Journal of Mathematical Sciences, Vol. 165, No. 6, 2010

AVERAGING OPERATORS AND REAL EQUATIONS OF HYDROMECHANICS

R. R. Aidagulov and M. V. Shamolin UDC 517.9+531.01

Abstract. Pseudodifferential operators occurring in real equations of continuous-medium mechanics are discussed.

In continuous-medium mechanics, one usually describes the medium evolution using differential equations (as a rule, partial differential equations). In [1], it is shown how, starting from only natural principles, one obtains a closed system of equations for the medium evolution. In this case, the main interphase forces and the closure of the equations with accuracy up to some parameters are obtained automatically. The latter are functions of some flow characteristics of the viscosity-coefficient type depending on the temperature and can be interpreted as closing relations.

Based on these principles, the authors constructed closed medium models. But here (as is noted in the work in what follows), the weakest and nonphysical principle is the principle of the possibility of describing the medium evolution as a solution of differential equations. In [1], the authors rejected some separate equations of impulse and energy conservation in each phase and preserved only general (total) conservation laws for all phases. However, in dissipative media, one also needs to reject the general impulse and energy conservation laws.

Only the mass conservation law in the description of the (one-phase) fluid motion remains:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} I = 0, \tag{1}$$

where I is the impulse-vector (mass flow) of the medium considered. The medium speed is defined as the ratio of the impulse to the mass:

$$v = \frac{I}{\rho}.$$

The medium *impulse* (calculated per volume unit) is exactly the mass flow.

The impulse and energy of a dissipative medium are not locally preserved, and a permanent interchange of the impulse and energy between neighboring fluid layers and with the material of the boundary on the boundary of the fluid flow occurs. In [2, 11], it was shown that the equations describing the medium evolution are pseudodifferential and are expressed (in an isotropic medium) through the operators

$$D = \frac{\partial}{\partial t} + v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v^3 \frac{\partial}{\partial z} \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In this case, the *pseudodifferentiality* of the equations (containing no arbitrary integro-differential expressions and written, for short, as PDE) follows from the fact that the evolution equations must be invariant with respect to the translation (shift) group in the space-time. As a subgroup, this group is contained in the Galileo group with respect to which the equations must be invariant. The Galileo group also contains arbitrary turns of the coordinate system considered; as a result, we obtain that in the expressions, there are no differentiations in an arbitrary direction, except for convective derivatives D and the Laplace operator Δ. In this case, the differential operations of divergence (applied to velocity components) transforming the objects from vectors into scalars are expressed through the operations applied to the density $D\varrho$.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 65, Mathematical Physics, Combinatorics, and Optimal Control, 2009.

The fact that in the expressions for differentiation in the direction of the velocity there also exists the differentiation in time is related to the possibility of an invariant passage to another inertial coordinate system. In general, such conclusions are obtained in the application to linear (quasilinear) operators.

It seems that it is not possible to classify strongly nonlinear operators, since there exist possibilities like the (possibly fractional) differentiation of one parameter in the direction of the gradient of another parameter, etc. However, it is necessary to keep in mind that the gradient is a covector, and the differentiation along a covector is invariant with respect to turns (in this case, we assume that the Euclidean metric such that to covectors, it puts in correspondence vectors is given) but not invariant with respect to the scaling. Therefore, such operators that are invariant under the Galileo transformation group (including turns) are not invariant with respect to a wider group that is necessary for their physical justification.

In mechanics, one usually considers only quasilinear operators. When there exist nonlinear terms, they somehow enter the operator "linear" in the differentiation as other parameters that are not differentiated. All this is analogous to the medium going away from its equilibrium state being locally rapidly reduced to the equilibrium state, and, therefore, at each point and at each instant of time, it is considered to be in the equilibrium state and is characterized by the mass, impulse, and energy related to the volume unit.

Without taking account of the absence of equilibrium between neighboring layers, the conservation equations yield the *Euler equations*. Taking account of the impulse and energy interchange between neighboring layers in the first approximation with respect to the violated equilibrium (the so-called quasi-equilibrium) leads to quasilinear systems of equations with respect to these parameters. In what follows, we restrict ourselves to the consideration of only quasilinear operators.

Scientific exchanges among some scientists indicate that they often encounter difficulties in understanding the concept of pseudodifferential operator without simple analogies. It is better to show these analogies dealing with matrices of finite order. Also, we need to do that since in numerically solving problems, any operator is often approximated to be a matrix operator.

A pseudodifferential operator is an arbitrary function of a differential operator, and the action of such an operator is calculated in the same way as that of an arbitrary function of a certain matrix representing a differential operator as an analogy. Note that for a polynomial or an arbitrary function of a matrix, the eigenvectors are the same as for the initial matrix. Arbitrary functions of matrices are respectively not arbitrary matrices but only those which commute with the initial matrix.

In calculating the action of the matrix $f(A)$ that is a certain function of a matrix, the analog of a linear differential operator A, on a vector-column (analog of a function), we expand this vector into a linear combination of eigenvectors:

$$\chi = \sum_{\omega} \omega \chi_{\omega}$$

(χ_{ω} is an eigenvector of the operator A with eigenvalue ω) and calculate the result:

$$f(A)\chi = \sum_{\omega} f(\omega)\chi_{\omega}.$$

The function applied to the eigenvalues to the right is the symbol (or the amplitude) of the operator $f(A)$. In the case of pseudodifferential operators, it is better to write this as

$$f(A)\chi = \sum_{\omega,i} f(\omega)\chi_{i\omega},$$

i.e., in calculation of the action, Fourier transforms are used. This has its own advantages when the symbol of a homogeneous adjoint operator is obtained by the complex conjugation of the symbol of the initial operator, and this writing is concordant with the conventional writing in quantum mechanics.

The Fourier transform of a function $f(x)$ can be represented as finding the amplitudes in the representation as the sum of waves of the following form:

$$f(x) = \int F(k) \exp(ik, x) dk;$$

in this case, the amplitude function $F(k)$ is obtained from the initial function as the Fourier transform

$$F(k) = (2\pi)^{-n} \int f(x) \exp(-ik, x) dx,$$

where n is the dimension of the space of variables x and k.

Also, to the operators used in mechanics, we can put in correspondence their amplitude $a(k)$ (which is usually called the *operator symbol* [12]), so that the action of this operator A on a function $f(x)$ is calculated by multiplying the function amplitude by the operator amplitude, which corresponds to the convolution of the function with a certain distribution (more precisely, with a *Sato hyperfunction* [8, 10]).

Passing to the symbols of pseudodifferential operators (PDOs) [12], we obtain that the evolution system of equations is defined by waves (which are eigenvectors of pseudodifferential operators) that propagate in the medium. By applying only the principles presented in [1], the system of equations is defined not with accuracy up to parameters but with accuracy up to wave-dispersion relations, which can also depend on other flow parameters. Correspondingly, the experimental "measurement" (definition) of the equations reduces to a more cumbersome measuring of the dispersion relations and their dependence on the parameters. Also, the order of the PDO plays an important role; it is defined as

$$\lim_{|k| \to \infty} \frac{\ln |f(k)|}{\ln |k|}.$$

This relation defines the operator type and roughly shows the number of additional boundary conditions that must be given. In what follows, we encounter only operators of negative order, which do not require the assignment of additional boundary conditions.

Now let us enumerate the main operators used in mechanics and calculate their amplitudes starting from the simplest operators.

1. Differentiation along the trajectory or the convective derivative:

$$D = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}$$

(here, x is a three-dimensional vector and time t is one-dimensional).

The amplitude of this operator is

$$ik_0 + iv_1 k_1 + iv_2 k_2 + iv_3 k_3.$$

2. The shift T_s of a function by a vector s:

$$T_s : f(\chi) \to f(x - s).$$

The amplitude

$$\exp(-ik, s)$$

corresponds to it.

In mechanics, one often encounters shifts along the trajectory; in this case,

$$s = \tau(1, v_1, v_2, v_3).$$

3. The averagings over the sphere of radius R:

$$M_R(f(x)) = \frac{1}{\omega_{n-1} \mathbb{R}^{n-1}} \int\limits_{|x-y|=R} f(y)\, dy.$$

Let us consider the dimensions $n = 1, 2$, and 3 separately. In the case $n = 1$, the sphere over which the integration is performed is zero-dimensional, and the averaging reduces to

$$M_R = \frac{f(x+R) + f(x-R)}{2},$$

whereas the operator amplitude is equal to the amplitude of a half-sum of shifts:

$$\frac{1}{2}(\exp(ikR) + \exp(-ikR)) = \cos(kR).$$

Calculating the volume [4]

$$\omega_{n-1} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

of the sphere in the n-dimensional space, we obtain the amplitude of the averaging operator M_R as the averaging of the amplitudes $\exp(-ik, s)$ of the shift over the sphere $|s| = R$:

$$P_\vartheta(Rk), \quad k = \sqrt{k_1^2 + \cdots + k_n^2},$$

where

$$\vartheta = \frac{n-2}{2}, \qquad P_\vartheta(s) = 2^\vartheta \Gamma(\vartheta + 1) s^{-\vartheta} J_\vartheta(s).$$

Since for half-integer subscripts, the Bessel function is expressed through elementary functions, for odd values of n a simpler form is obtained; precisely, for $n = 2$, we have the Bessel function

$$J_0(Rk),$$

and for $n = 3$, we have

$$\frac{\sin(Rk)}{Rk}.$$

The main operators of mechanics are interchange operators of the form

$$A = \int_0^\infty \int_0^\infty f(R,t)(T_\tau M_R - 1)\, dt\, dR, \tag{2}$$

where T_τ is the operator of shift by τ along the trajectory having the symbol $\exp(-\tau D)$.

The operators of such a type used in mechanics are differential operators in the sense that they transform constant functions into the identical zero, i.e., the relation

$$A[\text{const}] = 0$$

holds.

In the authors' opinion, by the term PDO, it would be desirable to mean only such "differential" convolution operators.

Owing to the fact that the velocity in the operator D is variable, such operators become nonlinear operators. In many problems of mechanics, it is possible to use the simplified shift operator (for example, when the velocity in the operator D is equal to zero or to a certain known constant quantity, which makes the operator be linear). In calculating the integrals for a point x, we assume that the velocity in the operator D is the same and is equal to $v(x)$ at all shifted points, which leads to the quasilinearity of this operator.

In fact, any reasonable differential operator can be represented in the form (2), and there exists a one-to-one correspondence between operators of the type $F(\Delta, D)$ and the functions $f(R,t)$, which can be called the *relaxation* (*rheology*) *functions*. However, in contrast to the function used in elasticity theory, here, such a function depends not only on the delay in time t but on the so-called "spatial delay."

In stationary problems, the operator reduces to a particular case of the form (2) where it is represented without time delay:

$$A = \int_0^\infty g(R)(M_R - 1)\, dR, \qquad g(R) = \int f(R,\tau)\, d\tau, \quad f(R,\tau) = g(R)\delta(\tau). \tag{3}$$

In this case, the linear operator A is expressed as a certain function of the Laplace operator and represents the Newton viscosity. In the presence of explicit dependence on the "delay time" τ, the visco-elasticity effects manifest themselves.

The nonlinearity of the shift operator

$$T_\tau = \exp(-i\tau D)$$

also introduces plasticity effects into the operator. In this case, the passage from the "elasticity" regime to the "viscous flow" regime can manifest itself in continuous way, as well as jump-like. The latter can happen when the velocity v entering the operator D passes from the "subsonic" regime to the "supersonic" regime with respect to a certain small creeping rate.

In what follows, we assume that the dimension of the space is equal to 3 (i.e., $n = 3$) and that the averagings are performed over two-dimensional spheres of radius R. The symbol of the averaging operator over a shifted sphere has the form

$$T_\tau M_R = \exp\left\{ i\tau(k_0 + v_1k_1 + v_2k_2 + v_3k_3)\frac{\sin(kR)}{kR} \right\}, \quad k = \sqrt{k_1^2 + k_2^2 + k_3^2}.$$

Now let us try to find the form of the equations of fluid motion. As in the viscous fluid (Navier–Stokes) theory, we write the equations of motion in the form

$$DI = -\frac{\partial p}{\partial x} + F + AI + \frac{\partial B\rho}{\partial x}, \tag{4}$$

$$D\varepsilon = R + C\varepsilon. \tag{5}$$

Here, $I = \rho v$ is the impulse, the first two terms on the right-hand side of (4) are the known functions (the gradient of pressure and the exterior force), A and C are interchange operators of the type (2), B is the operator characterizing the reaction to the compression, and R is the known function involving the viscous-force work and the heat radiation at a point.

The operators A, B, and C can be found experimentally by observing three types of waves. To find the operator A, it is necessary to observe and find the spectral properties of the friction wave (or transversal waves). By this we mean the wave-like solutions depending on only two coordinates, time t and one spatial coordinate x, when all impulse components are identically equal to zero, except for one coordinate that is perpendicular to the x axis; denote this coordinate by v. Taking into account that

$$\operatorname{div} I = 0 = \operatorname{div} \frac{I}{\rho}$$

and assuming that the exterior force in the direction of the velocity v is equal to zero, we obtain the following simplified equation of the type (4):

$$\frac{\partial v}{\partial t} = Av, \qquad D = \frac{\partial}{\partial t}.$$

This allows us to find the form of the linear operator A by using the results of measuring the dispersion relations of such waves. Such a possibility exists under the condition that the operator A is independent of the operator D; in this case, the latter operator coincides with the operator $\partial/\partial t$. In the case of the presence of such a dependence, we need to study these dispersion relations under the known exterior forces varying in time, which essentially complicates the reconstruction problem of the system of equations by the measurement results. The reconstruction problem of the system of equations without measurement with varying exterior forces cannot be solved.

There exist many equations that lead to the same dispersion relations; however, they all lead to the same single-valued solution. Therefore, it is the authors' opinion that not the equations themselves but the wave-dispersion relations are primary. Correspondingly, the continuous medium is determined by its own waves, which can propagate in it. This especially concerns the multiphase media when a medium is not defined by the presence or absence of additional components but is determined by whether or not new components create new waves in the medium. When new waves are created, the medium is defined by the character of these waves, i.e., it is more correct to speak about the "multiwave" medium rather than about the "multiphase" medium.

In a similar way, the operator C is defined by "temperature" waves. The operator B is defined after the "measurement" of the operator A by longitudinal waves that are "compression" waves. In [2, 11], some reasons for the form of the operator B are presented. When they are expressed through the density, they lead to the form

$$B\rho = \text{const} \cdot \Delta^{-3/4} D^{3/2} \ln \rho.$$

Here, we are interested only in the form of the operator A. Analogously to heat-conduction theory, we can conclude that the operator C has the same form as the operator A but with a new parameter instead of the coefficient of interchange between the layer impulses. This is indeed so in the case of a gas where the energy and the impulse are transferred by the same rarefied molecules.

The experiments in measuring waves can also be performed numerically. For this purpose, the authors have modeled the motion of approximately a million ($N = 2^{20}$) molecules interacting according to a given potential of the Lennard–Jones type. More precisely, the molecules are situated between two walls such that the distance between them is equal to c along the perpendicular direction (x axis). To reduce the boundary-influence effects, we assume that there exist infinitely many molecules, and they all are situated in directions of the axes y and z with period a. Correspondingly, N is the number of molecules in the fundamental domain.

In separating the wave propagation along the x axis into approximately 100 stages, approximately 10,000 molecules remain per stage. Taking into account that the temperature (mean-quadratic) molecule speed is a quantity of the sound speed V_T order, we see that the speed variance of 10,000 molecules is equal to $V_T/100$, which is approximately 10 times less than the averaged speed of the medium on a wave propagating in the medium (under the corresponding medium perturbation).

In calculations, many difficulties arise: with sufficient accuracy, it is required to know the density in order that the molecule system not be separated from the walls in the course of evolution, and it is required to assign the step in time to be sufficiently small even for integrating via the 4th-order Runge–Kutta method because of large gradient values of the interaction potentials at the repelling stage. When the computation steps are insufficiently small, there arises an artificial heat flow mainly from a wall that additionally distorts the wave character of the averaged quantity evolution.

Using such a method, we can indeed study the spectral properties of the compression waves. However, in studying the spectral properties of friction waves and the temperature wave, additional difficulties arise. Denote the complex speed by

$$c(k) = c(\omega) = \frac{\omega}{k}, \quad \omega = \omega(k), \quad k = k(\omega).$$

For these waves, the ratio

$$\frac{\operatorname{Re}(c(\omega))}{\operatorname{Im}(c(\omega))}$$

is of order 1, which leads to a decrease in the amplitude approximately by $e^{2\pi} \approx 535$ times per wave unit length. Respectively, the wave form becomes non-wave-like. For compression waves, this parameter is of order 100 or larger, which allows us to communicate through sound waves at a sufficient distance, and, in particular, whales in the water can communicate at the distance of the order of a hundred kilometers by using low-frequency waves.

We can construct an imaginary experiment for finding the wave speed

$$c(L), \quad L = \frac{2\pi}{k},$$

for large wave lengths L.

All this must be considered in a molecule system having the characteristic mean-quadratic (temperature) molecule speed v_T, the characteristic distance

$$l_0 = \frac{1}{\sqrt[3]{n}}$$

(n is the number of molecules in the volume unit) between the molecules, and the characteristic length l. The latter quantity is found from the condition that the potential of interaction between two molecules attains its minimum value:

$$U(l) = U_{\min} < 0$$

(this is the distance at which the interaction force is equal to zero). For a fixed interaction potential and for a wave length satisfying the conditions

$$L \gg l \quad \text{and} \quad L \gg l_0$$

(i.e., it is considerably larger than microscales), we obtain that the quantity

$$c(L) = v_T c_0\left(\frac{l}{l_0}, \frac{l}{L}, \frac{l_0}{L}\right) = v_T c_0\left(\frac{l}{l_0}, 0, 0\right)$$

is independent of the wave length.

For a compression wave, this limit speed is different from zero and allows us to find the operator B from the assumption of constancy of the complex speed for the compression wave.

A sharp decay (up to zero) of the wave complex speed $c(L)$ for friction and temperature waves when the wave length passes from the microscale to the macroscale makes it necessary to study the way in which the (complex) speed tends to zero. Precisely in these scales is the impulse and heat interchange between neighboring domains concentrated.

We begin by considering the problem on the way in which two molecules having initial speeds v_1 and v_2 and that are at a sufficient distance from one another interchange their impulses when the interaction forces can be neglected. Passing to the coordinate system related to the center of gravity of the interacting molecules, we reduce the problem to the problem of a mutual impact for molecules with impulses $\mathbf{p}$ and $-\mathbf{p}$ parallel to the direction of the x axis. The molecule trajectories remain in the fixed plane xy, whereas the interaction force remains parallel to the direction $\mathbf{r}_2 - \mathbf{r}_1$. After the interaction and going away, the impulse of the first molecule is obtained by the formula

$$|\mathbf{p}|(c_1(b, |\mathbf{p}|)\mathbf{i} + c_2(b, |\mathbf{p}|)\mathbf{j}),$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the axes x and y and

$$b = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{p}|}{|\mathbf{p}|}$$

is the minimal distance at which the molecules go away from one another when there is no interaction force.

Taking into account that the probability of the molecule going away to different sides at the same distance is the same, we obtain that the mean impulse variation is equal to

$$-2a\mathbf{p} = a(\mathbf{p}_2 - \mathbf{p}_1).$$

Averaging in all mutual impact speeds, we can assume that the coefficient a depends only on the interaction potential and is the *impulse-interchange coefficient*.

Now considering the interaction of the first molecule with all molecules that are at a distance R, we obtain that instead of the second molecule impulse, it suffices to take the mean impulse of all

molecules from this sphere. In this case, the number of molecules grows proportionally to the square of R, but the number of interacting molecules (flying near the first molecule) decays proportionally to the square of R, which leads to the same replacement with the same coefficient. Analogously, if we replace the first molecule with a system of molecules that are situated near it, then instead of the first molecule impulse, the total impulse of these molecules must be taken. If we assume that the "second" molecules also interact with the "third" ones before they fly to the first molecule, then, anyway, all of this does not influence the change of the first molecule (molecules) impulse, i.e.,

$$a(\mathbf{p}_2 + a(\mathbf{p}_3 - \mathbf{p}_2) - \mathbf{p}_1) + a(\mathbf{p}_3 - a(\mathbf{p}_3 - \mathbf{p}_2) - \mathbf{p}_1) = a(\mathbf{p}_2 - \mathbf{p}_1) + a(\mathbf{p}_3 - \mathbf{p}_1).$$

Molecules chaotically move in a medium, but their distribution satisfies the following diffusion law: the speeds are distributed by the Maxwell law, and for time τ, the displacements (y_1, y_2, y_3) from the initial position are distributed according to a similar law:

$$\frac{1}{(\pi\tau\vartheta)^{3/2}} \exp\left\{\frac{-y_1^2 - y_2^2 - y_3^2}{\tau\vartheta}\right\}.$$

This also holds for fluids and rigid bodies. Using computer modeling, one can verify that these laws initially obtained from the equilibrium state become steady in several units of time and are related to the ratio of the mean distance between the molecules to the mean-square molecule speed. The quantity ϑ is represented in the form

$$\vartheta = v_T l_0 \vartheta_0 \left(\frac{l}{l_0}\right)$$

and is of the kinetic viscosity dimension.

Correspondingly, the interchange amplitude in (2) has the form

$$f(R, \tau) = \frac{4R^2}{(\tau\vartheta)^{3/2}\sqrt{\pi}} \exp\left\{-\frac{R^2}{\tau\vartheta}\right\}.$$

However, in such an approach, the impulse interchange between molecules is repeatedly taken into account. If we take account of the impulse interchange only one time or take account of the subsequent interchanges with a certain diminution (by c times in each subsequent one), then in the expression for the exponential, there arises the additional summand

$$\exp\left\{-\frac{R^2}{\tau\vartheta} - b\tau\right\}.$$

An attempt to take this into account more correctly starting from the Boltzmann equations was undertaken in [2, 11]. However, for real gases, the Boltzmann equations are not appropriate at all.

For example, the probability of the molecule flight at a distance of $1.5l$ is 4 times greater than that at a distance of $0.75l$. According to Boltzmann, in the first case, the molecule scattering is executed without interaction, whereas in the second case, there is a certain impulse and energy interchange. In reality, under the interaction according to a certain potential and for a large mutual impact speed, in the first case, the molecules approach and go away from each other (respectively interchanging their impulses) but without mutual impact. For a lesser speed, the molecules can go away from each other after their mutual impact repelling from each other as in the second case. But under the assumption that the molecule distribution in speeds is Maxwell and the molecule displacement is executed according to the law presented above but with parameters slightly changed, we can more correctly justify the heuristic reasons for the form of the operator A presented above:

$$a\int_0^\infty \frac{\exp(-b\tau)}{(\pi\tau\vartheta)^{3/2}}\, d\tau \int \exp\left\{\frac{-y_1^2 - y_2^2 - y_3^2}{\tau\vartheta}\right\}(I(t-\tau, x-y-\tau v) - I(t,x))\, dy, \tag{6}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and $dy = dy_1 dy_2 dy_3$. In this case, the symbol of this operator has the form

$$A(k,\omega) = 4\pi a \int\limits_0^\infty \frac{\exp(-b\tau)}{(\pi k^2 \tau\vartheta)^{3/2}}\, d\tau \int\limits_0^\infty \exp\left\{\frac{-r^2}{\tau\vartheta k^2}\right\}\left(\exp(-i\tau\omega)\frac{\sin r}{r} - 1\right) r^2\, dr,$$

where $i\omega$ is the symbol of the operator D.

In fact, the Maxwell law of speed distribution is not even required in the latter case. For any walk without primary direction (all motion directions are equiprobable at the next moment) and for step-length distribution being independent of a given direction, we arrive at the probability of the molecule position at a distance R in time τ expressed by the formula

$$\exp\left\{-\frac{R^2}{\tau\vartheta}\right\}.$$

This allows us to apply a law of the type (6) to random motions, the chaotic motions in a turbulent flow. Depending on the parameters in expression (6), the characteristic time of delay,

$$T_0 = \frac{1}{b},$$

is related to the characteristic length scale R_0 by the formula

$$R_0 = \frac{1}{2}\sqrt{\vartheta T_0}.$$

When the ratio

$$\frac{T_0}{t_*}, \quad t_* = \frac{\|I\|}{\left\|\dfrac{\partial I}{\partial t}\right\|}$$

(t_* is the characteristic time of the impulse variation) is negligibly small, the operator in (6) is expressed through the deviation of the mean over the sphere from the unity. Correspondingly, for small values of R_0/r (r is the characteristic size of the problem), the symbol of the operator has the following approximate form:

$$a\left\{\frac{\sin kR_0}{kR_0} - 1\right\} \sim \frac{aR_0^2}{6}(-k^2) + \frac{aR_0^4}{120}k^4;$$

in this case, the first term corresponds to the Navier–Stokes viscosity, the second to the Barnett viscosity, and so on.

However, under the replacement of the operator with the symbol operator expansion for small k or long waves, we increase the order, violate the type of main operator, and need to assign additional boundary conditions. Nevertheless, as is seen from the calculations presented below, such a (Navier–Stokes) approximation is appropriate for stationary noncapillary flows.

A form of rheologic functions of two arguments that are the coefficients for the impulse (energy) interchange which is more general than that in (6) is obtained when the function from (2) has the form

$$f(R,t) = f_1(R)\exp\left\{-\frac{R^2}{\tau\vartheta}\right\} f_2(t).$$

The first function yields a correction with respect to the spatial coordinate, and the second improves the relaxation-medium characteristics.

In mechanics, the pseudodifferential equations also appeared previously. In the works of the Novosibirsk and Leningrad schools, equations of such a type appeared in the form of nonlocal equations of hydromechanics without dependence of the operator D related to the heredity [9]. In fact, from the deduction of the nonlocal equations from the Boltzmann equations [2, 11], it is obtained that even in the case of a gas, it is better to neglect the nonlocality in space rather than to neglect the relaxation

in time. For a gas, the characteristic size of integral-kernel variation is obtained to be of the order of the free molecule run length, which is essentially less than the characteristic problem sizes.

For speeds close to the speed of sound, the characteristic delay time $T_0 = 1/b$ can be comparable with the flow-parameter variation time. In the authors' opinion, for fluids, this quantity is of the nonspreading drop size and is measured by quantities from several microns up to several millimeters. In this case where there is a turbulence flow in which the vortex sizes approach the channel diameter size, the characteristic distance R_0 approaches the diameter size.

Here, we consider this quantity to be constant and study only the Poiselle laminar flows. The characteristic delay time is measured from fractions of a second up to days (and nights). In the authors' opinion, for rigid bodies, the distance R_0 is of the same order as for fluids, i.e., it is a quantity of the grain size; at the same time, time T_0 is measured from fractions of a month up to tens of years.

In the Moscow schools, scientists focused their attention on the dependence on heredity [6] for rigid bodies exclusively. Also, there was an attempt to explain the "nonlocality" appearance in the stress tensor [7]. However, the latter work is not completely true. In this work, two assumptions that do not hold are made. From these assumptions, under a more correct deduction, a conclusion opposite to that obtained by the author of [7] follows. A more accurate analysis leads to the conclusion that the main reason for the appearance of "nonlocality" in the equations of mechanics is the absence of a local equilibrium in speeds.

When we consider a medium near a certain point in more detail, the speeds are not stabilized near one quantity (corresponding to the speed of the medium at this point), but, on the contrary, the scattering even increases. In this case, the "nonlocality" simultaneously appears both in time and space. This fact is important in studying the flow stability.

In mechanics, one often encounters the situation where, to describe a complex continuous-medium model, new terms and new parameters are introduced. In this case, the parameter that was previously assumed to be a constant quantity well describes the mechanical process (corresponds to experimental data) for one regime and badly does so in another if one does not replace the value of this parameter with a new value (in other regimes). In this case, the replacement of the values of this parameter depending on the process regime is exactly the replacement of the quantity assumed to be constant with a pseudodifferential operator, and the dependence on the regime is the dependence of the operator symbol on wave numbers (frequency and wave vector).

The impossibility of introducing the parameter for a wide range of the process is a consequence of the bad spectral approximation in describing the process itself in the framework of differential equations.

To solve concrete problems, it is necessary to know how to pose the boundary conditions and understand how the "nonlocal" boundary conditions differ from the usual ones. In mathematics, one considers pseudodifferential operators with boundary condition having no physical meaning, although, the necessity (in the physical sense) manifests itself already in the usual partial differential equations in studying discontinuous solutions.

In fact, this also concerns the continuous solutions when the boundary is represented as a discontinuity of the flow parameters. For example, the typical boundary condition

$$v_n = 0$$

has a quite concrete meaning and means the impermeability of the immovable boundary. In contrast to this, the condition of vanishing velocity on the boundary (uniquely reasonable for the Navier–Stokes equations) for a viscous fluid is the invented condition, and the necessity of such a condition itself is a forced condition related to a nonjustified increase (which is not necessary in the framework of the PDO theory) in the order of differential equations for the viscous-medium description.

The nonjustified boundary conditions can lead to unexpected results. For example, in calculating the resonant phenomena in filtration in [3], under false boundary conditions, the authors obtain resonances that are hundreds of times greater than under those for true (equilibrium) boundary

conditions. Under the correct approach to boundary conditions based on the wave reflection, the results are almost independent of the concrete form of the boundary conditions.

In some sense, the wave approach to the boundary conditions is also applied in [5]. Such an approach to pseudodifferential equations and even to the usual differential equations is connected with the consideration of a wave near (in a small neighborhood of) the boundary or a jump, where the parameters can sharply change. When the equations are of divergence form, the jumps are calculated, but possibly not uniquely because of the nonlinearity. In the latter case, one also uses a selection neglecting needless entropy growth criteria, etc.

When the equations are not of divergence form, to compensate for the missing conservation laws, one uses a continuous solution of equations with introduced small dissipation that sharply varies. This way is justified when the result is independent of the arbitrariness of the introduced small dissipative term. However, this (i.e., the independence of the arbitrariness) occurs extremely rarely.

Pseudodifferential operators of the form (6) are convolutions over rays whose integral kernel is a function independent of the ray direction. To define the operator with boundary condition means to find how it is possible to extend the integration to the boundary. The simplest way is to stop the integration when the boundary is attained. Another way is to find a falling wave that reflects in the direction of integration; if there is no such wave (for example, in attaining the boundary at a small angle), then we need to stop the integration. Also, we can extend the integration along the falling ray, reducing the amplitude (integral kernel) as many times (k_0 times) as the reflected wave reduces.

Such an approach is also appropriate for nonlinear pseudodifferential operators. In this case, the boundary conditions are not only the continuous-medium attribute but are also determined by (reflection) properties of the material of which the boundary is composed. This allows us to explain the grinding effects when a small ship with the same boundaries has different resistances depending on whether or not the boundary is well polished.

It is natural that the boundary conditions influence the solution of the problem. As was mentioned above, this especially manifests itself in calculating instability resonances. However, for PDO boundary conditions, there also exists a qualitative distinction that leads to new effects. Owing to the pseudodifferential character of the connection between the stresses and the deformations, creeping and relaxation effects arise. When the wave runs near the material cracks or near the joining of distinct earth layers, residual deformations arise.

This effect is also possible owing to the nonlinearity of the connection between the tensors. However, in this case, this effect manifests itself as a quantity of the second order of smallness. In the media where these tensors are related to pseudodifferential relations, this effect becomes a quantity of the first order of smallness. This is enough for an explanation of the crack-growth effects, the blow-up owing to the accumulated residual deformations near cracks, and earthquakes owing to blow-ups near the joinings of layers because of the accumulated residual deformations after the flow action. The authors intend to present this in more detail in a separate work.

In our problem, A is a differential operator, i.e., together with the quantity

$$I(x, y, z, t),$$

the quantity

$$I(x, y, z, t) + I_0(t)$$

is also a solution for an arbitrary vector $I_0(t)$.

Moreover, this requires the boundary condition

$$I_n|_\Gamma = 0$$

in addition to the boundary impermeability condition, which is the boundary condition for the equation

$$\operatorname{div} v = 0.$$

Correspondingly, in our problem, we can calculate the quantity $I(y)$ under the condition that the velocity vanishes on the boundary (in the general case, with one arbitrary parameter equated to zero) and find the sliding velocity on the boundary (indeterminate arbitrary parameter) from the condition of impulse interchange between the fluid and the immovable boundary.

Note that one possible method for finding the sliding velocity in the Poiselle problem is the equating to zero of the impulse interchange between the fluid and the boundary. This means zero friction on the boundary or a full sliding.

Such an approach, as well as the equating to zero of the velocity on the boundary, is inconsistent. When the fluid moves between two surfaces, it is not possible to simultaneously satisfy the vanishing condition on both boundaries for the velocity as well as for the wall friction.

One method for finding the sliding velocity is the condition that the impulse interchange between the fluid molecules and the boundary be proportional to that between the fluid molecules situated on the boundary:

$$\iint g(|x-y|)(I(x,t)+I_0(t))dxdy = k\int\limits_{y\in\Gamma} g(|x-y|)(I(x,t)-I(y,t))\,dxdy.$$

Here, x runs through the fluid volume and y runs through the body volume or (that of the surface) of the boundary.

Since the integral of $I_0(t)$ on the left-hand side is positive, it is correctly defined.

Here, the impulse interchange between the fluid molecules and the boundary molecules is proportional to that between the fluid molecules with a certain proportionality coefficient. This condition contradicts the condition of fluid velocity vanishing on the body boundary, but the difference from zero of the velocity on the boundary is negligibly small when the body sizes are considerably greater than the characteristic scales of integral kernel variations, which are measured in microns. Therefore, the influence of this condition manifests itself only in capillary flows.

In calculations, this condition is used as follows:

$$I_0 = -\frac{k_0\int g(|x-y|)I(x)\,dx}{\int g(|x-y|)\,dx}, \quad k_0 = 1-k,$$

with k_0 being the same as was used in extending the integration after the ray reflection.

Now let us consider the process of solving concrete problems on the basis of the obtained equations (6). The simplest solutions are Poiselle-type solutions where the flow is stationary (there is no dependence on time) and the velocities are everywhere directed parallel to the x axis (and do not depend on this direction). All other flow parameters, including the pressure gradient, are independent of x. We can compare the obtained solutions with exact solutions of the Navier–Stokes equations for flows between two planes and the flow in a cylindrical tube when the flow parameters depend only on one coordinate y.

Owing to the flow parameters being independent of time and the coordinate x, in calculating the action of the operator A, we can use its form (3):

$$P_0 = \int g(z)(I(x)-I(x-z))\,dz, \qquad g(z) = \frac{1}{\pi^{3/2}}\int\limits_0^\infty \tau^{-3/2}\exp\left\{-\tau-\frac{z^2}{\tau}\right\}d\tau. \tag{7}$$

Here,

$$P_0 = \frac{\Delta p}{\rho v_0 a\tau_0}$$

is the dimensionless pressure overfall per length unit.

Additionally integrating the expressions in (7) in the first coordinate and the inessential second coordinate (or in the angle for motion in the cylinder), we reduce this to the form of an integral equation for functions of one variable. In the plane case, the following function slightly changes:

$$g(z) = \int\limits_0^\infty \exp\left\{ -\tau - \frac{z^2}{\tau} \right\} \frac{d\tau}{\sqrt{\pi\tau}} \le g(0) = 1.$$

In the cylindrical case, we have

$$g_1(z) = \frac{1}{\pi} \int\limits_0^\infty \exp\left\{ -\tau - \frac{z^2}{\tau} \right\} \frac{d\tau}{\tau}$$

$$= \frac{2}{\pi} \int\limits_0^z \frac{\exp\left\{ -\tau - \dfrac{z^2}{\tau} \right\}}{\tau}\, d\tau = \frac{2\exp(-2z)}{\pi} \int\limits_0^\infty \frac{\exp(-2t|z|)\, dt}{\sqrt{t^2+2t}},$$

i.e., the integral kernel has a weak singularity at zero:

$$-\frac{2}{\pi} \ln z + \text{const}\,.$$

Taking account of the reflection of the indicated form in the boundary conditions reduces to integrating in z with reflection in the section with the corresponding factor.

In the subsequent calculations, we consider the reflections with three coefficients 0.2, 0.5, and 0.8. In this case, we admit multiple reflections for flows in thin domains. After sampling of the integral calculation (100 values with respect to the radius or on half of the section), from (7) we obtain a system of equations for calculating the velocity distribution with respect to the section. Owing to the pseudodifferential property of the operator mentioned above, we first find the solution vanishing on the boundary and then, calculating the impulse interchange with the boundary for this solution, we find a correction to the flow velocity.

However, there exists another treatment in which there is no operator pseudodifferential property. For this purpose, we immediately add the impulse interchange with the boundary to relation (7).

When the calculation of the integral over the rays attains the boundary, as previously, we continue the integration over the fluid with reflection having coefficient k_0, but here we take account of the impulse interchange with the boundary (having zero impulse) having the coefficient $1 - k_0$. Physically, such a treatment corresponds to the impulse interchange owing to the diffusion motion of molecules, which lose (interchange the impulses) a part of the impulse not only under the mutual impact but under the reflection from the boundary. In this case, taking account of the impulse interchange with the boundary is distributed among the flow layers taking account of the going away from the boundary but is not concentrated only on a nonzero sliding velocity of a fluid near the boundary. In the calculations presented, precisely this approach was used.

One more approach to boundary conditions is possible. We can calculate the impulse interchange between the fluid layers and the boundary using another integral kernel and integrating the impulse interchange for each fluid layer on the bo undary, more precisely, outside the fluid-flow domain. Such a way corresponds to the interaction (of capillary type) between the fluid and boundary body.

In concrete cases, the boundary conditions can be respectively improved so that the solutions better correspond to the experiment. We do not consider such an approach in this work.

For small diameters, we can obtain a certain property of solutions not even using calculations. When the diameter is small as compared with the characteristic size, the entries of the matrix of the system tend to a constant quantity, i.e., in the limit, the rank of the matrix is equal to 1. The determinant of the matrix respectively tends to zero. This leads to a bad conditionality of the numerical solution

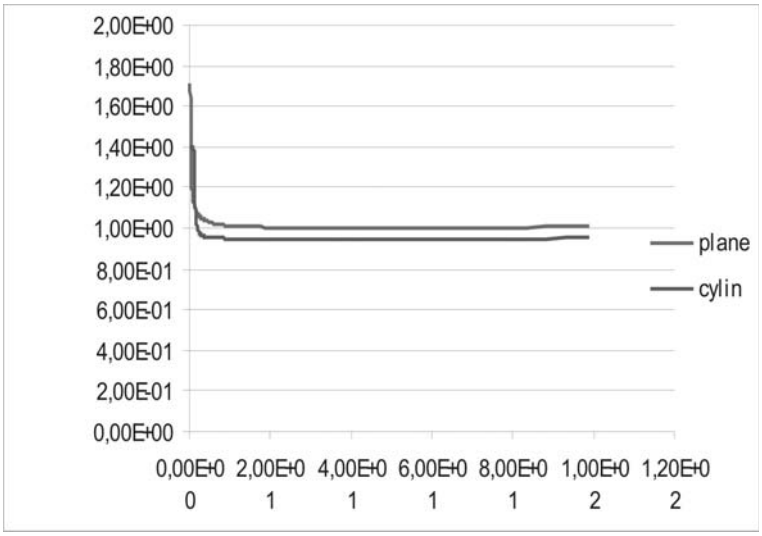

Fig. 1. Dependence of the quantity s on the radius.

for small radii. In this case, we obtain that the solution is almost independent of the distance of the axis to the boundary, i.e., the velocities are almost the same for all layers. This property is practically independent of the integral kernel.

To describe the profile type by one number, we introduce the number

$$s = \frac{x_*}{r}a,$$

where x_* is the coordinate (measured from the flow symmetry axis); in this case, the velocity is equal to the mean-expenditure fluid velocity, r is the section radius, and $a = \sqrt{3}$ in the plane case and $a = \sqrt{2}$ in the cylindrical case. This quantity is uniquely found when the velocity profile monotonically depends on the distance from the symmetry axis.

In the case of an arbitrary continuous profile, we can take the first (minimal) value of x_*. For example, for the velocity profile given by the formula

$$u(x) = b - cx^d$$

with fixed parameters b, c, and d, the number s depends only on d. In this case,

$$s = \frac{\sqrt{3}}{\sqrt[d]{d+1}}$$

in the plane case and

$$s = \sqrt{2}\sqrt[d]{\frac{2}{d+2}}$$

in the cylindrical case.

We choose the normalization so that for the parabolic profile ($d = 2$), this number is equal to 1. For a constant profile ($d \to \infty$), this number is equal to $\sqrt{3}$ in the plane case and $\sqrt{2}$ in the cylindrical case.

Figure 1 (the graph of the dependence of the number s on the radius) demonstrates the passage from the constant profile for small r to the parabolic profile for large r.

As is seen from Fig. 1, in the plane case, this quantity tends to 1 to the left ($d > 2$) slightly better. In the cylindrical case, the right convergence ($d < 2$) is slightly worse. In fact, for small values of r, the mean-expenditure velocity does not tend to zero proportionally to the square of the radius as in the

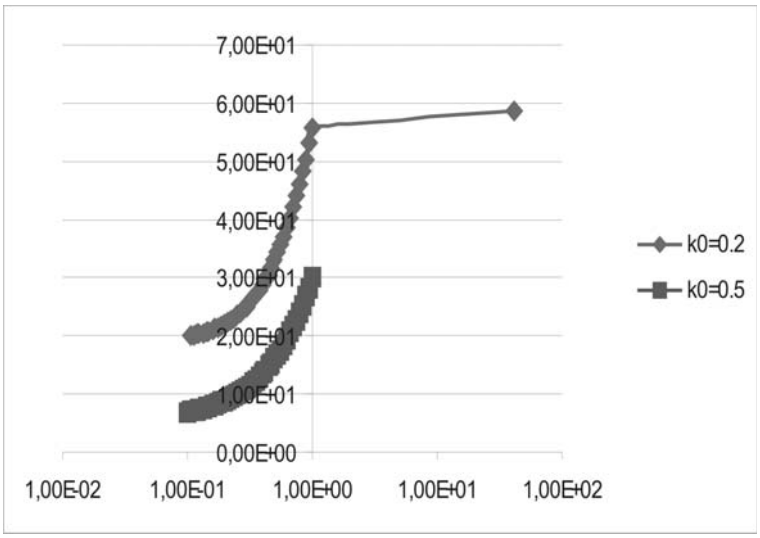

Fig. 2. Dependence of the mean-expenditure velocity on r for $r < 1$.

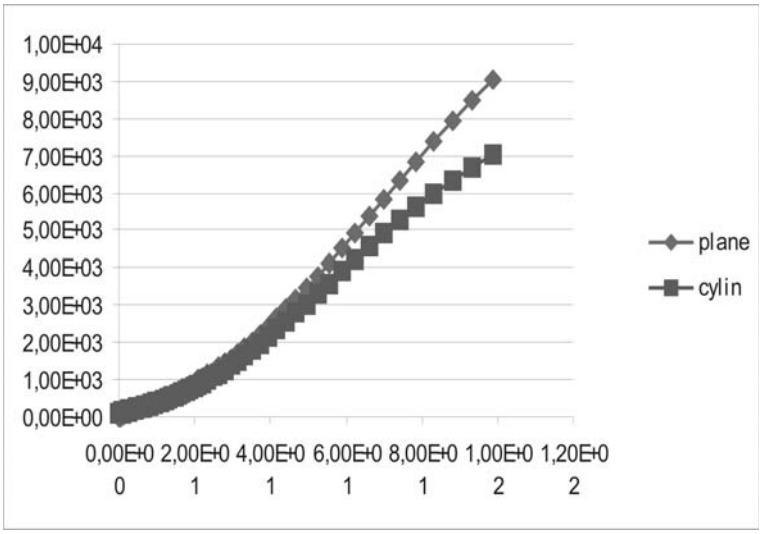

Fig. 3. Mean-expenditure velocity depending on r.

case of Navier–Stokes solutions, but it tends to a constant nonzero value depending on the geometry and the reflection coefficient k_0. This is demonstrated in Fig. 2.

For this reason, the qualitative feature of fluid flow in thin capillary tubes (when the fluid viscosity is seemingly lowered by many times) manifests itself.

The latter effect was experimentally detected in radioelectronics in studying the possibility of water cooling of an electronic board with flow in microchannels, and it has not been satisfactorily explained up to now. It was observed that the resistance is lowered by hundreds and thousands times as compared with the Poiselle one, which is calculated according to Navier–Stokes. For example, for the cases presented in the graph (plane case), when the radius decreases from 100 to 0.1, the velocity decreases not by one million times but approximately by 500 and 1500 times. For large values of r, the mean velocity enters the growth regime, is proportional to the square of the radius, and is practically independent of the reflection coefficient. For example, comparing the values of the mean-expenditure

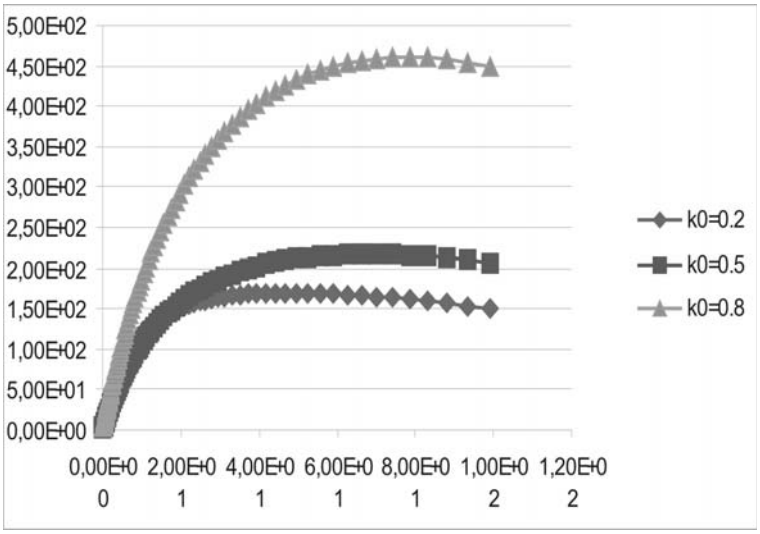

Fig. 4. Fluid velocity on the boundary.

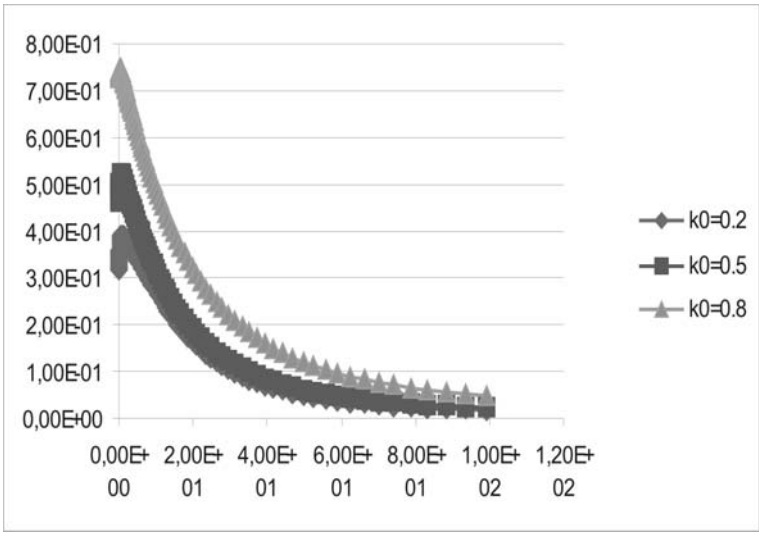

Fig. 5. Ratio of the velocity on the boundary to the fluid mean-expenditure velocity.

velocity for the values of the radii $r = 70.1$ and $r = 99$ (which differ approximately by $\sqrt{2}$ times), we obtain the following values: 5830 and 9050 for $k_0 = 0.2$ in the plane case and 5820 and 9060 for $k_0 = 0.5$.

The corresponding values for the cylindrical case are as follows: 4900, 7010 ($k_0 = 0.2$) and 4920, 7040 ($k_0 = 0.5$). This means that even for sufficiently large radii, there is no exact entrance to the quadratic dependence of the mean-expenditure velocity on the radius.

At the same time, the dependence of the velocity on the reflection coefficient disappears for such large r, and, therefore, an analog of Fig. 2 cannot be demonstrated for large r.

In Fig. 3, we present the graph of the dependence of this velocity on r for the cylindrical and plane cases.

The distinction of the flow from the Navier–Stokes regime starts from $r < 100$. It was experimentally detected that for the water flow, the distinction begins to manifest itself from $r = 100\ \mu$. Respectively,

the characteristic quantity with respect to which the radii are dimensionless is a quantity of the order of one micron.

In stationary flows, the effects of delay, relaxation, and "memory," which can be essential for nonstationary flows, especially in the stability study, do not manifest themselves. For fluids of the petroleum type, this characteristic size is probably substantially larger than that for water.

Correspondingly, a flow in layer holes can be substantially different from the Navier–Stokes flow.

In Fig. 4, we present the dependence of the velocity on the boundary for various values of the reflection coefficient (plane case).

Although the maximum value of the velocity on the boundary is attained for large radii of order 50, the ratio of this velocity to the mean-expenditure rapidly decays to zero. This is shown in Fig. 5.

Acknowledgments. The authors express their gratitude to O. R. Ganiev for a useful discussion of the results.

This work was supported by the Russian Foundation for Basic Research, Grant No. 08–01–00231-a.

REFERENCES

1. R. R. Aidagulov and M. V. Shamolin, "A phenomenological approach to finding interphase forces," *Dokl. Ross. Akad. Nauk*, **412**, No. 1, 44–47 (2007).
2. R. R. Aidagulov and M. V. Shamolin, "A general spectral approach to continuous medium dynamics," *Sovremennaya Matematika. Fundamental'nye Napravleniya*, **23**, 52–70 (2007).
3. R. F. Ganiev, L. E. Ukrainskii, and O. R. Ganiev "Resonant filtration flows in a porous medium saturated with a fluid," *Dokl. Ross. Akad. Nauk*, **412**, No. 1 (2007).
4. F. John, *Plane Waves and Spherical Means Applied to Partial Differential Equations* [Russian translation], IL, Moscow (1958).
5. A. G. Kulikovskii and E. I. Sveshnikova, *Nonlinear Waves in Elastic Media* [in Russian], Moskovskii Lizei, Moscow (1998).
6. A. A. Lokshin and Yu. V. Suvorova, *Mathematical Theory of Wave Propagation in Media With Memory* [in Russian], MGU, Moscow (1982).
7. A. N. Osiptsov, "On accounting for volume finiteness and hydrodynamic interaction of particles in gas suspensions," *Dokl. Akad. Nauk SSSR*, **275**, No. 5, 1073–1076 (1984).
8. R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* [Russian translation], Reg. Khaotic Dinamik, Moscow–Izhevsk (2007).
9. V. Ya. Rudyak, *Statistical Theory of Dissipative Processes in Gases and Fluids* [in Russian], Nauka, Novosibirsk (1987).
10. M. Sato, "Theory of hyperfunctions. I, II," *J. Fac. Shi. Univ. Tokyo, Sect. I*, 139–193 (1959); 387–437 (1960).
11. M. V. Shamolin, *Some Problems of Differential and Topological Diagnosis* [in Russian], 2nd Revised and Supplemented Edition, Ekzamen, Moscow (2007), pp. 240–281.
12. M. A. Shubin, *Pseudodifferential Operators and Spectral Theory* [in Russian], Nauka, Moscow (1979).

R. R. Aidagulov
Institute of Mechanics,
Moscow State University, Moscow, Russia
M. V. Shamolin
Institute of Mechanics,
Moscow State University, Moscow, Russia
E-mail: shamolin@imec.msu.ru