8z_2\frac{\partial}{\partial z_2} - 4z_1\frac{\partial}{\partial z_1} - 4z_3\frac{\partial}{\partial z_3}. \end{aligned} \tag{8.5}$$

Thus, if $\gamma = 0$,

$$\mathcal{C} = 2D_1 + D_2 - D_3 = 0. \tag{8.6}$$

This proves the theorem. □

Theorem 8.2. *Assume $\gamma \neq 0$, then with the same constants as in (8.2), we have*

$$\sum_j A_j \phi_j L_j = 0. \tag{8.7}$$

Proof. From (E_1) and (E_2),

$$T_1 = \phi(\tau_1)L(\tau_1) + \phi(\tau_2)L(\tau_2) + \widetilde{\phi(\tau_1)}\widetilde{L(\tau_1)} + \widetilde{\phi(\tau_2)}\widetilde{L(\tau_2)} = 2\gamma\left(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3}\right).$$

From (E_3) and (E_4)

$$T_2 = \phi(\tau)L(\tau) + \widetilde{\phi(\tau)}\widetilde{L(\tau)} = 4\gamma z_2\frac{\partial}{\partial z_2}.$$

From (E_5)

$$T_4 = \phi(\alpha)L(\alpha) + \phi(\delta)L(\delta) = 2\gamma\left[z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}\right].$$

Adding, we find $2T_1 - 2T_4 + T_2 = 0$. □

Corollary 8.3.

$$\sum_j A_j(\rho_j + \overline{\rho_j})\overline{L_j} = \sum_j A_j L_j \overline{L_j} + \gamma \sum_j A_j \phi_j \overline{L_j}. \tag{8.8}$$

9. Laplacian and O–U operator on $\mathcal{D}_2$ in terms of the infinitesimal representation of $Sp(2n)$, $n = 2$

With the same constants as in (8.1)–(8.2), we calculate $\sum_j A_j L_j \overline{L_j}$ and prove that it is the Laplacian on $\mathcal{D}_2$. Then we calculate the first order operator $U = \sum_j A_j \phi_j \overline{L_j}$, the "drift".

9.1. The Laplacian

Theorem 9.1. *Assume that $\gamma = 0$. Consider the vector fields as in (8.1), then*

$$\begin{aligned}&2\big[\rho(\tau_1)\overline{\rho(\tau_1)} + \widetilde{\rho}(\tau_1)\overline{\widetilde{\rho}(\tau_1)} + \rho(\tau_2)\overline{\rho(\tau_2)} + \widetilde{\rho}(\tau_2)\overline{\widetilde{\rho}(\tau_2)} - \big(\rho(\alpha)\overline{\rho(\alpha)} + \rho(\delta)\overline{\rho(\delta)}\big)\big]\\ &\quad + \rho(\tau)\overline{\rho(\tau)} + \widetilde{\rho}(\tau)\overline{\widetilde{\rho}(\tau)} - \big(\rho_m(\tau)\overline{\rho_m(\tau)} + \widetilde{\rho}_m(\tau)\overline{\widetilde{\rho}_m(\tau)}\big)\end{aligned} \tag{9.1}$$

is the Laplace–Beltrami Δ on $\mathcal{D}_2$, see (5.8).

Proof. Taking $\gamma = 0$, let

$$H_1 = \rho(\tau_1)\overline{\rho(\tau_1)} + \widetilde{\rho}(\tau_1)\overline{\widetilde{\rho}(\tau_1)} + \rho(\tau_2)\overline{\rho(\tau_2)} + \widetilde{\rho}(\tau_2)\overline{\widetilde{\rho}(\tau_2)} - \big(\rho(\alpha)\overline{\rho(\alpha)} + \rho(\delta)\overline{\rho(\delta)}\big),$$

$$\begin{aligned}
H_1 = {} & 2\big[(1-z_1\bar z_1)^2 + z_2^2\overline{z_2}^2\big]\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 2\big[(1-z_3\bar z_3)^2 + z_2^2\overline{z_2}^2\big]\frac{\partial^2}{\partial z_3\partial\overline{z_3}} \\
& + 2z_2\overline{z_2}[-1 + z_1\bar z_1 + z_3\overline{z_3}]\frac{\partial^2}{\partial z_2\partial\overline{z_2}} + 2\big[z_1\overline{z_2}(z_1\bar z_1 - 1) + z_2^2\overline{z_2}\bar z_3\big]\frac{\partial^2}{\partial z_1\partial\overline{z_2}} \\
& + 2\big[z_3\overline{z_2}(z_3\bar z_3 - 1) + z_2^2\overline{z_2}\bar z_1\big]\frac{\partial^2}{\partial z_3\partial\overline{z_2}} + 2\big[z_2\overline{z_1}(z_1\bar z_1 - 1) + z_2z_3\overline{z_2}^2\big]\frac{\partial^2}{\partial z_2\partial\overline{z_1}} \\
& + 2\big[z_2\overline{z_3}(z_3\bar z_3 - 1) + z_2z_1\overline{z_2}^2\big]\frac{\partial^2}{\partial z_2\partial\overline{z_3}} + 2\big[z_1^2\overline{z_2}^2 + z_2^2\overline{z_3}^2\big]\frac{\partial^2}{\partial z_1\partial\overline{z_3}} \\
& + 2\big[z_3^2\overline{z_2}^2 + z_2^2\overline{z_1}^2\big]\frac{\partial^2}{\partial z_3\partial\overline{z_1}}.
\end{aligned} \tag{9.2}$$

Then

$$\begin{aligned}
H_2 = {} & \rho(\tau)\overline{\rho(\tau)} + \widetilde{\rho}(\tau)\overline{\widetilde{\rho}(\tau)} \\
= {} & 8z_1\overline{z_1}z_2\overline{z_2}\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 2\big(1 + \big(z_1z_3 + z_2^2\big)\big(\overline{z_1}\,\overline{z_3} + \overline{z_2}^2\big)\big)\frac{\partial^2}{\partial z_2\partial\overline{z_2}} + 8z_3\overline{z_3}z_2\overline{z_2}\frac{\partial^2}{\partial z_3\partial\overline{z_3}} \\
& + 4z_1z_2\big(\overline{z_1}\,\overline{z_3} + \overline{z_2}^2\big)\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + 4\overline{z_1}\,\overline{z_2}\big(z_1z_3 + z_2^2\big)\frac{\partial^2}{\partial\overline{z_1}\partial z_2} + 8z_1\overline{z_3}z_2\overline{z_2}\frac{\partial^2}{\partial z_1\partial\overline{z_3}} \\
& + 8z_3\overline{z_1}z_2\overline{z_2}\frac{\partial^2}{\partial z_3\partial\overline{z_1}} + 4\overline{z_2}\,\overline{z_3}\big(z_1z_3 + z_2^2\big)\frac{\partial^2}{\partial z_2\partial\overline{z_3}} \\
& + 4z_2z_3\big(\overline{z_1}\,\overline{z_3} + \overline{z_2}^2\big)\frac{\partial^2}{\partial z_3\partial\overline{z_2}}
\end{aligned} \tag{9.3}$$

and

$$\begin{aligned}
H_3 = {} & \rho_m(\tau)\overline{\rho_m(\tau)} + \widetilde{\rho}_m(\tau)\overline{\widetilde{\rho}_m(\tau)} \\
= {} & 8z_2\overline{z_2}\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 8z_2\overline{z_2}\frac{\partial^2}{\partial z_3\partial\overline{z_3}} + 2(z_3\overline{z_3} + z_1\overline{z_1})\frac{\partial^2}{\partial z_2\partial\overline{z_2}} + 4z_2\overline{z_3}\frac{\partial^2}{\partial z_1\partial\overline{z_2}} \\
& + 4z_3\overline{z_2}\frac{\partial^2}{\partial z_2\partial\overline{z_1}} + 4z_1\overline{z_2}\frac{\partial^2}{\partial z_2\partial\overline{z_3}} + 4z_2\overline{z_1}\frac{\partial^2}{\partial z_3\partial\overline{z_2}}.
\end{aligned} \tag{9.4}$$

Adding all, we find $\Delta = 2H_1 + H_2 - H_3$,

$$\begin{aligned}
\Delta = {} & 4(1 - z_1\overline{z_1} - z_2\overline{z_2})^2\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 4(1 - z_3\overline{z_3} - z_2\overline{z_2})^2\frac{\partial^2}{\partial z_3\partial\overline{z_3}} \\
& + \big[(4z_2\overline{z_2} - 2)(-1 + z_1\overline{z_1} + z_3\overline{z_3}) + 2\big(z_1z_3 + z_2^2\big)\big(\overline{z_1}\,\overline{z_3} + \overline{z_2}^2\big)\big]\frac{\partial^2}{\partial z_2\partial\overline{z_2}} \\
& + 4(z_1\overline{z_1} + z_2\overline{z_2} - 1)\left[(z_1\overline{z_2} + z_2\overline{z_3})\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + (z_2\overline{z_1} + z_3\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_1}}\right] \\
& + 4(z_3\overline{z_3} + z_2\overline{z_2} - 1)\left[(z_3\overline{z_2} + z_2\overline{z_1})\frac{\partial^2}{\partial z_3\partial\overline{z_2}} + (z_2\overline{z_3} + z_1\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_3}}\right] \\
& + 4(z_1\overline{z_2} + z_2\overline{z_3})^2\frac{\partial^2}{\partial z_1\partial\overline{z_3}} + 4(z_3\overline{z_2} + z_2\overline{z_1})^2\frac{\partial^2}{\partial z_3\partial\overline{z_1}}.
\end{aligned} \tag{9.5}$$

As in (5.1)–(5.2), consider the matrices $\mathcal{Z}$ and $I-\mathcal{Z}\overline{\mathcal{Z}}$ with $u=1-z_1\overline{z_1}-z_2\overline{z_2}$, $v=1-z_3\overline{z_3}-z_2\overline{z_2}$ and $w=z_1\overline{z_2}+z_2\overline{z_3}$. We express the coefficients of the operator (9.5) with u, v, w. We see that Δ in (9.5) is equal to the Laplacian in (5.8). □

9.2. Case $\gamma\neq 0$. The O–U operator on $\mathcal{D}_2$

Theorem 9.2. *Assume that $\gamma\neq 0$, we write as in Section 7, $\rho=L+\gamma\phi I$. Then*

$$2\big[\rho(\tau_1)\overline{L(\tau_1)}+\widetilde{\rho}(\tau_1)\overline{\widetilde{L}(\tau_1)}+\rho(\tau_2)\overline{L(\tau_2)}+\widetilde{\rho}(\tau_2)\overline{\widetilde{L}(\tau_2)}-\big(\rho(\alpha)\overline{L(\alpha)}+\rho(\delta)\overline{L(\delta)}\big)\big]$$
$$+\rho(\tau)\overline{L(\tau)}+\widetilde{\rho}(\tau)\overline{\widetilde{L}(\tau)}-\big(\rho_m(\tau)\overline{L_m(\tau)}+\widetilde{\rho}_m(\tau)\overline{\widetilde{L}_m(\tau)}\big) \tag{9.6}$$

is equal to

$$\Delta-\gamma V \quad \text{where } V=\sum_{j,k}m_{jk}\frac{\partial}{\partial z_j}\log\det(I-\mathcal{Z}\overline{\mathcal{Z}})\frac{\partial}{\partial\overline{z_k}}. \tag{9.7}$$

The O–U operator $\Delta-\gamma V$ has $\mu^\gamma=\exp(-\gamma\log\det(I-\mathcal{Z}\overline{\mathcal{Z}}))\,dv$ for invariant measure, dv is the volume measure on $\mathcal{D}_2$. The measure μ^γ makes the representation unitary.

The proof will result from Theorem 5.4 and the following lemma where we relate U in (5.13)–(5.14) to the infinitesimal representation. With (A_j) as in (8.1)–(8.2)–(9.1)–(9.6), we put

$$\Delta^\gamma=\sum_j A_j(\rho_j+\overline{\rho_j})\overline{L_j}=\sum_j A_jL_j\overline{L_j}+\gamma\sum_j A_j\phi_j\overline{L_j}. \tag{9.8}$$

We have proved that $\sum_j A_jL_j\overline{L_j}$ is the Laplacian on $\mathcal{D}_2$.

Lemma 9.3. *The first order part in (9.8) is $\gamma\sum_j A_j(\phi_j+\overline{\phi_j})\overline{L_j}$ and is equal to $-4\gamma U$,*

$$\sum_j A_j\phi_j\overline{L_j}=4(\overline{z_1}u-\overline{w}\overline{z_2})\frac{\partial}{\partial\overline{z_1}}+4(\overline{z_3}v-w\overline{z_2})\frac{\partial}{\partial\overline{z_3}}+4(\overline{z_2}u-\overline{w}\overline{z_3})\frac{\partial}{\partial\overline{z_2}}. \tag{9.9}$$

Proof. By (8.7), we have $\sum_j A_j\overline{\phi_j}\overline{L_j}=0$. On the other hand, we calculate (9.9),

$$\sum_j A_j\phi_j\overline{L_j}=2U_1-2U_4+U_2 \tag{9.10}$$

with $U_1=\phi(\tau_1)\overline{H(\tau_1)}+\widetilde{\phi}(\tau_1)\overline{\widetilde{H(\tau_1)}}+\phi(\tau_2)\overline{H(\tau_2)}+\widetilde{\phi}(\tau_2)\overline{\widetilde{H(\tau_2)}}$,

$$U_1=-2z_1\overline{z_1}\left[\overline{z_1}\frac{\partial}{\partial\overline{z_1}}+\overline{z_2}\frac{\partial}{\partial\overline{z_2}}\right]-2z_3\overline{z_3}\left[\overline{z_2}\frac{\partial}{\partial\overline{z_2}}+\overline{z_3}\frac{\partial}{\partial\overline{z_3}}\right]-2z_1\overline{z_2}^2\frac{\partial}{\partial\overline{z_3}}-2z_3\overline{z_2}^2\frac{\partial}{\partial\overline{z_1}},$$
$$U_4=\phi(\alpha)\overline{H(\alpha)}+\phi(\delta)\overline{H(\delta)}=-2\left[\overline{z_1}\frac{\partial}{\partial\overline{z_1}}+\overline{z_2}\frac{\partial}{\partial\overline{z_2}}+\overline{z_3}\frac{\partial}{\partial\overline{z_3}}\right],$$
$$U_2=\phi(\tau)\overline{H(\tau)}+\widetilde{\phi}(\tau)\overline{\widetilde{H(\tau)}}=-8z_2\overline{z_2}\left[\overline{z_1}\frac{\partial}{\partial\overline{z_1}}+\overline{z_3}\frac{\partial}{\partial\overline{z_3}}\right]-4z_2\big(\overline{z_1}\overline{z_3}+\overline{z_2}^2\big)\frac{\partial}{\partial\overline{z_2}}.$$

Part 4: Kähler geometry on $\mathcal{D}_n$. Action of $Sp(2n)$

10. Metric and Laplacian, O–U operators on $\mathcal{D}_n$

We recall and prove for completeness the formulae relative to the metric and the expressions of the Laplacian on $\mathcal{D}_n$. The proofs are different from the specific calculations done for $\mathcal{D}_2$ in Section 6.

10.1. Metric and Laplacian on $\mathcal{D}_n$

The manifold $\mathcal{D}_n$ has complex dimension $n(n+1)/2$. We denote the coefficients of the matrix $\mathcal{Z}$

$$\mathcal{Z} = (z_1, z_2, \dots, z_{n(n+1)/2}).$$

Let

$$K(\mathcal{Z}, \overline{\mathcal{Z}}) = \det(I - \mathcal{Z}\overline{\mathcal{Z}}) \quad \text{and} \quad U(\mathcal{Z}, \overline{\mathcal{Z}}) = \log K(\mathcal{Z}, \overline{\mathcal{Z}}). \tag{10.1}$$

Since $\mathcal{Z}$ is symmetric, the transposed matrix $(I - \mathcal{Z}\overline{\mathcal{Z}})^t$ is equal to $I - \overline{\mathcal{Z}}\mathcal{Z}$ and

$$K(\mathcal{Z}, \overline{\mathcal{Z}}) = \overline{K(\mathcal{Z}, \overline{\mathcal{Z}})}. \tag{10.2}$$

This shows that $K(\mathcal{Z}, \overline{\mathcal{Z}})$ is a *real-valued function*. On the domain $\mathcal{D}_n$, we consider the Kählerian metric

$$ds^2 = -\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}})\, dz_j\, d\bar{z}_k = -\sum_{j,k} p_{jk}\, dz_j\, d\overline{z_k}. \tag{10.3}$$

Lemma 10.1. *Let*

$$P = (p_{jk}) \quad \text{with } p_{jk} = \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}). \tag{10.4}$$

Then

$$\frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) = -\operatorname{trace}\left[(I - \overline{\mathcal{Z}}\mathcal{Z})^{-1} \frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}} \times (I - \mathcal{Z}\overline{\mathcal{Z}})^{-1} \frac{\partial \mathcal{Z}}{\partial z_j}\right]. \tag{10.5}$$

In particular p_{jk} can be expressed in terms of the coefficients of the matrix $I - \mathcal{Z}\overline{\mathcal{Z}}$. We have ${}^t\overline{P} = P$ and

$$\det P = (-1)^{n(n+1)/2} 2^{n(n-1)/2} \times \det(I - \mathcal{Z}\overline{\mathcal{Z}})^{-(n+1)}. \tag{10.6}$$

Proof.

$$\log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) = \operatorname{trace}\log(I - \mathcal{Z}\overline{\mathcal{Z}}). \tag{10.7}$$

We deduce

$$\frac{\partial}{\partial z_j}\log\det(I-\mathcal{Z}\overline{\mathcal{Z}}) = -\mathrm{trace}\left[(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\frac{\partial \mathcal{Z}}{\partial z_j}\overline{\mathcal{Z}}\right],$$

$$\frac{\partial^2}{\partial z_j\partial \overline{z_k}}\log\det(I-\mathcal{Z}\overline{\mathcal{Z}}) = -\mathrm{trace}\left[\frac{\partial}{\partial \overline{z_k}}(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\frac{\partial \mathcal{Z}}{\partial z_j}\overline{\mathcal{Z}}\right] - \mathrm{trace}\left[(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\frac{\partial \mathcal{Z}}{\partial z_j}\frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}}\right]. \tag{10.8}$$

Then we use

$$\frac{\partial}{\partial \overline{z_k}}(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1} = (I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\mathcal{Z}\frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}}\times(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}.$$

Thus

$$p_{jk} = -\mathrm{trace}\big[Q(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\big] \quad \text{with } Q = \frac{\partial \mathcal{Z}}{\partial z_j}\overline{\mathcal{Z}}(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\mathcal{Z}\frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}} + \frac{\partial \mathcal{Z}}{\partial z_j}\frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}}.$$

Since

$$Q = \frac{\partial \mathcal{Z}}{\partial z_j}(I-\overline{\mathcal{Z}}\mathcal{Z})^{-1}\frac{\partial \overline{\mathcal{Z}}}{\partial \overline{z_k}}$$

this proves (10.5). To prove (10.6), we note that the matrices $\frac{\partial \mathcal{Z}}{\partial z_j}$ form a basis of the complex $n(n+1)/2$-dimensional vector space *Sym* of $n\times n$ complex symmetric matrices. On *Sym*, we consider the scalar product

$$(\mathcal{Z}_1|\mathcal{Z}_2) = \mathrm{trace}[\mathcal{Z}_1\overline{\mathcal{Z}}_2]. \tag{i}$$

We denote

$$E_j = \frac{\partial \mathcal{Z}}{\partial z_j} \quad \text{if } z_j \text{ is on the principal diagonal of } \mathcal{Z} \tag{ii}$$

and

$$E_j = \frac{1}{\sqrt{2}}\frac{\partial \mathcal{Z}}{\partial z_j} \quad \text{if } z_j \text{ is outside the principal diagonal of } \mathcal{Z}. \tag{iii}$$

With the scalar product (i), $(E_j)_{j=1,\dots,n(n+1)/2}$ is an orthonormal basis of *Sym*. In the following, *we fix the matrix* $\mathcal{Z}$. The matrix $H=(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}$ is hermitian ($\overline{H}={}^tH$), thus we diagonalize: $H={}^t\overline{U}DU$ where ${}^t\overline{U}U=I$ and D is an $n\times n$ diagonal matrix. Let $S\in Sym$ and consider the linear map $A: Sym\to Sym$,

$$A: S\to(I-\overline{\mathcal{Z}}\mathcal{Z})^{-1}S\times(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}. \tag{10.9}$$

Let $C: Sym\to Sym$ defined by

$$C: S\to{}^tUSU.$$

The inverse map of C is $C^{-1}: S\to\overline{U}S{}^t\overline{U}$. We denote $\Lambda: Sym\to Sym$ the map

$$\Lambda: S\to DSD. \tag{10.10}$$

Then $A=C\circ\Lambda\circ C^{-1}$. We deduce that $\det(A)=\det(\Lambda)$. Thus it is enough to calculate $\det(\Lambda)$. For this, denote $(\lambda_1,\lambda_2,\dots,\lambda_n)$ the elements of the diagonal of D. The matrix of Λ in the basis

$(E_j)_{1\leqslant j\leqslant n(n+1)/2}$ is diagonal and is constituted of

$$\left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2, \lambda_1\lambda_2, \ldots, \lambda_1\lambda_n, \lambda_2\lambda_3, \ldots\right).$$

Its determinant is $\prod_j \lambda_j^{n+1} = [\det(D)]^{n+1}$. Thus $\det(\Lambda) = [\det(D)]^{n+1}$. This gives

$$\det(A) = \left[\det(I - \mathcal{Z}\overline{\mathcal{Z}})\right]^{-(n+1)}. \tag{10.11}$$

In the basis $(E_j)_{j=1,\ldots,n(n+1)/2}$, the matrix of A is $A = (A_{kj})$ with $A_{kj} = (AE_j|E_k)$. From (ii) and (iii), we have

$$(AE_j|E_k) = -p_{kj} \quad \text{if } z_j \text{ and } z_k \text{ are both on the diagonal of } \mathcal{Z}, \tag{iv}$$

$$(AE_j|E_k) = -(1/\sqrt{2})\,p_{kj} \quad \text{if only one of } z_j \text{ and } z_k \text{ is on the diagonal of } \mathcal{Z} \tag{v}$$

and the other is outside the diagonal, finally

$$(AE_j|E_k) = -(1/2)\,p_{kj} \quad \text{if none of } z_j \text{ and } z_k \text{ are on the diagonal of } \mathcal{Z}. \tag{vi}$$

This gives

$$\det A = (-1)^{n(n+1)/2}\left(\frac{1}{2}\right)^{n(n-1)/2} \det P. \tag{vii}$$

From (10.11) and (vii), we deduce (10.6) and we have proved the lemma. □

Let

$$\omega = i\sum_{j,k} p_{jk}\, dz_j \wedge d\overline{z_k} = i\bar{\partial}\partial U \tag{10.12}$$

then $d\omega = 0$, $\partial\omega = 0$, $\bar{\partial}\omega = 0$. We put

$$M = -4\overline{P}^{-1}, \qquad M = (m_{jk}), \qquad \Delta = \sum_{j,k} m_{jk}\frac{\partial^2}{\partial z_j \partial \overline{z_k}} \tag{10.13}$$

then Δ is the Laplacian on $\mathcal{D}$ associated to the metric (10.3). The volume element is given by

$$dv = \frac{dz_1 \wedge d\overline{z_1} \wedge \cdots}{|\det(I - \mathcal{Z}\overline{\mathcal{Z}})|^{n+1}}. \tag{10.14}$$

Theorem 10.2. *The volume measure dv is an invariant measure for Δ ($\int \Delta\psi\, dv = 0$). We have*

$$\sum_j \frac{\partial}{\partial z_j}(m_{jk}\det P) = 0. \tag{10.15}$$

Proof. By (10.15), integration by parts shows that the volume is an invariant measure for Δ. Now we prove (10.15). The condition

$$\sum_j \frac{\partial}{\partial z_j}\left[\left(\overline{P}^{-1}\right)_{jk}\det P\right] = 0, \quad \forall k$$

is equivalent to

$$\sum_j \frac{\partial}{\partial z_j}\big[\big(\overline{P}^{-1}\big)_{jk}\big] + \sum_j \big(\overline{P}^{-1}\big)_{jk} \frac{\partial}{\partial z_j} \log[\det P] = 0, \quad \forall k. \tag{i}$$

Multiplying on the right by the matrix $\overline{P}$, we deduce from (i),

$$\sum_{j,k} \frac{\partial}{\partial z_j}\big[\big(\overline{P}^{-1}\big)_{jk}\big]\overline{P}_{kq} + \sum_{j,k} \big(\overline{P}^{-1}\big)_{jk} \overline{P}_{kq} \frac{\partial}{\partial z_j} \log[\det P] = 0.$$

Equivalently

$$-\sum_{j,k} \big(\overline{P}^{-1}\big)_{jk} \frac{\partial}{\partial z_j} \overline{P}_{kq} + \frac{\partial}{\partial z_q} \log[\det P] = 0. \tag{ii}$$

Thus to obtain (10.15), we prove (ii). By (10.4),

$$\frac{\partial}{\partial z_j}\overline{P}_{kq} = \frac{\partial}{\partial z_j}\frac{\partial^2}{\partial \bar{z}_k \partial z_q} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) = \frac{\partial}{\partial z_q} P_{jk} = \frac{\partial}{\partial z_q} \overline{P}_{kj}.$$

This gives

$$\sum_{j,k} \big(\overline{P}^{-1}\big)_{jk} \frac{\partial}{\partial z_j} \overline{P}_{kq} = \sum_{j,k} \big(\overline{P}^{-1}\big)_{jk} \frac{\partial}{\partial z_q} \overline{P}_{kj} = \operatorname{trace}\left[\overline{P}^{-1} \frac{\partial}{\partial z_q} \overline{P}\right]$$
$$= \operatorname{trace}\left[\frac{\partial}{\partial z_q} \log \overline{P}\right] = \frac{\partial}{\partial z_q} \operatorname{trace} \log \overline{P} = \frac{\partial}{\partial z_q} \log\det(P)$$

where we have used (10.7) at the last step. This proves (ii). □

Lemma 10.3. *(Compare with (10.5).) The matrix $\mathcal{Z}$ being fixed, we denote $z_j \in \operatorname{diag}(\mathcal{Z})$ if z_j is on the principal diagonal of $\mathcal{Z}$ and $z_j \notin \operatorname{diag}(\mathcal{Z})$ if not. We have*

$$\big(P^{-1}\big)_{jk} = -\operatorname{trace}\left[(I - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial \mathcal{Z}}{\partial z_j} (I - \overline{\mathcal{Z}}\mathcal{Z}) \frac{\partial \overline{\mathcal{Z}}}{\partial \bar{z}_k}\right] \quad \text{if } z_j \in \operatorname{diag}(\mathcal{Z}),\ z_k \in \operatorname{diag}(\mathcal{Z}),$$

$$\big(P^{-1}\big)_{jk} = -\frac{1}{2}\operatorname{trace}\left[(I - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial \mathcal{Z}}{\partial z_j} (I - \overline{\mathcal{Z}}\mathcal{Z}) \frac{\partial \overline{\mathcal{Z}}}{\partial \bar{z}_k}\right] \quad \text{if only one of } z_j, z_k \in \operatorname{diag}(\mathcal{Z}),$$

$$\big(P^{-1}\big)_{jk} = -\frac{1}{4}\operatorname{trace}\left[(I - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial \mathcal{Z}}{\partial z_j} (I - \overline{\mathcal{Z}}\mathcal{Z}) \frac{\partial \overline{\mathcal{Z}}}{\partial \bar{z}_k}\right] \quad \text{if } z_j \notin \operatorname{diag}(\mathcal{Z}),\ z_k \notin \operatorname{diag}(\mathcal{Z}).$$

Proof. As in the proof of Lemma 10.1, let *Sym* be the complex vector space of $n \times n$ complex symmetric matrices. Then we have the direct sum

$$Sym = Sym_1 \oplus Sym_2 \tag{i}$$

where Sym_1 is the subspace of $n \times n$ diagonal matrices and Sym_2 is the subspace of matrices with 0 on the diagonal. In the decomposition (i), let $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_2 & A_3 \end{smallmatrix}\right)$ be the matrix of the A defined in (10.9), $A(S) = (I - \overline{\mathcal{Z}}\mathcal{Z})^{-1} S \times (I - \mathcal{Z}\overline{\mathcal{Z}})^{-1}$. From (iv)–(v)–(vi) in the proof of Lemma 10.1, the matrix P is given by

$$P = -\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

and the inverse of P is

$$P^{-1} = -\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} A^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We deduce that $(P^{-1})_{kj} = -(A^{-1}E_j|E_k)$ if z_j and z_k are both on the diagonal of $\mathcal{Z}$, then $(P^{-1})_{kj} = -\frac{1}{\sqrt{2}}(A^{-1}E_j|E_k)$ if only one of z_j and z_k is on the diagonal of $\mathcal{Z}$ and the other is outside the diagonal, finally $(P^{-1})_{kj} = -\frac{1}{2}(A^{-1}E_j|E_k)$ if none of z_j and z_k are on the diagonal of $\mathcal{Z}$. Since

$$A^{-1} : S \to (I - \overline{\mathcal{Z}}\mathcal{Z})S(I - \mathcal{Z}\overline{\mathcal{Z}})$$

we obtain

$$\left(A^{-1}E_j \middle| E_k\right) = \operatorname{trace}\left[(I - \overline{\mathcal{Z}}\mathcal{Z})E_j(I - \mathcal{Z}\overline{\mathcal{Z}})E_k\right] = \left(A^{-1}\right)_{kj}.$$

Replacing E_j and E_k with (ii)–(iii) in the proof of Lemma 10.1, we finish the proof. □

With Notation 6.2, given two $n \times n$ symmetric matrices $U = (U_{jk})$ and $V = (V_{rs})$, we have for $j \leqslant k$, $s \leqslant r$,

$$\operatorname{trace}\big(U(\delta_{jk})V(\delta_{rs})\big) = U_{sj}V_{kr}. \tag{10.16}$$

From Lemma 10.3, we deduce

Corollary 10.4. *Let $M = -4\overline{P}^{-1}$ and the Laplacian Δ be as in (10.13), then the coefficient of*

$$\frac{\partial^2}{\partial \mathcal{Z}_{jk}\partial \overline{\mathcal{Z}_{rs}}} \quad \text{in } \Delta \text{ is equal to } \operatorname{trace}\big[(I - \overline{\mathcal{Z}}\mathcal{Z})\big((\delta_{jk}) + (\delta_{kj})\big)(I - \mathcal{Z}\overline{\mathcal{Z}})\big((\delta_{rs}) + (\delta_{sr})\big)\big]. \tag{10.17}$$

Proof. From Lemma 10.3 and (10.13). □

Remark 10.5. The lemmas of the present subsection extend if we consider the set of $\mathcal{Z}$ such that $I - p\mathcal{Z}\overline{\mathcal{Z}} > 0$ and functions $K(\mathcal{Z})$ of the form $K(\mathcal{Z}) = \log\det(I - p\mathcal{Z}\overline{\mathcal{Z}})$ where $-1 \leqslant p \leqslant 1$, $p \neq 0$, p is a real number. See [4]. The associated metrics as in (10.3) are Kählerian. In the following, we restrict ourselves to the case where $K(\mathcal{Z})$ is given by (10.1).

10.2. O–U operators on $\mathcal{D}_n$

As in Definition 2.1, consider the complex O–U operator on $\mathcal{D}_n$,

$$\Delta^\gamma = \sum_{jk} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} - \gamma \sum_{j,k} m_{jk} \frac{\partial}{\partial z_j} \log\det(I - \mathcal{Z}\overline{\mathcal{Z}}) \frac{\partial}{\partial \overline{z_k}} \tag{10.18}$$

where $M = (m_{jk})$, Δ are given in (10.13) and K in (10.1).

Theorem 10.6 *(First order part in Δ^γ). Let $\mathcal{Z} = (\mathcal{Z}_{jk}) = (z_1, z_2, \ldots)$. We have*

$$V = \sum_{j,k} m_{jk}\left[\frac{\partial}{\partial z_j}\log K\right]\frac{\partial}{\partial \overline{z_k}} = 4\sum_{r\leqslant s}\big(\overline{\mathcal{Z}}(I-\mathcal{Z}\overline{\mathcal{Z}})\big)_{rs}\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}. \tag{10.19}$$

The matrix $\overline{\mathcal{Z}}(I-\mathcal{Z}\overline{\mathcal{Z}})$ is symmetric and is equal to $(I-\overline{\mathcal{Z}}\mathcal{Z})\overline{\mathcal{Z}}$. Compare with (5.10)–(5.13).

Proof. For $\mathcal{Z} = (\mathcal{Z}_{jk}) = (z_1, z_2, \ldots)$, we deduce from (10.8) that

$$V = 4\sum\big(\overline{P}^{-1}\big)_{jk}\,\mathrm{trace}\left[(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}\left(\frac{\partial \mathcal{Z}}{\partial z_J}\right)\overline{\mathcal{Z}}\right]\frac{\partial}{\partial \overline{z_\kappa}}. \tag{i}$$

We pass to double indices for the coefficients of the matrix $\mathcal{Z}$: we put $J = (jk)$ and $\kappa = (rs)$. By (10.17) and (i), we have

$$V = 4\sum(I-\overline{\mathcal{Z}}\mathcal{Z})_{sj}(I-\mathcal{Z}\overline{\mathcal{Z}})_{kr}(I-\mathcal{Z}\overline{\mathcal{Z}})^{-1}_{lj}\overline{\mathcal{Z}}_{jl}\frac{\partial}{\partial \overline{\mathcal{Z}}_{rs}}.$$

After simplification, we obtain (10.19). □

Corollary 10.7. *Let Φ be a differentiable function on $\mathcal{D}_n$ depending only on the coefficients of $R = I - \mathcal{Z}\overline{\mathcal{Z}} = (R_{jk})$, then for the vector field V in (10.19), we have*

$$V = \sum_{j,k} m_{jk}\left[\frac{\partial}{\partial z_j}\log K\right]\frac{\partial}{\partial \overline{z_k}} = 4\sum_{r,s}\big[R(I-R)\big]_{rs}\frac{\partial \Phi}{\partial R_{rs}}. \tag{10.20}$$

Proof. Consider the function Φ of $(I-\mathcal{Z}\overline{\mathcal{Z}})$. We have

$$\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}\Phi(I-\mathcal{Z}\overline{\mathcal{Z}}) = \sum\frac{\partial \Phi}{\partial R_{jk}}\times\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}(I-\mathcal{Z}\overline{\mathcal{Z}})_{jk}. \tag{i}$$

Since

$$\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}(I-\mathcal{Z}\overline{\mathcal{Z}})_{jk} = -\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}(\mathcal{Z}\overline{\mathcal{Z}})_{jk} = -\sum_l\frac{\partial}{\partial \overline{\mathcal{Z}_{rs}}}(\mathcal{Z}_{jl}\overline{\mathcal{Z}}_{lk}) = -\mathcal{Z}_{jr}1_{k=s}, \tag{ii}$$

replacing (ii) in (i), we obtain (10.20). This is also explicited in (5.18). □

Remark 10.8. We call radial coordinates in $\mathcal{D}_n$ the coefficients of the characteristic polynomial $\det(R-\lambda I)$ of the matrix $R = I - \mathcal{Z}\overline{\mathcal{Z}}$. Radial coordinates are given by the eigenvalues of the hermitian matrix $R = I - \mathcal{Z}\overline{\mathcal{Z}}$. In Section 5, the Laplacian and O–U operators on $\mathcal{D}_2$ have been explicited in radial coordinates. If Φ is a function on $\mathcal{D}_n$ depending only on $I - \mathcal{Z}\overline{\mathcal{Z}}$ and Δ is the Laplacian in (10.18), we cannot express $\Delta\Phi$ only with the coefficients of $I - \mathcal{Z}\overline{\mathcal{Z}}$. But if Φ depends only on the eigenvalues of $R = I - \mathcal{Z}\overline{\mathcal{Z}}$, then $\Delta\Phi$ is obtained with the radial expression of Δ, see [8, p. 64] and [1, pp. 639–650].

11. Infinitesimal representation on $\mathcal{D}_n$

Consider a complex symmetric matrix $\mathcal{Z} = (\mathcal{Z}_{pk})$ of order n. Extending (6.7)–(6.8), Notation 7.1, we obtain for the vectors in E_1 and E_2, see (6.9)–(6.10),

$$\rho(\tau_p)\Phi = \sum_{k,j,k\leqslant j} \big[(\delta_{pp}) - \mathcal{Z}(\delta_{pp})\mathcal{Z}\big]_{kj} \frac{\partial}{\partial \mathcal{Z}_{kj}}\Phi + \gamma z_{pp}\Phi,$$

$$\widetilde{\rho}(\tau_p)\Phi = i\sum_{k,j,k\leqslant j} \big[(\delta_{pp}) + \mathcal{Z}(\delta_{pp})\mathcal{Z}\big]_{kj} \frac{\partial}{\partial \mathcal{Z}_{kj}}\Phi - i\gamma z_{pp}\Phi. \tag{11.1}$$

As in Theorem 8.1, assume that $\gamma = 0$. We deduce

$$\rho(\tau_p)^2 + \widetilde{\rho}(\tau_p)^2 = -4\sum_{k,j,k\leqslant j} \big(\mathcal{Z}(\delta_{pp})\mathcal{Z}\big)_{kj} \frac{\partial^2}{\partial \mathcal{Z}_{kj}\partial \mathcal{Z}_{pp}} - 2\big[(\delta_{pp})\mathcal{Z} + \mathcal{Z}(\delta_{pp})\big]_{kj} \frac{\partial}{\partial \mathcal{Z}_{kj}}. \tag{11.2}$$

If $\gamma = 0$, as in (8.3), let $D_1 = \sum_p[\rho(\tau_p)^2 + \widetilde{\rho}(\tau_p)^2] - \sum_p \rho(\alpha_p)^2$, $D_2 = \sum[\rho(\tau)^2 + \widetilde{\rho}(\tau)^2]$ where $\rho(\tau)$, $\widetilde{\rho}(\tau)$ come from vectors $v_\tau \in E_4, E_5$, then $D_3 = \sum[\rho(\tau)^2 + \widetilde{\rho}(\tau)^2]$ where $\rho(\tau)$, $\widetilde{\rho}(\tau)$ come from vectors $v_\tau \in E_6, E_7$. By expressing the coefficients of $\frac{\partial^2}{\partial \mathcal{Z}_{pj}\partial \mathcal{Z}_{ks}}$ in terms of the coefficients of the matrices $\mathcal{Z}(\delta_{pp})\mathcal{Z}$, we find that (8.6) holds true.

To obtain Theorem 9.1 for general n, we proceed as follows:

1. *Vectors like in $\mathcal{E}_1$, $\mathcal{E}_2$.*

$$\rho(\tau_p)\overline{\rho(\tau_p)} + \widetilde{\rho}(\tau_p)\overline{\widetilde{\rho}(\tau_p)} = 2\sum_{k\leqslant j}\sum_{r\leqslant s}\big[(\delta_{pp})_{kj}(\delta_{pp})_{rs} + \big(\mathcal{Z}(\delta_{pp})\mathcal{Z}\big)_{kj}\big(\overline{\mathcal{Z}}(\delta_{pp})\overline{\mathcal{Z}}\big)_{rs}\big]\frac{\partial^2}{\partial \mathcal{Z}_{kj}\partial \overline{\mathcal{Z}_{rs}}}. \tag{11.3}$$

2. *Vectors coming from $\mathcal{E}_5$, see* (6.13). The matrix a_1 is diagonal with $e^{i\alpha_p}$, $p = 1, \ldots, n$ on the diagonal. With (6.17), we obtain

$$\rho(\alpha_p)\Phi = i\sum_{k,j,k\leqslant j} \big[(\delta_{pp})\mathcal{Z} + \mathcal{Z}(\delta_{pp})\big]_{kj} \frac{\partial}{\partial \mathcal{Z}_{kj}}\Phi - i\gamma\Phi. \tag{11.4}$$

From (11.3)–(11.4), the coefficient of $\frac{\partial^2}{\partial \mathcal{Z}_{kj}\partial \overline{\mathcal{Z}_{rs}}}$ in $\rho(\tau_p)\overline{\rho(\tau_p)} + \widetilde{\rho}(\tau_p)\overline{\widetilde{\rho}(\tau_p)} - \rho(\alpha_p)\overline{\rho(\alpha_p)}$ is equal to

$$\begin{aligned} h_1 &= 2\big[(\delta_{pp})_{kj}(\delta_{pp})_{rs} + \big(\mathcal{Z}(\delta_{pp})\mathcal{Z}\big)_{kj}\big(\overline{\mathcal{Z}}(\delta_{pp})\overline{\mathcal{Z}}\big)_{rs}\big] \\ &\quad - \big[(\delta_{pp})\mathcal{Z} + \mathcal{Z}(\delta_{pp})\big]_{kj}\big[(\delta_{pp})\overline{\mathcal{Z}} + \overline{\mathcal{Z}}(\delta_{pp})\big]_{rs} \\ &= 2(\delta_{pp})_{kj}(\delta_{pp})_{rs} + 2\mathcal{Z}_{kp}\mathcal{Z}_{jp}\overline{\mathcal{Z}}_{rp}\overline{\mathcal{Z}}_{sp} - \alpha(p;k,j,r,s)\mathcal{Z}_{kj}\overline{\mathcal{Z}}_{rs} \end{aligned} \tag{11.5}$$

with $\alpha(p;k,j,r,s) = 1_{p=k=r} + 1_{p=k=s} + 1_{p=j=r} + 1_{p=j=s}$. We put $1_{p=k=s} = 1$ if $p = k = s$, otherwise $1_{p=k=s} = 0$.

3. *Vectors in the Lie algebra as in $\mathcal{E}_3$, $\mathcal{E}_4$, see* (6.11)–(6.12). Denote

$$(\widehat{\delta_q})(p) = (\delta_{pq}) + (\delta_{qp}) \quad \text{with } p \neq q.$$

Let $\rho_q(p)$, $\widetilde{\rho_q}(p)$ be the vector fields of the representation and associated to $(\widehat{\delta_q}(p))$ as in $\mathcal{E}_3$, $\mathcal{E}_4$. Then

$$\rho_q=\sum_{k\leqslant j}\big[(\widehat{\delta_q})-\mathcal{Z}(\widehat{\delta_q})\mathcal{Z}\big]_{kj}\frac{\partial}{\partial\mathcal{Z}_{kj}}+\gamma\,\mathrm{trace}\big[(\widehat{\delta_q})\mathcal{Z}\big],$$

$$\widetilde{\rho_q}=i\Bigg(\sum_{k\leqslant j}\big[(\widehat{\delta_q})+\mathcal{Z}(\widehat{\delta_q})\mathcal{Z}\big]_{kj}\frac{\partial}{\partial\mathcal{Z}_{kj}}-\gamma\,\mathrm{trace}\big[(\widehat{\delta_q})\mathcal{Z}\big]\Bigg). \tag{11.6}$$

The coefficient of $\frac{\partial^2}{\partial\mathcal{Z}_{kj}\partial\overline{\mathcal{Z}_{rs}}}$ in $\rho_q\overline{\rho_q}+\widetilde{\rho_q}\overline{\widetilde{\rho_q}}$ is

$$\begin{aligned}h_2(q)&=2\big[(\widehat{\delta_q})_{kj}(\widehat{\delta_q})_{rs}+\big(\mathcal{Z}(\widehat{\delta_q})\mathcal{Z}\big)_{kj}\big(\overline{\mathcal{Z}}(\widehat{\delta_q})\overline{\mathcal{Z}}\big)_{rs}\big]\\&=2\big[(\widehat{\delta_q})_{kj}(\widehat{\delta_q})_{rs}+\mathcal{Z}_{kp}\overline{\mathcal{Z}}_{rp}\mathcal{Z}_{jq}\overline{\mathcal{Z}}_{sq}+\mathcal{Z}_{kp}\overline{\mathcal{Z}}_{sp}\mathcal{Z}_{jq}\overline{\mathcal{Z}}_{rq}\\&\quad+\mathcal{Z}_{jp}\overline{\mathcal{Z}}_{rp}\mathcal{Z}_{kq}\overline{\mathcal{Z}}_{sq}+\mathcal{Z}_{jp}\overline{\mathcal{Z}}_{sp}\mathcal{Z}_{kq}\overline{\mathcal{Z}}_{rq}\big].\end{aligned} \tag{11.7}$$

4. *Vectors in the Lie algebra as in $\mathcal{F}_6$, $\mathcal{E}_7$, see* (6.14)–(6.15). We denote

$$(\delta_q)=(\delta_{pq})-(\delta_{qp})\quad\text{with } p<q.$$

Let $\rho_m(q)$, $\widetilde{\rho_m(q)}$ the vector fields of the representation as in $\mathcal{F}_6$, $\mathcal{F}_7$ associated to (δ_q) and $(\widehat{\delta_q})$. Then

$$\rho_m(q)=\sum_{k\leqslant j}\big[(\delta_q)\mathcal{Z}-\mathcal{Z}(\delta_q)\big]_{kj}\frac{\partial}{\partial\mathcal{Z}_{kj}},\qquad\widetilde{\rho_m}(q)=i\sum_{k\leqslant j}\big[(\widehat{\delta_q})\mathcal{Z}+\mathcal{Z}(\widehat{\delta_q})\big]_{kj}\frac{\partial}{\partial\mathcal{Z}_{kj}}.$$

The coefficient of $\frac{\partial^2}{\partial\mathcal{Z}_{kj}\partial\overline{\mathcal{Z}_{rs}}}$ in $\rho_m(q)\overline{\rho_m(q)}+\widetilde{\rho_m}(q)\overline{\widetilde{\rho_m}(q)}$ is given by summing

$$\begin{aligned}h_3(p,q)&=2(1_{p=k}\mathcal{Z}_{jq}+1_{p=j}\mathcal{Z}_{kq})(1_{p=r}\overline{\mathcal{Z}}_{sq}+1_{p=s}\overline{\mathcal{Z}_{rq}})\\&\quad+2(1_{q=k}\mathcal{Z}_{jp}+1_{q=j}\mathcal{Z}_{kp})(1_{q=r}\overline{\mathcal{Z}}_{sp}+1_{q=s}\overline{\mathcal{Z}_{rp}}).\end{aligned} \tag{11.8}$$

We extend Theorem 9.1 to the case of $Sp(2n)$ acting on $\mathcal{D}_n$ for arbitrary n.

Theorem 11.1. *We take the basis as in Definition 6.3 and we assume that $\gamma=0$, then*

$$2\Bigg[\sum_{v\in\mathcal{E}_1\cup\mathcal{E}_2}\rho(v)\overline{\rho(v)}-\sum_{v\in\mathcal{F}_5}\rho(v)\overline{\rho(v)}\Bigg]+\sum_{v\in\mathcal{E}_3\cup\mathcal{E}_4}\rho(v)\overline{\rho(v)}-\sum_{v\in\mathcal{E}_6\cup\mathcal{E}_7}\rho(v)\overline{\rho(v)} \tag{11.9}$$

is equal to the Laplacian on $\mathcal{D}_n$. The coefficients of the Laplacian on $\mathcal{D}_n$ are given by Corollary 10.4.

Proof. It is a consequence of (11.5)–(11.7)–(11.8). If we do the proof on 3×3 matrices, we use the identities

$$\mathcal{Z}=\begin{pmatrix}z_1&z_2&z_4\\z_2&z_3&z_5\\z_4&z_5&z_6\end{pmatrix},\quad\text{then }\mathcal{Z}\delta_{11}\mathcal{Z}=\begin{pmatrix}z_1^2&z_1z_2&z_1z_4\\z_1z_2&z_2^2&z_2z_4\\z_1z_4&z_2z_4&z_4^2\end{pmatrix},$$

$$\mathcal{Z}\delta_{22}\mathcal{Z}=\begin{pmatrix}z_2^2&z_3z_2&z_5z_2\\z_3z_2&z_3^2&z_5z_3\\z_2z_5&z_3z_5&z_5^2\end{pmatrix},\qquad\mathcal{Z}\delta_{33}\mathcal{Z}=\begin{pmatrix}z_4^2&z_5z_4&z_6z_4\\z_4z_5&z_5^2&z_6z_5\\z_6z_4&z_6z_5&z_6^2\end{pmatrix}, \tag{i}$$

$$\delta_{11}\mathcal{Z}+\mathcal{Z}\delta_{11}=\begin{pmatrix}2z_1&z_2&z_4\\z_2&0&0\\z_4&0&0\end{pmatrix},\qquad\delta_{22}\mathcal{Z}+\mathcal{Z}\delta_{22}=\begin{pmatrix}0&z_2&0\\z_2&2z_3&z_5\\0&z_5&0\end{pmatrix},$$

$$\delta_{33}\mathcal{Z}+\mathcal{Z}\delta_{33}=\begin{pmatrix}0&0&z_4\\0&0&z_5\\z_4&z_5&2z_6\end{pmatrix}, \tag{ii}$$

$$\mathcal{Z}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\mathcal{Z}=\begin{pmatrix}2z_1z_2&z_2^2+z_1z_3&z_2z_4+z_1z_5\\z_1z_3+z_2^2&2z_2z_3&z_3z_4+z_2z_5\\z_2z_4+z_1z_5&z_3z_4+z_2z_5&2z_4z_5\end{pmatrix}, \tag{iii}$$

$$\mathcal{Z}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\mathcal{Z}=\begin{pmatrix}2z_1z_4&z_4z_2+z_1z_5&z_4^2+z_1z_6\\z_1z_5+z_2z_4&2z_2z_5&z_5z_4+z_2z_6\\z_1z_6+z_4^2&z_2z_6+z_4z_5&2z_4z_6\end{pmatrix}, \tag{iv}$$

$$\mathcal{Z}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\mathcal{Z}=\begin{pmatrix}2z_2z_4&z_4z_3+z_2z_5&z_4z_5+z_2z_6\\z_2z_5+z_3z_4&2z_3z_5&z_5^2+z_3z_6\\z_2z_6+z_4z_5&z_3z_6+z_5^2&2z_5z_6\end{pmatrix}, \tag{v}$$

$$\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\mathcal{Z}-\mathcal{Z}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}2z_2&z_3-z_1&z_5\\z_3-z_1&-2z_2&-z_4\\z_5&-z_4&0\end{pmatrix}, \tag{vi}$$

$$\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\mathcal{Z}-\mathcal{Z}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}=\begin{pmatrix}2z_4&z_5&z_6-z_1\\z_5&0&-z_2\\z_6-z_1&-z_2&-2z_4\end{pmatrix}, \tag{vii}$$

$$\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\mathcal{Z}-\mathcal{Z}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}=\begin{pmatrix}0&z_4&-z_2\\z_4&2z_5&z_6-z_3\\-z_2&z_6-z_3&-2z_5\end{pmatrix} \tag{viii}$$

and for $\widetilde{\rho}_m$ in (11.8), we need

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\mathcal{Z}+\mathcal{Z}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}2z_2&z_3+z_1&z_5\\z_3+z_1&2z_2&z_4\\z_5&z_4&0\end{pmatrix}, \tag{ix}$$

$$\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\mathcal{Z}+\mathcal{Z}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}=\begin{pmatrix}2z_4&z_5&z_6+z_1\\z_5&0&z_2\\z_6+z_1&z_2&2z_4\end{pmatrix}, \tag{x}$$

$$\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\mathcal{Z}+\mathcal{Z}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}=\begin{pmatrix}0&z_4&z_2\\z_4&2z_5&z_6+z_3\\z_2&z_6+z_3&2z_5\end{pmatrix}. \tag{xi}$$

This ends the proof. □

Similarly we extend Theorem 9.2 to the O–U operator on $\mathcal{D}_n$.

Theorem 11.2. *As in Theorem 9.2, let $\rho = L + \gamma I$.*

$$2\Big[\sum_{v\in\mathcal{E}_1\cup\mathcal{E}_2}\rho(v)\overline{L(v)}-\sum_{v\in\mathcal{F}_5}\rho(v)\overline{L(v)}\Big]+\sum_{v\in\mathcal{E}_3\cup\mathcal{E}_4}\rho(v)\overline{L(v)} - \sum_{v\in\mathcal{E}_6\cup\mathcal{E}_7}\rho(v)\overline{L(v)} \tag{11.10}$$

is equal to the O–U operator on $\mathcal{D}_n$, see (10.18).

Proof. By (11.1)–(11.4), we have

$$U_1 = \sum_{v\in\mathcal{E}_1\cup\mathcal{E}_2} \phi(v)\overline{L(v)} = -2\sum \mathcal{Z}_{pp}\big[\overline{\mathcal{Z}}(\delta_{pp})\overline{\mathcal{Z}}\big]_{kj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}} = -2\sum \mathcal{Z}_{pp}\overline{\mathcal{Z}}_{kp}\overline{\mathcal{Z}}_{pj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}},$$

$$U_4 = \sum_{v\in\mathcal{F}_5} \phi(v)\overline{L(v)} = -\sum\big((\delta_{pp})\overline{\mathcal{Z}} + \overline{\mathcal{Z}}(\delta_{pp})\big)_{kj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}} = -2\sum \overline{\mathcal{Z}}_{kj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}}.$$

With (11.7), we obtain

$$\begin{aligned}U_2 &= \sum_{v\in\mathcal{E}_3\cup\mathcal{E}_4} \phi(v)\overline{L(v)}\\ &= -2\,\mathrm{trace}\big[(\widehat{\delta_q})\mathcal{Z}\big]\times\big(\overline{\mathcal{Z}}(\widehat{\delta_q})\overline{\mathcal{Z}}\big)_{kj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}} = -4\sum_{p<q}\mathcal{Z}_{pq}(\overline{\mathcal{Z}}_{kp}\overline{\mathcal{Z}}_{qj} + \overline{\mathcal{Z}}_{kq}\overline{\mathcal{Z}}_{pj})\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}},\end{aligned}$$

$$2U_1 + U_2 = -4(\mathcal{Z}\overline{\mathcal{Z}})_{pk}\overline{\mathcal{Z}}_{pj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}}$$

and

$$2U_1 + U_2 - 2U_4 = 4(I - \mathcal{Z}\overline{\mathcal{Z}})_{pk}\overline{\mathcal{Z}}_{pj}\frac{\partial}{\partial\overline{\mathcal{Z}}_{kj}}.$$

We identify $U = 2U_1 - 2U_4 + U_2$ with the first order part in (10.18). □

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