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THE INTERIOR LAYER FOR A NONLINEAR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATION*

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Abstract In this article, the interior layer for a second order nonlinear singularly perturbed differential-difference equation is considered. Using the methods of boundary function and fractional steps, we construct the formula of asymptotic expansion and point out that the boundary layer at $t = 0$ has a great influence upon the interior layer at $t = \sigma$. At the same time, on the basis of differential inequality techniques, the existence of the smooth solution and the uniform validity of the asymptotic expansion are proved. Finally, an example is given to demonstrate the effectiveness of our result. The result of this article is new and it complements the previously known ones.

Key words Differential-difference equation; interior layer; asymptotic expansion; boundary function

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1 Introduction

The boundary-value problems for singulary perturbed differential-difference equations are often used as mathematical models describing processes in biomechanics and physics [1, 2]. In recent years, more and more attention was paid to the study of singularly perturbed differential-difference problems, especially for linear problems [3–11]. For nonlinear problems, now, we also have a few results [12–15]. Most of these works are related to boundary layer, numerical solution, or the existence of the solution, while few of them concern interior layer and the uniform validity of the asymptotic expansion [16]. In this article, we will discuss the interior layer for a nonlinear singularly perturbed differential-difference equation and construct its asymptotic

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expansion formula. Meanwhile, the existence of the smooth interior layer solution and the uniform validity of the asymptotic expansion will be proved.

In articles [9, 10], a mathematical model about neural network was presented:

$$\frac{\sigma^2}{2}y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E - \lambda_I)y(x) = -1,$$

where the values $x = x_1$ and $x = x_2$ corresponds to the inhibitory reversal potential and threshold value of membrane potential for action potential generation, respectively. σ and μ are variance and drift parameters, respectively, y is the expected first-exit time and the first order derivative term $-xy'$ corresponds to the exponential decay between synaptic inputs. The undifferentiated terms corresponds to excitatory and inhibitory synaptic inputs, modeled as Poisson process with mean rates λ_E and λ_I, respectively, and produce jumps in the membrane potential of amounts a_E and a_I, which are small quantities and could depend on voltage.

Considering the complexity of the neural network and the small parameter in front of $y'(x)$ in this model, we propose the following weak-nonlinear differential-difference problem.

2 Statement of the Problem

We consider a boundary-value problem for a weak nonlinear singulary perturbed differential-difference equation, which only contains negative shift,

$$\begin{cases} \mu^2 y''(t) = F(\mu y'(t), y(t), y(t-\sigma), t), \quad 0 \le t \le T; \\ y(t,\mu) = \alpha(t), -\sigma \le t \le 0, \quad y(T,\mu) = y^T, \end{cases} \tag{1}$$

where $0 < \mu \ll 1$ is a small parameter and σ is a delay argument. $\alpha(t)$ defined in $[-\sigma, 0]$ is a smooth function. T is a positive constant satisfying $\frac{4}{3}\sigma \le T \le 2\sigma$. The restriction on T will not influence the essence of the problem and it is only convenient for our discussion.

First of all, we use the transformation $\mu y' = z$, which results in the system

$$\begin{cases} \mu y'(t) = z(t), \\ \mu z'(t) = F(z(t), y(t), y(t-\sigma), t); \end{cases} \tag{2}$$

$$y(t,\mu) = \alpha(t), -\sigma \le t \le 0, \quad y(T,\mu) = y^T, 0 < \mu \ll 1. \tag{3}$$

This article is organized as follows. In Section 3, we construct the asymptotic expansion in the interval $[0, \sigma]$. In Section 4, we construct the asymptotic expansion in the interval $[\sigma, T]$. In Section 5, The existence of the smooth interior layer solution and the uniform validity of the asymptotic expansion are obtained. Finally, an example is given to illustrate the effectiveness of our results.

First, we impose two conditions on equation (2). When additional hypotheses are required, they will be stated.

H_1 Suppose that $F(z, y, u, t)$ is sufficiently smooth with respect to each argument and, for $0 \le t \le T$, the reduced equation $F(0, \overline{y}(t), \overline{y}(t-\sigma), t) = 0$ has an isolate root $\overline{y}(t) = \varphi(t)$, $0 \le t \le \sigma$, $\overline{y}(t) = \psi(t), \sigma \le t \le T$ (see Fig.1), where $u = y(t-\sigma)$.

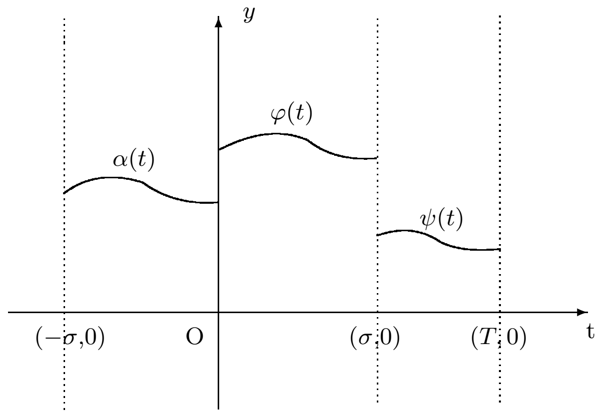

Fig.1

H_2 Suppose that $F_y(0,\overline{y}(t),\overline{y}(t-\sigma),t)>0$, $F_u(0,\overline{y}(t),\overline{y}(t-\sigma),t)\leq 0$, $t\in[0,T]$, where u has the same definition as above.

3 Construction of the Asymptotic Expansion in $[0,\sigma]$

Letting $x=(y,z)^T$ and using the method of boundary function [17], we construct a series formally satisfying (2),(3) in $[0,\sigma]$:

$$x^{(-)}(t,\mu)=\overline{x}(t,\mu)+\Pi x(\tau_0,\mu)+Q^{(-)}x(\tau,\mu),\quad \tau_0=\frac{t}{\mu},\ \tau=\frac{t-\sigma}{\mu}, \tag{4}$$

where

$$\overline{x}(t,\mu)=\overline{x}_0(t)+\mu\overline{x}_1(t)+\cdots+\mu^k\overline{x}_k(t)+\cdots \tag{5}$$

is called the regular series of (4), while

$$\Pi x(\tau_0,\mu)=\Pi_0x(\tau_0)+\mu\Pi_1x(\tau_0)+\cdots+\mu^k\Pi_kx(\tau_0)+\cdots \tag{6}$$

is called the boundary series for $t=0$,

$$Q^{(-)}x(\tau,\mu)=Q_0^{(-)}x(\tau)+\mu Q_1^{(-)}x(\tau)+\cdots+\mu^kQ_k^{(-)}x(\tau)+\cdots \tag{7}$$

is called the left boundary series for $t=\sigma$. $\Pi_kx(\tau_0)$, $Q_k^{(-)}x(\tau)$ are called boundary functions, and $\lim\limits_{\tau_0\to+\infty}\Pi_kx(\tau_0)=0$, $\lim\limits_{\tau\to-\infty}Q_k^{(-)}x(\tau)=0$ hold.

Specially, we assume that

$$y(\sigma,\mu)=p(\mu)=p_0+\mu p_1+\mu^2p_2+\cdots+\mu^kp_k+\cdots,$$

where $p_k, k=0,1,\cdots$, are unknown constants and they are determined by the smooth connection at $t=\sigma$. By the boundary function method, we obtain

$$\begin{cases}\mu\dfrac{\mathrm{d}\overline{y}}{\mathrm{d}t}=\overline{z}(t,\mu),\\[2mm] \mu\dfrac{\mathrm{d}\overline{z}}{\mathrm{d}t}=F(\overline{z}(t,\mu),\overline{y}(t,\mu),\overline{y}(t-\sigma,\mu),t);\end{cases} \tag{8}$$

$$\begin{cases} \dfrac{\mathrm{d}\Pi y}{\mathrm{d}\tau_0} = \Pi z(\tau_0, \mu), \\ \dfrac{\mathrm{d}\Pi z}{\mathrm{d}\tau_0} = F(\overline{z}(\mu\tau_0, \mu) + \Pi z(\tau_0, \mu), \overline{y}(\mu\tau_0, \mu) + \Pi y(\tau_0, \mu), \overline{y}(\mu\tau_0 - \sigma, \mu), \mu\tau_0) \\ \qquad -F(\overline{z}(\mu\tau_0, \mu), \overline{y}(\mu\tau_0, \mu), \overline{y}(\mu\tau_0 - \sigma, \mu), \mu\tau_0); \end{cases} \tag{9}$$

$$\begin{cases} \dfrac{\mathrm{d}Q^{(-)}y}{\mathrm{d}\tau} = Q^{(-)}z(\tau, \mu), \\ \dfrac{\mathrm{d}Q^{(-)}z}{\mathrm{d}\tau} = F(\overline{z}(\sigma + \mu\tau, \mu) + Q^{(-)}z(\tau, \mu), \overline{y}(\sigma + \mu\tau, \mu) + Q^{(-)}y(\tau, \mu), \\ \qquad \overline{y}(\mu\tau, \mu), \sigma + \mu\tau) - F(\overline{z}(\sigma + \mu\tau, \mu), \overline{y}(\sigma + \mu\tau, \mu), \overline{y}(\mu\tau, \mu), \sigma + \mu\tau), \end{cases} \tag{10}$$

Substituting (5)–(7) into (8)–(10) and equating terms with same powers of μ for $\overline{x}_k(t)$, we obtain

$$\overline{z}_0(t) = 0, \quad F(\overline{z}_0(t), \overline{y}_0(t), \alpha(t-\sigma), t) = 0; \tag{11}$$

$$\frac{\mathrm{d}\overline{y}_{k-1}}{\mathrm{d}t} = \overline{z}_k(t), \qquad \frac{\mathrm{d}\overline{z}_{k-1}}{\mathrm{d}t} = \overline{F}_z \overline{z}_k(t) + \overline{F}_y \overline{y}_k(t) + \overline{h}_k(t); \tag{12}$$

where $\overline{F}_z, \overline{F}_y$ take their values at $(0, \overline{y}_0(t), \alpha(t-\sigma), t)$ and $\overline{h}_k(t)$ are determined functions. (11) coincides with the reduced equation of (2), so, we have $\overline{y}_0(t) = \varphi(t), \overline{z}_0(t) = 0$. By H_2 and (12), $\overline{x}_k(t)$ $(k \geq 1)$ can be obtained completely.

For $\Pi_0 x(\tau_0)$, we have

$$\frac{\mathrm{d}\Pi_0 y}{\mathrm{d}\tau_0} = \Pi_0 z, \quad \frac{\mathrm{d}\Pi_0 z}{\mathrm{d}\tau_0} = F(\Pi_0 z, \varphi(0) + \Pi_0 y, \alpha(-\sigma), 0); \tag{13}$$

$$\Pi_0 y(0) = \alpha(0) - \varphi(0), \ \Pi_0 y(+\infty) = 0. \tag{14}$$

Let $\varphi(0) + \Pi_0 y = \widetilde{y}, \ \widetilde{y}' = \widetilde{z}$. Then, the problem (13)–(14) can be transformed into the problem as follows:

$$\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\tau_0} = \widetilde{z}, \quad \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}\tau_0} = F(\widetilde{z}, \widetilde{y}, \alpha(-\sigma), 0); \tag{15}$$

$$\widetilde{y}(0) = \alpha(0), \ \widetilde{y}(+\infty) = \varphi(0). \tag{16}$$

Because the eigenvalues of (15) at the point $(\varphi(0), 0)$ are

$$\lambda_{1,2} = \frac{F_{\widetilde{z}} \pm \sqrt{F_{\widetilde{z}}^2 + 4F_{\widetilde{y}}}}{2}\bigg|_{(\varphi(0),0)},$$

by H_2, the equilibrium $M_1(\varphi(0), 0)$ is a saddle point on the phase plane $(\widetilde{y}, \widetilde{z})$. Thus, passing through M_1, there exists a steady manifold $\Sigma_0 : \widetilde{z} = \Phi_0(\widetilde{y})$.

H_3 Suppose that the line $\widetilde{y}(0) = \alpha(0)$ intersects with the manifold $\Sigma_0 : \widetilde{z} = \Phi_0(\widetilde{y})$.

Lemma 1 Under conditions $H_1 - H_3$, the following inequalities holds:

$$C_{1_0} e^{-\overline{k}_0 \tau_0} \leq \Pi_0 x(\tau_0) \leq C_{2_0} e^{-\underline{k}_0 \tau_0}, \quad \tau_0 \geq 0,$$

where C_{1_0}, C_{2_0}, $\overline{k}_0$, and $\underline{k}_0$ are all positive constants.

Proof By H_3, in the neighborhood of the saddle point $M_1(\varphi(0), 0)$, there exists a manifold $\widetilde{z} = \widetilde{z}(\widetilde{y})$ passing through $M_1(\varphi(0), 0)$. We expand it into Taylor series at the point $\widetilde{y} = \varphi(0)$

(suppose that $\widetilde{y} < \varphi(0)$), $\widetilde{z} = \widetilde{z}(\varphi(0)) + \frac{\mathrm{d}\widetilde{z}}{\mathrm{d}\widetilde{y}}(\varphi(0))(\widetilde{y} - \varphi(0)) + \mathrm{o}(\widetilde{y} - \varphi(0))$. In contrast, by (15), (16), we obtain $\frac{\mathrm{d}\widetilde{z}}{\mathrm{d}\widetilde{y}} = \frac{F(\widetilde{z},\widetilde{y},\alpha(-\sigma),0)}{\widetilde{z}}$. According to the L'Hospital rule, we obtain

$$\frac{\mathrm{d}\widetilde{z}}{\mathrm{d}\widetilde{y}}(\varphi(0)) = \frac{F_{\widetilde{z}} - \sqrt{F_{\widetilde{z}}^2 + 4F_{\widetilde{y}}}}{2}\Big|_{\widetilde{y}=\varphi(0)} = \lambda_1 < 0.$$

Because $\widetilde{z}(\varphi(0)) = 0$, we have $\widetilde{z} = \lambda_1(\widetilde{y} - \varphi(0)) + \mathrm{o}(\widetilde{y} - \varphi(0))$. Thus, there exists τ^*, positive constants $\underline{k}_0$ and $\overline{k}_0$, such that, for τ_0 large enough and $\tau_0 > \tau^*$, the inequality $-\underline{k}_0(\widetilde{y} - \varphi(0)) \leq \widetilde{z} \leq -\overline{k}_0(\widetilde{y} - \varphi(0))$ hold. Integrating the above inequality with respect to τ_0 from τ^* to τ_0, by Gronwall inequality, we obtain

$$C_{1_0} e^{-\overline{k}_0\tau_0} \leq \Pi_0 y(\tau_0) \leq C_{2_0} e^{-\underline{k}_0\tau_0}, \quad \tau_0 \geq 0, \tag{17}$$

where $C_{1_0} = \Pi_0 y(\tau^*)e^{\overline{k}_0(\tau^*)}$, $C_{2_0} = \Pi_0 y(\tau^*)e^{\underline{k}_0(\tau^*)}$. The estimate of $\Pi_0 z(\tau_0)$ can be obtained if we differentiate (17) with respect to τ_0. The proof of Lemma 1 is completed.

Here, the constants may be different from C_{1_0}, C_{2_0}, but they have no essential impact on the problem. So, we still denote them by C_{1_0}, C_{2_0}. In the following, we will deal with the constants which we meet in the same way.

For $\Pi_k x(\tau_0)$, we have

$$\frac{\mathrm{d}\Pi_k y}{\mathrm{d}\tau_0} = \Pi_k z, \quad \frac{\mathrm{d}\Pi_k z}{\mathrm{d}\tau_0} = \widetilde{F}_z\Pi_k z + \widetilde{F}_y\Pi_k y + G_k(\tau_0); \tag{18}$$

$$\Pi_k y(0) = -\overline{y}_k(0),\ \Pi_k y(+\infty) = 0, \tag{19}$$

where $\widetilde{F}_z, \widetilde{F}_y$ take their values at the point $(\Pi_0 z, \varphi(0) + \Pi_0 y, \alpha(-\sigma), 0)$. $G_k(\tau)$ are functions compound formed by $\overline{x}_i(t)$ and $\Pi_i x(\tau_0) (i = 0, 1, \cdots, k-1)$.

In fact, $\frac{d^2\Pi_k y}{\mathrm{d}\tau_0^2} = \widetilde{F}_z\frac{\mathrm{d}\Pi_k y}{\mathrm{d}\tau_0} + \widetilde{F}_y\Pi_k y$ has a particular solution $\widetilde{z}(\tau_0) = \frac{\mathrm{d}\Pi_0 y}{\mathrm{d}\tau_0}$. According to Liouville formulas and constant-change method, we obtain

$$\Pi_k y = \frac{\widetilde{z}(\tau_0)}{\widetilde{z}(0)}(-\overline{y}_k(0)) + \widetilde{z}(\tau_0)\int_0^{\tau_0}\frac{1}{\widetilde{z}^2(\eta)p(\eta)}\int_{+\infty}^{\eta}\widetilde{z}(s)p(s)G_k(s)\mathrm{d}s d\eta. \tag{20}$$

where $p(\tau_0) = \exp(-\int_0^{\tau_0}\widetilde{F}_z\mathrm{d}\tau_0)$. Thus, $\Pi_k x(\tau_0)$ is completely determined. The exponential decay of $\Pi_k x(\tau_0)$ can be obtained from (20).

Lemma 2 Under conditions $H_1 - H_3$, the following inequalities

$$C_{1_k} e^{-\overline{k}_k\tau_0} \leq \Pi_k x(\tau_0) \leq C_{2_k} e^{-\underline{k}_k\tau_0}, \quad \tau_0 \geq 0,$$

are valid, where C_{1_k}, C_{2_k}, $\overline{k}_k$, and $\underline{k}_k$ are all positive constants.

$Q_0^{(-)}x(\tau)$ is determined by the following system:

$$\frac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = Q_0^{(-)}z, \quad \frac{\mathrm{d}Q_0^{(-)}z}{\mathrm{d}\tau} = F(Q_0^{(-)}z, \varphi(\sigma) + Q_0^{(-)}y, \alpha(0), \sigma); \tag{21}$$

$$Q_0^{(-)}y(0) = p_0 - \varphi(\sigma),\ Q_0^{(-)}y(-\infty) = 0. \tag{22}$$

Let $\varphi(\sigma) + Q_0^{(-)}y(\tau) = y^l$, $Q_0^{(-)}z(\tau) = z^l$, then, (21) and (22) can be written as

$$\frac{\mathrm{d}y^l}{\mathrm{d}\tau} = z^l, \quad \frac{\mathrm{d}z^l}{\mathrm{d}\tau} = F(z^l, y^l, \alpha(0), \sigma); \tag{23}$$

$$y^l(0) = p_0,\ y^l(-\infty) = \varphi(\sigma). \tag{24}$$

By H_2, the equilibrium $(\varphi(\sigma), 0)$ is a saddle point on the phase plane (y^l, z^l), so when passing through $(\varphi(\sigma), 0)$, there exists a steady manifold $\Sigma_l : z^l = \Phi_l(y^l)$.

H_4 Suppose that the line $\widetilde{y}^l(0) = p_0$ intersects with the manifold Σ_l.

Lemma 3 Under conditions H_1, H_2, and H_4, the following inequalities hold,

$$C_{1_0} e^{\overline{k}_0 \tau} \le Q_0^{(-)} x(\tau) \le C_{2_0} e^{\underline{k}_0 \tau}, \quad \tau \le 0,$$

where C_{1_0}, C_{2_0}, $\overline{k}_0$, and $\underline{k}_0$ are all positive constants.

$Q_k^{(-)} x(\tau)$ is determined by the following system:

$$\frac{\mathrm{d}Q_k^{(-)} y}{\mathrm{d}\tau} = Q_k^{(-)} z, \quad \frac{\mathrm{d}Q_k^{(-)} z}{\mathrm{d}\tau} = \widetilde{F}_z^{(-)} Q_k^{(-)} z + \widetilde{F}_y^{(-)} Q_k^{(-)} y + H_k^{(-)}(\tau); \tag{25}$$

$$Q_k^{(-)} y(0) = P_k - \overline{y}_k(\sigma),\ Q_k^{(-)} y(-\infty) = 0, \tag{26}$$

where $\widetilde{F}_z^{(-)}, \widetilde{F}_y^{(-)}$ take their values at $(Q_0^{(-)} z, \varphi(\sigma) + Q_0^{(-)} y, \alpha(0), \sigma)$. $H_k^{(-)}(\tau)$ is a known function.

In the following, we will seek the solution of equations (25)–(26). In fact, the homogeneous system, corresponding to (25),

$$\frac{\mathrm{d}Q_k^{(-)} y}{\mathrm{d}\tau} = Q_k^{(-)} z, \quad \frac{\mathrm{d}Q_k^{(-)} z}{\mathrm{d}\tau} = \widetilde{F}_z^{(-)} Q_k^{(-)} z + \widetilde{F}_y^{(-)} Q_k^{(-)} y \tag{27}$$

is the variational equation of (21). Thus, it has a steady manifold $Q_k^{(-)} z = \frac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l} Q_k^{(-)} y$. Combining this manifold with $\frac{\mathrm{d}Q_k^{(-)} y}{\mathrm{d}\tau} = Q_k^{(-)} z$, we obtain $\frac{\mathrm{d}Q_k^{(-)} y}{\mathrm{d}\tau} = \frac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l} Q_k^{(-)} y$. Now, letting the general solution of $\frac{\mathrm{d}Q_k^{(-)} y}{\mathrm{d}\tau} = \frac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l} Q_k^{(-)} y$ be $Q_k^{(-)} y = C\Phi_1(\tau)$, under the boundary condition (26), we obtain a solution of (27):

$$\begin{cases} (Q_k^{(-)} y(\tau))^G = (p_k - \overline{y}_k(\sigma))\Phi_1(\tau)\Phi_1^{-1}(0), \\ (Q_k^{(-)} z(\tau))^G = \dfrac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l}(p_k - \overline{y}_k(\sigma))\Phi_1(\tau)\Phi_1^{-1}(0). \end{cases} \tag{28}$$

Next, Set $Q_k^{(-)} y^*$, $Q_k^{(-)} z^*$ be a particular solution of (25). Introducing a new transformation $Q_k^{(-)} y^* = \delta_1, \quad Q_k^{(-)} z^* = \frac{\mathrm{d}\Phi_l(y^l)}{\partial y^l} Q_k^{(-)} y^* + \delta_2$ and substituting them into (25), we obtain

$$\begin{cases} \dfrac{\mathrm{d}\delta_1}{\mathrm{d}\tau} = \dfrac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l}\delta_1 + \delta_2, \\ \dfrac{\mathrm{d}\delta_2}{\mathrm{d}\tau} = \left(\widetilde{F}_z^{(-)} - \dfrac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l}\right)\delta_2 + H_k^{(-)}(\tau). \end{cases}$$

Let $\delta_2 = C\Psi_1(\tau)$ be the general solution of $\frac{\mathrm{d}\delta_2}{\mathrm{d}\tau} = (\widetilde{F}_z^{(-)} - \frac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l})\delta_2$, then, we obtain a particular solution $\delta_2 = \int_{-\infty}^{\tau} \Psi_1(\tau)\Psi_1^{-1}(s) H_k^{(-)}(s)\mathrm{d}s$ of $\frac{\mathrm{d}\delta_2}{\mathrm{d}\tau} = (\widetilde{F}_z^{(-)} - \frac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l})\delta_2 + H_k^{(-)}(\tau)$. Further more, we have

$$\delta_1 = \int_{\tau}^{0} \Phi_1(\tau)\Phi_1^{-1}(s)\left[\int_{-\infty}^{s} \Psi_1(s)\Psi_1^{-1}(p) H_k^{(-)}(p)\mathrm{d}p\right]\mathrm{d}s.$$

So, a particular solution of (25) is given in the following form:

$$\begin{cases} Q_k^{(-)}y^*(\tau) = \displaystyle\int_\tau^0 \Phi_1(\tau)\Phi_1^{-1}(s)\Big[\int_{-\infty}^s \Psi_1(s)\Psi_1^{-1}(p)H_k^{(-)}(p)\mathrm{d}p\Big]\mathrm{d}s, \\ Q_k^{(-)}z^*(\tau) = \dfrac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l}\cdot Q_k^{(-)}y^*(\tau) + \displaystyle\int_{-\infty}^\tau \Psi_1(\tau)\Psi_1^{-1}(s)H_k^{(-)}(s)\mathrm{d}s, \end{cases}$$

Thus, we obtain

$$\begin{cases} Q_k^{(-)}y(\tau) = (p_k - \overline{y}_k(\sigma))\Phi_1(\tau)\Phi_1^{-1}(0) + Q_k^{(-)}y^*(\tau), \\ Q_k^{(-)}z(\tau) = \dfrac{\mathrm{d}\Phi_l(y^l)}{\mathrm{d}y^l}(p_k - \overline{y}_k(\sigma))\Phi_1(\tau)\Phi_1^{-1}(0) + Q_k^{(-)}z^*(\tau). \end{cases} \tag{29}$$

Now, $Q_k^{(-)}x(\tau)$ is completely determined, but it contains the unknown number p_k. Obviously, $Q_k^{(-)}x(\tau)$ decays exponentially as $\tau \to -\infty$.

Lemma 4 Under conditions H_1, H_2, and H_4, the following inequalities hold,

$$C_{1_k}e^{\overline{k}_k\tau} \le Q_k^{(-)}x(\tau) \le C_{2_k}e^{\underline{k}_k\tau}, \quad \tau \le 0,$$

where C_{1_k}, C_{2_k}, $\overline{k}_k$, $\underline{k}_k$ are all positive constants.

4 Construction of the Asymptotic Expansion in $[\sigma, T]$

Using the method of boundary function, we construct a series formally satisfying (2), (3) in $[\sigma, T]$:

$$x^{(+)}(t,\mu) = \overline{\overline{x}}(t,\mu) + Q^{(+)}x(\tau,\mu) + Rx(\tau_T,\mu), \quad \tau = \frac{t-\sigma}{\mu},\ \tau_T = \frac{t-T}{\mu}, \tag{30}$$

where

$$\overline{\overline{x}}(t,\mu) = \overline{\overline{x}}_0(t) + \mu\overline{\overline{x}}_1(t) + \cdots + \mu^k\overline{\overline{x}}_k(t) + \cdots \tag{31}$$

is called the regular series of (30), while

$$Q^{(+)}x(\tau,\mu) = Q_0^{(+)}x(\tau) + \mu_1 Q_1^{(+)}x(\tau) + \cdots + \mu^k Q_k^{(+)}x(\tau) + \cdots \tag{32}$$

is called the right boundary series for $t = \sigma$.

$$Rx(\tau_T,\mu) = R_0x(\tau_T) + \mu R_1x(\tau_T) + \cdots + \mu^k R_kx(\tau_T) + \cdots \tag{33}$$

is called the boundary series for $t = T$. $R_kx(\tau_T)$, $Q_k^{(+)}x(\tau)$ are called boundary functions, and $\lim\limits_{\tau\to+\infty} Q_k^{(+)}x(\tau) = 0$, $\lim\limits_{\tau_T\to-\infty} R_kx(\tau_T) = 0$ hold.

Substituting (30)–(33) into (2), (3), separating t, τ, τ_T and equating terms with same powers of μ, for $\overline{\overline{x}}(t)$, we obtain

$$\overline{\overline{z}}_0(t) = 0, \quad F(\overline{\overline{z}}_0(t), \overline{\overline{y}}_0(t), \varphi(t-\sigma), t) = 0; \tag{34}$$

$$\frac{\mathrm{d}\overline{\overline{y}}_{k-1}}{\mathrm{d}t} = \overline{\overline{z}}_k(t), \quad \frac{\mathrm{d}\overline{\overline{z}}_{k-1}}{\mathrm{d}t} = \overline{\overline{F}}_z\overline{\overline{z}}_k(t) + \overline{\overline{F}}_y\overline{\overline{y}}_k(t) + \overline{\overline{h}}_k(t); \tag{35}$$

where $\overline{\overline{F}}_z, \overline{\overline{F}}_y$ take their values at $(0, \overline{\overline{y}}_0(t), \varphi(t-\sigma), t)$ and $\overline{\overline{h}}_k(t)$ are determined functions. (34) coincides with the reduced equation of (2), so, by H_1, we have $\overline{\overline{y}}_0(t) = \psi(t), \overline{\overline{z}}_0(t) = 0$. By H_2 and (35), $\overline{\overline{x}}_k(t)$ can be completely determined.

Due to the deviation of arguments, the equations determining $Q_0^{(+)}x(\tau)$ will be relevant to $\Pi_0 y(\tau_0)$. Namely,

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = Q_0^{(+)}z, \\ \dfrac{\mathrm{d}Q_0^{(+)}z}{\mathrm{d}\tau} = F(Q_0^{(+)}z, \psi(\sigma) + Q_0^{(+)}y, \varphi(0) + \Pi_0 y(\tau), \sigma); \end{cases} \tag{36}$$

$$Q_0^{(+)}y(0) = p_0 - \psi(\sigma),\ Q_0^{(+)}y(+\infty) = 0. \tag{37}$$

Let $\psi(\sigma) + Q_0^{(+)}y(\tau) = y^r,\ Q_0^{(+)}z(\tau) = z^r$, then, the above system can be written as

$$\frac{\mathrm{d}y^r}{\mathrm{d}\tau} = z^r, \quad \frac{\mathrm{d}z^r}{\mathrm{d}\tau} = F(z^r, y^r, \varphi(0) + \Pi_0 y(\tau), \sigma); \tag{38}$$

$$y^r(0) = p_0,\ y^r(+\infty) = \psi(\sigma). \tag{39}$$

Combining (15),(16) with (38),(39), we have

$$\begin{cases} \dfrac{\mathrm{d}y^r}{\mathrm{d}\tau} = z^r, \\ \dfrac{\mathrm{d}z^r}{\mathrm{d}\tau} = F(z^r, y^r, \widetilde{y}, \sigma); \\ \dfrac{\mathrm{d}\widetilde{y}}{\mathrm{d}\tau_0} = \widetilde{z}, \\ \dfrac{\mathrm{d}\widetilde{z}}{\mathrm{d}\tau_0} = F(\widetilde{z}, \widetilde{y}, \varphi(-\sigma), 0); \end{cases} \tag{40}$$

$$\widetilde{y}(0) = \alpha(0),\ \widetilde{y}(+\infty) = \varphi(0),\ y^r(0) = p_0,\ y^r(+\infty) = \psi(\sigma). \tag{41}$$

Here, the phase space $(y^r, z^r, \widetilde{y}, \widetilde{z})$ is the direct sum of (y^r, z^r) and $(\widetilde{y}, \widetilde{z})$. The equilibrium $\mathrm{M}(\psi(\sigma), 0, \varphi(0), 0)$ is a hyperbolic saddle point because the characteristic equation at $\mathrm{M}(\psi(\sigma), 0, \varphi(0), 0)$ is $[\lambda(\lambda - F_{z^r}) - F_{y^r}][\lambda(\lambda - F_{\widetilde{z}}) - F_{\widetilde{y}}] = 0$ and its eigenvalues satisfy $\lambda_1\lambda_2 = -F_{y^r} < 0$, $\lambda_3\lambda_4 = -F_{\widetilde{y}} < 0$. Thus, there exist a two-dimensional stable manifold $W^s(M)$ and a two-dimensional unstable manifold $W^u(M)$ of the system (40). Set $W^s(M) : Z = \overline{\Phi}(Y)$, where $Y = (y^r, \widetilde{y})^T, Z = (z^r, \widetilde{z})^T, \overline{\Phi} = (\Phi_0, \Phi_r)^T$. Obviously, the projection of $W^s(M)$ on the phase plane $(\widetilde{y}, \widetilde{z})$ is Σ_0. Namely, $(W^s(M))^{\perp}_{(\widetilde{y},\widetilde{z})} = \Sigma_0$. Set $(W^s(M))^{\perp}_{(y^r,z^r)} = \Sigma_r$, then,

$$z^r = \Phi_r(y^r, \widetilde{y}).$$

H_5 Suppose that the plane $y^r(0) = p_0$ intersects with the steady manifold Σ_r in the phase space.

Lemma 5 Under conditions H_1, H_2, and H_5, the following inequalities hold,

$$C_{1_0} e^{-\overline{k}_0 \tau} \leq Q_0^{(+)}x(\tau) \leq C_{2_0} e^{-\underline{k}_0 \tau}, \quad \tau \geq 0,$$

where C_{1_0}, C_{2_0}, $\overline{k}_0$, $\underline{k}_0$ are all positive constants.

$Q_k^{(+)}x(\tau)$ satisfies the following boundary value problem:

$$\frac{\mathrm{d}Q_k^{(+)}y}{\mathrm{d}\tau}=Q_k^{(+)}z,\quad \frac{\mathrm{d}Q_k^{(+)}z}{\mathrm{d}\tau}=\widetilde{F}_z^{(+)}Q_k^{(+)}z+\widetilde{F}_y^{(+)}Q_k^{(+)}y+H_k^{(+)}(\tau); \tag{42}$$

$$Q_k^{(+)}y(0)=P_k-\overline{\overline{y}}_k(\sigma),\ Q_k^{(+)}y(+\infty)=0, \tag{43}$$

where $\widetilde{F}_z^{(+)},\widetilde{F}_y^{(+)}$ take their values at $(Q_0^{(+)}z,\psi(\sigma)+Q_0^{(+)}y,\varphi(0)+\Pi_0y(\tau),\sigma)$. $H_k^{(+)}(\tau)$ are determined functions.

In a similar manner for solving $Q_k^{(-)}x(\tau)$, we obtain

$$\begin{cases} Q_k^{(+)}y(\tau)=(p_k-\overline{\overline{y}}_k(\sigma))\Phi_2(\tau)\Phi_2^{-1}(0)+Q_k^{(+)}y^*(\tau),\\ Q_k^{(+)}z(\tau)=\dfrac{\mathrm{d}\Phi_r(y^r)}{\mathrm{d}y^r}(p_k-\overline{\overline{y}}_k(\sigma))\Phi_2(\tau)\Phi_2^{-1}(0)+Q_k^{(+)}z^*(\tau),\end{cases} \tag{44}$$

where

$$Q_k^{(+)}y^*(\tau)=\int_0^\tau\Phi_2(\tau)\Phi_2^{-1}(s)\Big[\int_{+\infty}^s\Psi_2(s)\Psi_2^{-1}(p)H_k^{(+)}(p)\mathrm{d}p\Big]\mathrm{d}s$$

and

$$Q_k^{(+)}z^*(\tau)=\frac{\mathrm{d}\Phi_r(y^r)}{\mathrm{d}y^r}\cdot Q_k^{(+)}y^*(\tau)+\int_{+\infty}^\tau\Psi_2(\tau)\Psi_2^{-1}(s)H_k^{(+)}(s)\mathrm{d}s.$$

Thus, $Q_k^{(+)}x(\tau)$ can be completely determined.

In the same way, we have the following lemma.

Lemma 6 Under conditions H_1,H_2, and H_5, the following inequalities hold,

$$C_{1_k}e^{-\overline{k}_k\tau}\le Q_kx(\tau)\le C_{2_k}e^{-\underline{k}_k\tau},\quad \tau\ge 0,$$

where C_{1_k}, C_{2_k}, $\overline{k}_k$, $\underline{k}_k$ are all positive constants.

Now, $Q_k^{(\pm)}x(\tau)$ are all known, but they contain unknown numbers p_k which are determined by the smooth connection at $t=\sigma$:

$$\frac{\mathrm{d}y^{(-)}(\sigma,\mu)}{\mathrm{d}t}=\frac{\mathrm{d}y^{(+)}(\sigma,\mu)}{\mathrm{d}t}.$$

Namely,

$$\frac{d}{\mathrm{d}\tau}Q_0^{(-)}y(0)=\frac{d}{\mathrm{d}\tau}Q_0^{(+)}y(0),$$

$$\varphi'(\sigma)+\frac{d}{\mathrm{d}\tau}Q_1^{(-)}y(0)=\psi'(\sigma)+\frac{d}{\mathrm{d}\tau}Q_1^{(+)}y(0),$$

$$\vdots$$

$$\overline{y}'_{k-1}(\sigma)+\frac{d}{\mathrm{d}\tau}Q_k^{(-)}y(0)=\overline{\overline{y}}'_{k-1}(\sigma)+\frac{d}{\mathrm{d}\tau}Q_k^{(+)}y(0). \tag{45}$$

First, we will seek the value of p_0. By H_4,H_5, the solutions of systems (23), (24) and (38), (39) exist. Let

$$H(p_0)=z^l(0,p_0)-z^r(0,p_0)=\Phi_l(p_0)-\Phi_r(p_0,\varphi(0))=0. \tag{46}$$

H_6 Suppose that (46) has a solution $p_0=\overline{p}_0$ and $\frac{dH}{dp_0}\big|_{p_0=\overline{p}_0}<0$.

For p_k, by virtue of (29), (44), and (45), we have

$$\left(\frac{\mathrm{d}\Phi_l(p_0)}{\mathrm{d}p_0}-\frac{\mathrm{d}\Phi_r(p_0)}{\mathrm{d}p_0}\right)p_k=(\overline{\overline{y}}'_{k-1}(\sigma)-\overline{y}'_{k-1}(\sigma))-\frac{\mathrm{d}\Phi_r(p_0)}{\mathrm{d}p_0}\overline{\overline{y}}_k(\sigma)+\frac{\mathrm{d}\Phi_l(p_0)}{\mathrm{d}p_0}\overline{y}_k(\sigma)$$
$$-\int_{-\infty}^{0}\Psi_1(0)\Psi_1^{-1}(s)H_k^{(-)}(s)\mathrm{d}s+\int_{+\infty}^{0}\Psi_2(0)\Psi_2^{-1}(s)H_k^{(+)}(s)\mathrm{d}s$$

By H_6, the coefficient of p_k is not equal to zero, so, p_k are all determined. Thus, $Q_k^{(\pm)}x(\tau)$ are all completely determined.

For boundary functions $R_kx(\tau_T)(k\geq 0)$, they have no essential influence on the interior layer, so, we will not discuss them in detail but only narrate their existing conditions.

H_7 Suppose that that the line $\widetilde{\widetilde{y}}(T)=y^T$ intersects with Σ_T, which is the steady manifold of system

$$\frac{\mathrm{d}\widetilde{\widetilde{y}}}{\mathrm{d}\tau_0}=\widetilde{\widetilde{z}},\quad \frac{\mathrm{d}\widetilde{\widetilde{z}}}{\mathrm{d}\tau_0}=F(\widetilde{\widetilde{z}},\widetilde{\widetilde{y}},\psi(T-\sigma),T).$$
$$\widetilde{\widetilde{y}}(T)=y^T,\ \widetilde{y}(-\infty)=\psi(T),$$

where $\widetilde{\widetilde{y}}=\psi(T)+R_0y(\tau_T)$. Thus, the boundary function $R_kx(\tau_T),k\geq 0$, are all determined. Similarly, $R_kx(\tau_T),k\geq 0$, decay exponentially as $\tau_T\to-\infty$. Now, the coefficients of (32), (33) are all known, so, the asymptotic expansions are constructed.

5 The Main Result

Let

$$X_n(t,\mu)=\begin{cases}\displaystyle\sum_{i=0}^{n}\mu^i(\overline{x}_i(t)+\Pi_ix(\tau_0)+Q_i^{(-)}x(\tau)), & 0\leq t\leq\sigma,\\ \displaystyle\sum_{i=0}^{n}\mu^i(\overline{\overline{x}}_i(t)+Q_i^{(+)}x(\tau)+R_ix(\tau_T)), & \sigma\leq t\leq T,\end{cases}$$

Theorem Under conditions H_1-H_7, there exist positive constants $\mu_0>0,\ c>0$, such that, for $0<\mu\leq\mu_0$, the solution $x(t,\mu)$ of the problem (2)–(3) exists in the interval $[0,T]$ and satisfies the inequality

$$\|x(t,\mu)-X_n(t,\mu)\|\leq c\mu^{n+1}. \tag{47}$$

Omitting details of the general statement, we give the version needed in the forthcoming considerations.

Lemma 7 Suppose that there exist two functions $\underline{\omega}(t)$ and $\overline{\omega}(t)$, such that the following assertions are valid:

(1) $\underline{\omega}(t)\leq\overline{\omega}(t),\ -\sigma\leq t\leq T$;

(2)

$$\mu^2\underline{\omega}''(t)\geq F(\mu\underline{\omega}'(t),\underline{\omega}(t),\underline{\omega}(t-\sigma),t),\quad 0\leq t\leq T;$$
$$\mu^2\overline{\omega}''(t)\leq F(\mu\overline{\omega}'(t),\overline{\omega}(t),\overline{\omega}(t-\sigma),t),\quad 0\leq t\leq T;$$

(3) $\underline{\omega}(t)\leq\alpha(t)\leq\overline{\omega}(t),\ -\sigma\leq t\leq 0;\quad \underline{\omega}(T)\leq y^T\leq\overline{\omega}(T)$

(4) The inequalities

$$\frac{\mathrm{d}\overline{\omega}}{dt}(-)\geq\frac{\mathrm{d}\overline{\omega}}{dt}(+),\quad \frac{d\underline{\omega}}{dt}(-)\leq\frac{d\underline{\omega}}{dt}(+)$$

hold and the function $f(z,y,u,t)$ belongs to Nagumo function class [12], where the subscripts + and - on the normal derivative mean that they are evaluated to the right and the left, respectively, at their discontinuous points. Then, there exists a solution $y(x)$ of problem (1), such that $\underline{\omega}(t)\le y(t)\le\overline{\omega}(t)$, where $\underline{\omega}(t)$ and $\overline{\omega}(t)$ are referred to as a lower solution and an upper solution of problem (1), respectively.

Next, we will proceed to the proof of the theorem.

Proof First, we express the upper solution as

$$\overline{\omega}(t)=\begin{cases}\alpha(t)+\mu^2, & -\sigma\le t\le 0,\\ \varphi(t)+\Pi_0y(\tau_0)+Q_{0\beta}^{(-)}y(\tau)+\mu(\overline{y}_1(t)+\Pi_{1\beta}y(\tau_0)+Q_{1\beta}^{(-)}y(\tau))+\mu^2, & 0\le t\le\sigma,\\ \psi(t)+Q_{0\beta}^{(+)}y(\tau)+R_0y(\tau_T)+\mu(\overline{\overline{y}}_1(t)+Q_{1\beta}^{(+)}y(\tau)+R_{1\beta}y(\tau_T))+\mu, & \sigma\le t\le T,\end{cases}$$

and the lower solution as

$$\underline{\omega}(t)=\begin{cases}\alpha(t)-\mu^2, & -\sigma\le t\le 0,\\ \varphi(t)+\Pi_0y(\tau_0)+Q_{0\alpha}^{(-)}y(\tau)+\mu(\overline{y}_1(t)+\Pi_{1\alpha}y(\tau_0)+Q_{1\alpha}^{(-)}y(\tau))-\mu^2, & 0\le t\le\sigma,\\ \psi(t)+Q_{0\alpha}^{(+)}y(\tau)+R_0y(\tau_T)+\mu(\overline{\overline{y}}_1(t)+Q_{1\alpha}^{(+)}y(\tau)+R_{1\alpha}y(\tau_T))-\mu, & \sigma\le t\le T.\end{cases}$$

The functions $Q_{0\beta}^{(-)}y(\tau)$, $Q_{1\beta}^{(-)}y(\tau)$ and $\Pi_{1\beta}y(\tau_0)$ can be found from the problem

$$\begin{cases}\dfrac{d^2Q_{0\beta}^{(-)}y}{\mathrm{d}\tau^2}=F\Big(\dfrac{\mathrm{d}Q_{0\beta}^{(-)}y}{\mathrm{d}\tau},\varphi(\sigma)+Q_{0\beta}^{(-)}y(\tau),\alpha(0),\sigma\Big),\\ Q_{0\beta}^{(-)}y(0)=(p_0+\delta)-\varphi(\sigma),\ Q_{0\beta}^{(-)}y(-\infty)=0.\end{cases}$$

$$\begin{cases}\dfrac{d^2Q_{1\beta}^{(-)}y}{\mathrm{d}\tau^2}=\widetilde{F}_z^{(-)}\dfrac{\mathrm{d}Q_{1\beta}^{(-)}y}{\mathrm{d}\tau}+\widetilde{F}_y^{(-)}Q_{1\beta}^{(-)}y(\tau)+H_{1\beta}^{(-)}(\tau)+\omega e^{k_1\tau},\\ Q_{1\beta}^{(-)}y(0)=p_1-\overline{y}_1(\sigma),\ Q_{1\beta}^{(-)}y(-\infty)=0.\end{cases}$$

and

$$\begin{cases}\dfrac{d^2\Pi_{1\beta}y}{\mathrm{d}\tau_0^2}=\widetilde{F}_z\dfrac{\mathrm{d}\Pi_{1\beta}y}{\mathrm{d}\tau_0}+\widetilde{F}_y\Pi_{1\beta}y(\tau_0)+G_{1\beta}(\tau_0)+\omega e^{-k_0\tau_0},\\ \Pi_{1\beta}y(0)=-\overline{y}_1(0),\ \Pi_{1\beta}y(+\infty)=0,\end{cases}$$

respectively, where δ,ω,k_0,k_1 are all positive constants. $Q_{0\beta}^{(+)}y(\tau)$, $Q_{1\beta}^{(+)}y(\tau)$ and $R_{1\beta}y(\tau_T)$ are given by

$$\begin{cases}\dfrac{d^2Q_{0\beta}^{(+)}y}{\mathrm{d}\tau^2}=F\Big(\dfrac{\mathrm{d}Q_{0\beta}^{(+)}y}{\mathrm{d}\tau},\psi(\sigma)+Q_{0\beta}^{(+)}y(\tau),\varphi(0)+\Pi_{0\beta}y(\tau),\sigma\Big),\\ Q_{0\beta}^{(+)}y(0)=(p_0+\delta)-\psi(\sigma),\ Q_{0\beta}^{(+)}y(+\infty)=0.\end{cases}$$

$$\begin{cases}\dfrac{d^2Q_{1\beta}^{(+)}y}{\mathrm{d}\tau^2}=\widetilde{F}_z^{(+)}\dfrac{\mathrm{d}Q_{1\beta}^{(+)}y}{\mathrm{d}\tau}+\widetilde{F}_y^{(+)}Q_{1\beta}^{(+)}y(\tau)+H_{1\beta}^{(+)}(\tau)+\omega e^{-k_1\tau},\\ Q_{1\beta}^{(+)}y(0)=p_1-\overline{\overline{y}}_1(\sigma),\ Q_{1\beta}^{(+)}y(+\infty)=0.\end{cases}$$

and

$$\begin{cases}\dfrac{d^2R_{1\beta}^{(-)}y}{\mathrm{d}\tau_T^2}=\widetilde{\widetilde{F}}_z\dfrac{dR_{1\beta}y}{\mathrm{d}\tau_T}+\widetilde{\widetilde{F}}_yR_{1\beta}y(\tau_T)+K_{1\beta}(\tau_T)+\omega e^{k_0\tau_T},\\ R_{1\beta}y(0)=-\overline{\overline{y}}_1(0),\ R_{1\beta}y(-\infty)=0.\end{cases}$$

Replace $p_0+\delta$, β and ω by $p_0-\delta$, α and $-\omega$, respectively, we can obtain all terms in $\underline{\omega}(t)$.

Next, we will show that the functions $\overline{\omega}(t)$ and $\underline{\omega}(t)$ satisfy all items of Lemma 7. For convenience, we divide $[0,T]$ into four parts, namely, $[0,\frac{2}{3}\sigma],[\frac{2}{3}\sigma,\sigma],[\sigma,\frac{4}{3}\sigma]$, and $[\frac{4}{3}\sigma,T]$.

Obviously, the third item holds.

In the following, we will verify the first item: $\overline{\omega}(t)\geq\underline{\omega}(t)$.

In $[0,\frac{2}{3}\sigma]$, because $\Pi_{1\beta}y(0)=\Pi_{1\alpha}y(0)=-\overline{y}_1(0)$ and $\Pi_{1\alpha}y(\tau_0)$ $\Pi_{1\beta}y(\tau_0)$ are both of exponential decay for $\tau_0>0$, we have

$$\overline{\omega}(t)-\underline{\omega}(t)=\mu(\Pi_{1\beta}y(\tau_0)-\Pi_{1\alpha}y(\tau_0))+2\mu^2>0.$$

The fact $\overline{\omega}(t)>\underline{\omega}(t)$ in $[\frac{2}{3}\sigma,\sigma]$ and $[\frac{4}{3}\sigma,T]$ can be treated similarly.

In $[\sigma,\frac{4}{3}\sigma]$, $Q_{0\beta}^{(+)}y(0)=p_0+\delta-\psi(\sigma)$, $Q_{0\alpha}^{(+)}y(0)=p_0-\delta-\psi(\sigma)$, $Q_{1\beta}^{(+)}y(0)-Q_{1\alpha}^{(+)}y(0)=0$ and $Q_{0\beta}^{(+)}y(\tau)$, $Q_{0\alpha}^{(+)}y(\tau)$, $Q_{1\beta}^{(+)}y(\tau)$, $Q_{1\alpha}^{(+)}y(\tau))$ are all of exponential decay for $\tau>0$, so

$$\overline{\omega}(t)-\underline{\omega}(t)=Q_{0\beta}^{(+)}y(\tau)-Q_{0\alpha}^{(+)}y(\tau)+\mu(Q_{1\beta}^{(+)}y(\tau)-Q_{1\alpha}^{(+)}y(\tau))+2\mu>0.$$

Thus, we finish the proof of the first item.

Next, we will verify the second item. Most results will be stated in terms of the upper solution and obvious analogous results for lower solution will not be stated. In $[0,\frac{2}{3}\sigma]$, we have

$$\begin{aligned}L\overline{\omega}(t)&=\mu^2\overline{\omega}''(t)-F(\mu\overline{\omega}'(t),\overline{\omega}(t),\alpha(t-\sigma),t)\\&=\mu^2\varphi''(t)+\mu^3\overline{y}_1''(t)+\frac{d^2\Pi_0y(\tau_0)}{\mathrm{d}\tau_0^2}+\mu\frac{\mathrm{d}\Pi_{1\beta}y(\tau_0)}{\mathrm{d}\tau_0}\\&\quad-\Big[F(\mu\varphi'(t)+\mu^2\overline{y}_1(t),\varphi(t)+\mu\overline{y}_1(t)+\mu^2,\alpha(t-\sigma),t)\\&\quad+(F(\mu\varphi'(\mu\tau_0)+\frac{\mathrm{d}\Pi_0y}{\mathrm{d}\tau_0}+\mu^2\overline{y}_1'(\mu\tau_0)+\mu\frac{\mathrm{d}\Pi_{1\beta}y}{\mathrm{d}\tau_0},\varphi(\mu\tau_0)+\Pi_0y(\tau_0)\\&\quad+\mu(\overline{y}_1(\mu\tau_0)+\Pi_{1\beta}y(\tau_0))+\mu^2,\alpha(\mu\tau_0-\sigma),\mu\tau_0)\\&\quad-F(\mu\varphi'(\mu\tau_0)+\mu^2\overline{y}_1(\mu\tau_0),\varphi(\mu\tau_0)+\mu\overline{y}_1(\mu\tau_0)+\mu^2,\alpha(\mu\tau_0-\sigma),\mu\tau_0))\Big]\\&=-2\mu^2F_y(0,\phi(t),\alpha(t-\sigma),t)+EST<0.\end{aligned}$$

Similarly, in $[\frac{2}{3}\sigma,\sigma]$ and $[\frac{4}{3}\sigma,T]$, we also have $L\overline{\omega}(t)<0$. In $[\sigma,\frac{4}{3}\sigma]$,

$$\begin{aligned}L\overline{\omega}(t)&=\mu^2\overline{\omega}''(t)-F(\mu\overline{\omega}'(t),\overline{\omega}(t),\overline{\omega}(t-\sigma),t)\\&=\mu^2\psi''(t)+\mu^3\overline{y}_1''(t)+\frac{d^2Q_{0\beta}^{(+)}y}{\mathrm{d}\tau^2}+\mu\frac{\mathrm{d}Q_{1\beta}^{(+)}y}{\mathrm{d}\tau}\\&\quad-F(\mu\psi'(t)+\mu^2\overline{\overline{y}}_1'(t),\psi(t)+\mu\overline{\overline{y}}_1(t)+\mu,\alpha(t-\sigma)+\mu\overline{y}_1(t-\sigma)+\mu^2,t)\\&\quad-\Big[F(\mu\psi'(\sigma+\mu\tau)+\frac{\mathrm{d}Q_{0\beta}^{(+)}y}{\mathrm{d}\tau}+\mu^2\overline{\overline{y}}_1'(\sigma+\mu\tau)+\mu\frac{\mathrm{d}Q_{1\beta}^{(+)}y}{\mathrm{d}\tau},\\&\quad\psi(\sigma+\mu\tau)+Q_{0\beta}^{(+)}y(\tau)+\mu(\overline{\overline{y}}_1(\sigma+\mu\tau)+Q_{1\beta}^{(+)}y(\tau))+\mu,\\&\quad\varphi(\mu\tau)+\pi_0y(\tau)+\mu(\overline{y}_1(\mu\tau)+\pi_{1\beta}y(\tau))+\mu^2,\sigma+\mu\tau)\\&\quad-F(\mu\psi'(\sigma+\mu\tau)+\mu^2\overline{\overline{y}}_1'(\sigma+\mu\tau),\psi(\sigma+\mu\tau)+\mu(\overline{\overline{y}}_1(\sigma+\mu\tau)\\&\quad+\mu,\varphi(\mu\tau)+\pi_0y(\tau)+\mu(\overline{y}_1(\mu\tau)+\pi_{1\beta}y(\tau))+\mu^2,\sigma+\mu\tau)\Big]\\&=-\mu F_y(0,\phi(t),\alpha(t-\sigma),t)+EST<0.\end{aligned}$$

Finally, we will prove the fourth item.

$$\begin{aligned}\mu\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}(\sigma-)-\mu\frac{\mathrm{d}\overline{\omega}}{\mathrm{d}t}(\sigma+)&=\Big[\mu\varphi'(\sigma-)+\mu^2\overline{y}_1'(\sigma-)+\frac{\mathrm{d}Q_{0\beta}^{(-)}y}{\mathrm{d}\tau}(0-)+\mu\frac{\mathrm{d}Q_{1\beta}^{(-)}y}{\mathrm{d}\tau}(0-)\Big]\\&\quad-\Big[\mu\psi'(\sigma+)+\mu^2\overline{\overline{y}}_1'(\sigma+)+\frac{\mathrm{d}Q_{0\beta}^{(+)}y}{\mathrm{d}\tau}(0+)+\mu\frac{\mathrm{d}Q_{1\beta}^{(+)}y}{\mathrm{d}\tau}(0+)\Big]\\&=\frac{\mathrm{d}Q_{0\beta}^{(-)}y}{\mathrm{d}\tau}(0-)-\frac{\mathrm{d}Q_{0\beta}^{(+)}y}{\mathrm{d}\tau}(0+)+O(\mu)\\&=H(p+\delta)+O(\mu)=H(p_0)+H'(p_0)\delta+O(\mu)\\&=H'(p_0)\delta+O(\mu).\end{aligned}$$

By H_6, the above formula is negative.

In the same way, we obtain $\frac{\mathrm{d}\omega}{\mathrm{d}t}(\sigma-)\leq\frac{\mathrm{d}\omega}{\mathrm{d}t}(\sigma+)$. In accordance with Lemma 7, the solution of (1) exists.

To obtain the asymptotic solution with $O(\mu)$, we should expand the upper solution and lower solution to $O(\mu^2)$, namely,

$$\overline{\omega}_1(t)=\begin{cases}\displaystyle\sum_{i=0}^{1}\mu^i(\overline{y}_i(t)+\Pi_iy(\tau_0)+Q_{i\beta}^{(-)}y(\tau))+\mu^2(\overline{y}_2(t)+\Pi_{2\beta}y(\tau_0)+Q_{2\beta}^{(-)}y(\tau)+\mu),\\ \hfill 0\leq t\leq\sigma,\\ \displaystyle\sum_{i=0}^{1}\mu^i(\overline{\overline{y}}_i(t)+Q_{i\beta}^{(+)}y(\tau)+R_iy(\tau_T))+\mu^2(\overline{\overline{y}}_2(t)+Q_{2\beta}^{(+)}y(\tau)+R_{2\beta}y(\tau_T)+1),\\ \hfill \sigma\leq t\leq T,\end{cases}$$

$$\underline{\omega}_1(t)=\begin{cases}\displaystyle\sum_{i=0}^{1}\mu^i(\overline{y}_i(t)+\Pi_iy(\tau_0)+Q_{i\alpha}^{(-)}y(\tau))+\mu^2(\overline{y}_2(t)+\Pi_{2\alpha}y(\tau_0)+Q_{2\alpha}^{(-)}y(\tau)-\mu),\\ \hfill 0\leq t\leq\alpha,\\ \displaystyle\sum_{i=0}^{1}\mu^i(\overline{\overline{y}}_i(t)+Q_{i\alpha}^{(+)}y(\tau)+R_iy(\tau_T))+\mu^2(\overline{\overline{y}}_2(t)+Q_{2\alpha}^{(+)}y(\tau)+R_{2\alpha}y(\tau_T)-1),\\ \hfill \alpha\leq t\leq T.\end{cases}$$

In a similar manner, we can prove that $\overline{\omega}_1(t)$, $\underline{\omega}_1(t)$ also satisfy the items of Lemma 7.

Thus, according to Lemma 7, we can not only prove the existence of the solution of (1) but also write out the asymptotic representation with accuracy $O(\mu)$, namely,

$$y(t,\mu)=\begin{cases}\varphi(t)+\Pi_0y(\tau_0)+Q_0^{(-)}y(\tau)+\mu(\overline{y}_1(t)+\Pi_1y(\tau_0)+Q_1^{(-)}y(\tau))+O(\mu),\\ \hfill 0\leq t\leq\sigma,\\ \psi(t)+Q_0^{(+)}y(\tau)+R_0y(\tau_T)+\mu(\overline{y}_1(t)+Q_1^{(+)}y(\tau)+R_1y(\tau_T))+O(\mu),\\ \hfill \sigma\leq t\leq T.\end{cases}$$

In the same way, we can obtain the asymptotic expansion with $O(\mu^n)$. The proof of the theorem is completed.

6 Example

Let us consider the problem

$$\begin{cases} \mu^2 y'' = y - y(t-1); \\ y(t) = t,\ t \in [-1, 0], \quad y(\frac{3}{2}) = 0. \end{cases} \tag{48}$$

For $0 \le t \le \frac{3}{2}$, the degenerate equation of (48) has an isolate root:

$$\overline{y}(t) = \begin{cases} t-1 & t \in [0,1]; \\ t-2, & T \in [1, \frac{3}{2}]. \end{cases}$$

Obviously, conditions H_2, H_3 hold. $\Pi_0 y(\tau_0)$, $R_0 y(\tau_T)$, $Q_0^{(-)} y(\tau)$ and $Q_0^{(+)} y(\tau)$ are given by the following systems respectively,

$$\frac{d^2 \Pi_0 y}{\mathrm{d}\tau_0^2} = \Pi_0 y, \quad \Pi_0 y(0) = 1,\ \Pi_0 y(+\infty) = 0;$$

$$\frac{d^2 R_0 y}{\mathrm{d}\tau_T^2} = R_0 y(\tau_T), \quad R_0 y(0) = \frac{1}{2},\ R_0 y(-\infty) = 0;$$

$$\frac{d^2 Q_0^{(-)} y}{\mathrm{d}\tau^2} = Q_0^{(-)} y(\tau), \quad Q_0^{(-)} y(0) = p_0,\ Q_0^{(-)} y(-\infty) = 0;$$

$$\begin{cases} \dfrac{d^2 Q_0^{(+)} y}{\mathrm{d}\tau^2} = Q_0^{(+)} y(\tau) - \Pi_0 y(\tau), \\ Q_0^{(+)} y(0) = p_0 + 1,\ Q_0^{(+)} y(-\infty) = 0. \end{cases}$$

After simple manipulations, we obtain $\Pi_0 y(\tau_0) = e^{-\tau_0}$, $Q_0^{(-)} y = p_0 e^{\tau}$, $Q_0^{(+)} y(\tau) = (p_0 + \frac{1}{2})e^{-\tau} + \tau e^{-\tau} + \frac{1}{2} e^{-\tau}$, $R_0 y(\tau_T) = \frac{1}{2} e^{\tau_T}$. By the smooth connection $\frac{\mathrm{d}Q_0^{(-)} y}{\mathrm{d}\tau}\big|_{\tau=0} = \frac{\mathrm{d}Q_0^{(+)} y}{\mathrm{d}\tau}\big|_{\tau=0}$, we have $H(p_0) = p_0 - (-p_0 - \frac{1}{2} - \frac{1}{2} + 1) = 0$. Obviously, $p_0 = 0$ and $\frac{\mathrm{d}H(p_0)}{\mathrm{d}p_0}\big|_{p_0=0} = 2 \neq 0$. Thus, we obtain the zero order asymptotic solution of (48):

$$y(t,\mu) = \begin{cases} t - 1 + e^{-\frac{t}{\mu}}, & t \in [0,1]; \\ t - 2 + \dfrac{t-\sigma}{\mu} e^{-\frac{t-\sigma}{\mu}} + e^{\frac{t-\sigma}{\mu}} + \dfrac{1}{2} e^{\frac{t-T}{\mu}}, & t \in [1, \frac{3}{2}]. \end{cases}$$

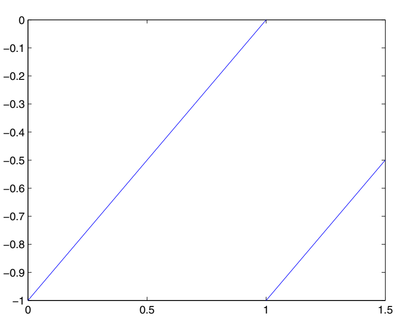

Fig.2

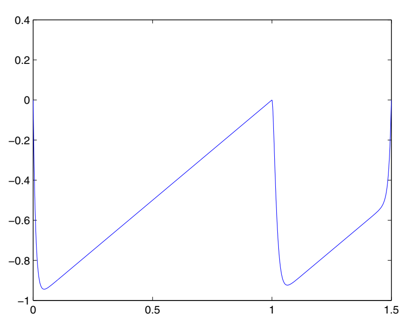

Fig.3

The following graphs are the degenerate solution (Fig.2) and the zero order asymptotic solution (Fig.3) respectively.

From the above two figures, we can see that the zero order asymptotic solution is a good approximation to the reduced solution.

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