KYUNGPOOK Math. J. 49(2009), 545-555

Multivalent Harmonic Uniformly Starlike Functions

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Abstract. In this paper, we investigate a generalized family of complex-valued harmonic functions that are multivalent, sense-preserving, and are associated with k-uniformly harmonic functions in the unit disk. The results obtained here include a number of known and new results as their special cases.

1. Introduction

A harmonic function f defined in a simply connected complex domain $D \subset \mathbb{C}$ can be expressed by $f(z) = h(z) + \overline{g(z)}, z \in D$. We call h the analytic part and g the co-analytic part of f. If the co-analytic part of f is zero, then f reduces to the analytic case. The mapping $z \to f(z)$ is sense-preserving and locally one-to-one in D if and only if the Jacobian of f is positive, that is, if and only if

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0,\ z \in D.$$

Denote by H the family of functions $f = h + \overline{g}$ which are harmonic, sense-preserving and univalent in the open unit disk $\triangle = \{z : |z| < 1\}$ with

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

The class H was defined and studied by Clunie and Sheil-Small [10]. Also, see

* Corresponding author.
Received July 17, 2008; accepted September 12, 2008.
2000 Mathematics Subject Classification: Primary 30C45, Secondary 30C50, 30C55.
Key words and phrases: harmonic functions, multivalent harmonic functions, convex combinations, distortion bounds, uniformly harmonic starlike.

excellent monograph entitled, 'Harmonic mapping in the plane' by Duren [11]. For a fixed positive integer $m \geq 1$, let $H(m)$ denote the family of all multivalent harmonic functions $f = h + \overline{g}$ which are sense-preserving in $\triangle$ and are of the form

$$(1.2) \qquad h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \ |b_m| < 1.$$

Recent interest in the study of multivalent harmonic functions in the plane prompted the publication of several articles, such as [3], [4], [5], [9] and [15]. Note that $H(1) \equiv H$. We say that $f \in H(m)$ is a multivalent harmonic starlike of order $\beta, 0 \leq \beta < 1$ if f satisfies the condition

$$\frac{\partial}{\partial \theta}(\arg(f(re^{i\theta}))) \geq m\beta$$

for each $z = re^{i\theta}, 0 \leq \theta < 2\pi$ and $0 \leq r < 1$. Denote this class of multivalent harmonic starlike functions of order β by $S_H^*(m, \beta)$. The classes $S_H^*(1, \beta)$ and $S_H^*(m, \beta)$ were studied in [3], [5] and [15].

Let $G_H(k, m, \beta, t)$ be the family of functions f in $H(m)$ satisfying the inequality

$$(1.3) \qquad Re\left(\frac{zf'(z)}{z'[(1-t)z^m + tf(z)]}\right) \geq k\left|\frac{zf'(z)}{z'[(1-t)z^m + tf(z)]} - m\right| + m\beta,$$

for some $k, (0 \leq k < \infty), m(m \geq 1), \beta(0 \leq \beta < 1), t(0 \leq t \leq 1), z \in \triangle$ and where

$$z' = \frac{\partial}{\partial \theta}(z = re^{i\theta}), \ f'(z) = \frac{\partial}{\partial \theta} f(re^{i\theta}) = i(zh'(z) - \overline{zg'(z)}).$$

Using the fact that $Rew > k|w - m| + m\beta \Leftrightarrow Re[(ke^{i\theta} + 1)w - kme^{i\theta}] \geq m\beta$, it follows from the condition (1.3) that f is in $G_H(k, m, \beta, t)$ if and only if

$$(1.4) \qquad Re\left[\frac{(ke^{i\theta} + 1)(zh'(z) - \overline{zg'(z)})}{(1-t)z^m + t(h(z) + \overline{g(z)})} - kme^{i\theta}\right] \geq m\beta.$$

The set $G_H(k, m, \beta, t)$ is a comprehensive family that contains several previously studied subclasses of $H(m)$ or H. For example,

$$G_H(0, m, \beta, 1) \equiv S_H^*(m, \beta); [3], [15]$$
$$G_H(0, m, 0, 1) \equiv S_H^*(m, 0); [5]$$
$$G_H(0, 1, \beta, 1) \equiv S_H^*(1, \beta) \equiv S_H^*(\beta); \ [16]$$
$$G_H(0, 1, 0, 1) \equiv S_H^*(0) \equiv S_H^*; \ [24], [25]$$

$$G_H(0,m,\beta,0) \equiv R_H(m,\beta) := \left\{ f \in H(m) : Re\left(\frac{f'(z)}{\frac{\partial}{\partial\theta}(z^m)}\right) \geq m\beta,\ 0 \leq \beta < 1 \right\};\ [4]$$

$$G_H(0,1,\beta,0) \equiv R_H(1,\beta) \equiv R_H(\beta);\ [2]$$

$$G_H(1,m,\beta,1) \equiv G_H(m,\beta) := \left\{ f \in H(m) : Re\left((1+e^{i\alpha})\frac{zf'(z)}{z'f(z)} - me^{i\alpha}\right) \geq m\beta \right\};\ [17]$$

$$G_H(1,1,\beta,1) \equiv G_H(1,\beta) \equiv G_H(\beta);\ [23]$$

$$G_H(k,1,\beta,t) \equiv G_H(k,\beta,t).\ [1]$$

Let $S(m)$ be the well known family of functions h in $H(m)$ that are analytic and univalent in $\triangle$ and are of the form $h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, z \in \triangle$. We observe that $S(1) \subset S, S(m) \subset H(m)$, and $G_s(k,m,\beta,t) \subset G_H(k,m,\beta,t)$. Also the family $G_s(k,m,\beta,t)$ contains several previously studied subclasses of analytic functions in $\triangle$. For example

$$G_s(0,1,\beta,t) := \left\{ h \in S : Re\frac{zh'(z)}{(1-t)z + th(z)} > \beta \right\};\ [7],[20]$$

$$G_s(1,1,\beta,1) := \left\{ h \in S : Re\left(\frac{zh'(z)}{h(z)}\right) \geq \left|\frac{zh'(z)}{h(z)} - 1\right| + \beta \right\};\ [8]$$

$$G_s(k,1,0,1) \equiv k - ST := \left\{ h \in S : Re\left(\frac{zh'(z)}{h(z)}\right) \geq k\left|\frac{zh'(z)}{h(z)} - 1\right| \right\};\ [18]$$

$$G_s(1,1,0,1) \equiv UST \equiv 1 - ST;\ [19],[21],[22].$$

Finally, we define the family

$$G_{\overline{H}}(k,m,\beta,t) := TH(m) \cap G_H(k,m,\beta,t),$$

where $TH(m), m \geq 1$ denote the class of functions $f = h + \overline{g}$ in $H(m)$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1},\ z \in \triangle. \tag{1.5}$$

The class $TH(m)$ was first studied in [5].

In this paper, we investigate coefficient conditions, extreme points, and distortion bounds for functions in the families $G_H(k,m,\beta,t)$ and $G_{\overline{H}}(k,m,\beta,t), m \geq 1$. We also examine their convolution and convex combination properties. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of S, H, $S(m)$, and $H(m)$ as listed previously in this section.

2. Main results

We first prove sufficient coefficient conditions for harmonic functions in $G_H(k,m,\beta,t)$. These conditions are shown to be necessary for the functions in $G_{\overline{H}}(k,m,\beta,t)$.

Theorem 1. *Let $f = h + \overline{g}$ be so that h and g are given by* (1.2)*. If*

$$\sum_{n=2}^{\infty} \frac{(n+m-1)(k+1) - tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{(n+m-1)(k+1) + tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|b_{n+m-1}| \leq \frac{1}{2}, \tag{2.1}$$

when $k \geq 0, m \geq 1, 0 \leq \beta < 1$ and $0 \leq t \leq 1$, then $f \in G_H(k,m,\beta,t)$.

Proof. Suppose that (2.1) holds. It suffices to prove that $Re\{A(z)/B(z)\} > 0$, where

$$A(z) = (ke^{i\theta}+1)(zh'(z) - \overline{zg'(z)}) - m(ke^{i\theta}+\beta)((1-t)z^m + th(z) + \overline{tg(z)}),$$
$$B(z) = (1-t)z^m + th(z) + \overline{tg(z)}.$$

Using the fact that $Re\omega \geq 0$ if and only if $|1+\omega| \geq |1-\omega|$ it suffices to show that

$$|A(z)+B(z)| - |A(z)-B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2), we obtain

$$\begin{aligned} & |A(z)+B(z)| - |A(z)-B(z)| \\ = & \Bigg|(m(1-\beta)+1)z^m \\ & + \sum_{n=2}^{\infty}[((n+m-1) - m\beta t + t) + ke^{i\theta}((n+m-1)-mt)]a_{n+m-1}z^{n+m-1} \\ & - \sum_{n=1}^{\infty}[((n+m-1) + m\beta t - t) + ke^{i\theta}((n+m-1)+mt)]\overline{b}_{n+m-1}(\overline{z})^{n+m-1}\Bigg| \\ & - \Bigg|(m(1-\beta)-1)z^m \\ & - \sum_{n=2}^{\infty}[((n+m-1) - m\beta t - t) + ke^{i\theta}((n+m-1)-mt)]a_{n+m-1}z^{n+m-1} \\ & - \sum_{n=1}^{\infty}[((n+m-1) + m\beta t + t) + ke^{i\theta}((n+m-1)+mt)]\overline{b}_{n+m-1}(\overline{z})^{n+m-1}\Bigg| \end{aligned}$$

$$\begin{aligned}
&\geq \big(m(1-\beta)+1-|m(1-\beta)-1|\big)|z|^m \\
&\quad\times\Big\{1-\sum_{n=2}^{\infty}\frac{2[(n+m-1)(k+1)-tm(k+\beta)]}{m(1-\beta)+1-|m(1-\beta)-1|}|a_{n+m-1}||z^{n-1}| \\
&\quad-\sum_{n=1}^{\infty}\frac{2[(n+m-1)(k+1)+tm(k+\beta)]}{m(1-\beta)+1-|m(1-\beta)-1|}|b_{n+m-1}||z^{n-1}|\Big\} \\
&\geq \big(m(1-\beta)+1-|m(1-\beta)-1|\big)|z|^m \\
&\quad\times\Big\{1-\sum_{n=2}^{\infty}\frac{2[(n+m-1)(k+1)-tm(k+\beta)]}{m(1-\beta)+1-|m(1-\beta)-1|}|a_{n+m-1}| \\
&\quad-\sum_{n=1}^{\infty}\frac{2[(n+m-1)(k+1)+tm(k+\beta)]}{m(1-\beta)+1-|m(1-\beta)-1|}|b_{n+m-1}|\Big\}.
\end{aligned}$$

This last expression is non-negative by the hypothesis and so the proof is complete. The functions

$$\begin{aligned}
f(z) = z^m &+ \sum_{n=2}^{\infty}\frac{m(1-\beta)+1-|m(1-\beta)-1|}{2[(n+m-1)(k+1)-tm(k+\beta)]}x_{n+m-1}z^{n+m-1} \\
&+ \sum_{n=1}^{\infty}\frac{m(1-\beta)+1-|m(1-\beta)-1|}{2[(n+m-1)(k+1)+tm(k+\beta)]}\overline{y}_{n+m-1}(\overline{z})^{n+m-1},
\end{aligned} \tag{2.3}$$

where $\sum_{n=2}^{\infty}|x_{n+m-1}| + \sum_{n=1}^{\infty}|y_{n+m-1}| = 1$, show that the coefficient bound given by (2.1) is sharp. □

Corollary 1. *Let $f = h + \overline{g}$ be so that h and g are given by* (1.2). *Also, let $m \geq 1/(1-\beta), 0 \leq \beta < 1$ and $0 \leq t \leq 1$. If the condition*

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n+m-1)(k+1)-tm(k+\beta)]|a_{n+m-1}| \\
&+\sum_{n=1}^{\infty}[(n+m-1)(k+1)+tm(k+\beta)]|b_{n+m-1}| \leq 1
\end{aligned}$$

is satisfied, then $f \in G_H(k,m,\beta,t)$.

Corollary 2. *Let $f = h + \overline{g}$ be so that h and g are given by* (1.2). *Also, suppose $1 \leq m \leq 1/(1-\beta), 0 \leq \beta < 1$ and $0 \leq t \leq 1$. If the condition*

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n+m-1)(k+1)-tm(k+\beta)]|a_{n+m-1}| \\
&+\sum_{n=1}^{\infty}[(n+m-1)(k+1)+tm(k+\beta)]|b_{n+m-1}| \leq m(1-\beta)
\end{aligned}$$

holds, then $f \in G_H(k,m,\beta,t)$.

Theorem 2. *Let* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.5). *Also, let* $k \geq 0$, $0 \leq t \leq 1$ *and* $0 \leq \beta < 1$. *Furthermore,*
(i) *if* $1 \leq m \leq 1/(1-\beta)$, *then* $f \in G_{\overline{H}}(k,m,\beta,t)$ *if and only if*

$$\sum_{n=2}^{\infty}[(n+m-1)(k+1) - tm(k+\beta)]|a_{n+m-1}| + \sum_{n=1}^{\infty}[(n+m-1)(k+1) + tm(k+\beta)]|b_{n+m-1}| \leq m(1-\beta);$$

(ii) *if* $m(1-\beta) \geq 1$, *then* $f \in G_{\overline{H}}(k,m,\beta,t)$ *if and only if*

$$\sum_{n=2}^{\infty}[(n+m-1)(k+1) - tm(k+\beta)]|a_{n+m-1}| + \sum_{n=1}^{\infty}[(n+m-1)(k+1) + tm(k+\beta)]|b_{n+m-1}| \leq 1. \tag{2.4}$$

Proof. In view of Corollary 1 and Corollary 2, it suffices to show that $f \in G_{\overline{H}}(k,m,\beta,t)$ if the condition (2.4) does not hold. We note that a necessary and sufficient condition for $f = h + \overline{g}$, given by (1.5), to be in $G_{\overline{H}}(k,m,\beta,t)$ is that the coefficient condition (1.4) to be satisfied. Equivalently, we must have

$$Re\left\{\frac{(ke^{i\theta}+1)(zh'(z) - \overline{zg'(z)}) - m(ke^{i\theta}+\beta)((1-t)z^m + th(z) + \overline{tg(z)})}{(1-t)z^m + th(z) + \overline{tg(z)}}\right\} \geq 0. \tag{2.5}$$

Upon choosing the value of z on the positive real axis and using $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\begin{aligned}&\{m(1-\beta) - \sum_{n=2}^{\infty}((n+m-1)(k+1) - mt(k+\beta))|a_{n+m-1}|r^{n-1}\\ &- \sum_{n=1}^{\infty}((n+m-1)(k+1) + mt(k+\beta))|b_{n+m-1}|r^{n-1}\}\\ &\times\{1 - \sum_{n=2}^{\infty}|a_{n+m-1}|r^{n-1} + t\sum_{n=1}^{\infty}|b_{n+m-1}|r^{n-1}\}^{-1} \geq 0.\end{aligned} \tag{2.6}$$

If condition (2.5) does not hold then the numerator of (2.6) is negative for r sufficiently close to 1 because of conditions (i) or (ii). Thus there exits $z_0 = r_0 > 1$, for which the left side of (2.6) is negative. This contradicts the required condition for $f \in G_{\overline{H}}(k,m,\beta,t)$. Using definition (1.3), and according to the arguments given in [2] and [16], we obtain distortion bounds for the functions in $G_{\overline{H}}(k,m,\beta,t)$ in

Theorem 3 and extreme points of the closed convex hulls of $G_{\overline{H}}(k,m,\beta,t)$, denoted by clco$G_{\overline{H}}(k,m,\beta,t)$, in Theorem 4. The proofs of Theorems 3 and 4 are similar to the corresponding results in [2] and [16] and so are omitted. □

Theorem 3. *If $f \in G_{\overline{H}}(k,m,\beta,t)$, then for $|z| = r < 1$*

$$|f(z)| \leq \begin{cases} (1+|b_m|)r^m + \left(\frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right) r^{m+1}, \\ \hfill \text{if } \ m(1-\beta)\leq 1 \\ (1+|b_m|)r^m + \left(\frac{1}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right) r^{m+1}, \\ \hfill \text{if } \ m(1-\beta)\geq 1 \end{cases}$$

$$|f(z)| \geq \begin{cases} (1-|b_m|)r^m - \left(\frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right) r^{m+1}, \\ \hfill \text{if } \ m(1-\beta)\leq 1 \\ (1-|b_m|)r^m - \left(\frac{1}{(m+1)(k+1)-tm(k+\beta)} - \frac{m(k+1)+tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m|\right) r^{m+1}, \\ \hfill \text{if } \ m(1-\beta)\geq 1 \end{cases}$$

Corollary 3. *If $f \in G_{\overline{H}}(k,m,\beta,t)$, then*

$$\left\{\omega : |\omega| < \begin{bmatrix} 1-\frac{m(1-\beta)}{(m+1)(k+1)-tm(k+\beta)} - \frac{(k+1)-2tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m| \ \text{if } m(1-\beta)\leq 1 \\ 1-\frac{1}{(m+1)(k+1)-tm(k+\beta)} - \frac{(k+1)-2tm(k+\beta)}{(m+1)(k+1)-tm(k+\beta)}|b_m| \ \text{if } m(1-\beta)\geq 1 \end{bmatrix}\right\} \subset f(\triangle).$$

Theorem 4. *A function f is in clco$G_{\overline{H}}(k,m,\beta,t)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty}(X_{n+m-1}h_{n+m-1} + Y_{n+m-1}g_{n+m-1})$$

where

$$h_m(z) = z^m,$$

$$h_{n+m-1}(z) = \begin{cases} z^m - \dfrac{m(1-\beta)}{(n+m-1)(k+1)-tm(k+\beta)}z^{n+m-1}(n=2,3,4,\cdots), \\ \hfill \text{if } \ m(1-\beta)\leq 1 \\ z^m - \dfrac{1}{(n+m-1)(k+1)-tm(k+\beta)}z^{n+m-1}(n=2,3,4,\cdots), \\ \hfill \text{if } \ m(1-\beta)\geq 1 \end{cases}$$

$$g_{n+m-1}(z) = \begin{cases} z^m + \dfrac{m(1-\beta)}{(n+m-1)(k+1)+tm(k+\beta)}(\overline{z})^{n+m-1}(n=1,2,3,\cdots), \\ \qquad\qquad\qquad\qquad\qquad\qquad if \ \ m(1-\beta) \leq 1 \\ z^m + \dfrac{1}{(n+m-1)(k+1)+tm(k+\beta)}(\overline{z})^{n+m-1}(n=1,2,3,\cdots), \\ \qquad\qquad\qquad\qquad\qquad\qquad if \ \ m(1-\beta) \geq 1 \end{cases}$$

and

$$\sum_{n=1}^{\infty}(X_{n+m-1}+Y_{n+m-1})=1, \ \ X_{n+m-1}\geq 0 \ and \ Y_{n+m-1}\geq 0.$$

In particular, the extreme points of $G_{\overline{H}}(k,m,\beta,t)$ *are* $\{h_{n+m-1}\}$ *and* $\{g_{n+m-1}\}$.

In the next two theorems, we prove that the class $G_{\overline{H}}(k,m,\beta,t)$ is invariant under convolution and convex combinations of its members. We first recall that for harmonic functions

$$f(z) = z^m - \sum_{n=2}^{\infty}|a_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty}|b_{n+m-1}|(\overline{z})^{n+m-1} \tag{2.7}$$

and

$$F(z) = z^m - \sum_{n=2}^{\infty}|A_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty}|B_{n+m-1}|(\overline{z})^{n+m-1} \tag{2.8}$$

in $TH(m)$, the convolution of f and F is defined as

$$\begin{aligned} & (f*F)(z) = f(z)*F(z) \\ = \ & z^m - \sum_{n=2}^{\infty}|a_{n+m-1}A_{n+m-1}|z^{n+m-1} + \sum_{n=1}^{\infty}|b_{n+m-1}B_{n+m-1}|(\overline{z})^{n+m-1}. \end{aligned}$$

Using this definition, we first show that $G_{\overline{H}}(k,m,\beta,t)$ is closed under convolution.

Theorem 5. *For* $0 \leq \alpha \leq \beta < 1$, *let* $f \in G_{\overline{H}}(k,m,\beta,t)$ *and* $F \in G_{\overline{H}}(k,m,\alpha,t)$, *then*

$$f*F \in G_{\overline{H}}(k,m,\beta,t) \subset G_{\overline{H}}(k,m,\alpha,t).$$

Proof. Let,$f, F \in G_{\overline{H}}(k,m,\beta,t)$ be given by (2.7) and (2.8), respectively. Note that the coefficients of f and F must satisfy the conditions similar to the inequality (2.4). For $F \in G_{\overline{H}}(k,m,\alpha,t)$ we observe that $|A_{n+m-1}| \leq 1$ and $|B_{n+m-1}| \leq 1$.

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(n+m-1)(k+1)-tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|a_{n+m-1}||A_{n+m-1}| \\
&\quad+\sum_{n=1}^{\infty}\frac{(n+m-1)(k+1)+tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|b_{n+m-1}||B_{n+m-1}| \\
\leq\; &\sum_{n=2}^{\infty}\frac{(n+m-1)(k+1)-tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|a_{n+m-1}| \\
&\quad+\sum_{n=1}^{\infty}\frac{(n+m-1)(k+1)+tm(k+\beta)}{m(1-\beta)+1-|m(1-\beta)-1|}|b_{n+m-1}|.
\end{aligned}$$

The right hand side of the above inequality is bounded by 1 because $f \in G_{\overline{H}}(k,m,\beta,t)$. Therefore the result follows.
Finally, we determine the convex combination of the members of $G_{\overline{H}}(k,m,\beta,t)$. □

Theorem 6. *The class $G_{\overline{H}}(k,m,\beta,t)$ is closed under convex combination.*

Proof. For $i=1,2,3,\cdots$ suppose $f_i \in G_{\overline{H}}(k,m,\beta,t)$, where f_i are given by

$$f_i(z)=z^m-\sum_{n=2}^{\infty}|a_{i_{n+m-1}}|z^{n+m-1}+\sum_{n=1}^{\infty}|b_{i_{n+m-1}}|(\overline{z})^{n+m-1}.$$

For $\sum_{i=1}^{\infty} t_i=1, 0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_i f_i(z)=z^m-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i_{n+m-1}}|\right)z^{n+m-1}+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i_{n+m-1}}|\right)(\overline{z})^{n+m-1}.$$

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n+m-1)(k+1)-tm(k+\beta)]|a_{i_{n+m-1}}| \\
&\qquad+\sum_{n=1}^{\infty}[(n+m-1)(k+1)+tm(k+\beta)]|b_{i_{n+m-1}}| \\
&\qquad\leq\begin{cases} m(1-\beta), & \text{if } \; m(1-\beta)\geq 1,\\ 1, & \text{if } \; m(1-\beta)\leq 1,\end{cases}
\end{aligned}$$

it follows from the above equation

$$\begin{aligned}
&\sum_{n=2}^{\infty}[(n+m-1)(k+1)-tm(k+\beta)]\sum_{i=1}^{\infty}t_i|a_{i_{n+m-1}}| \\
&\qquad\qquad + \sum_{n=1}^{\infty}[(n+m-1)(k+1)+tm(k+\beta)]\sum_{i=1}^{\infty}t_i|b_{i_{n+m-1}}| \\
= &\sum_{i=1}^{\infty}t_i\Bigg\{\sum_{n=2}^{\infty}[(n+m-1)(k+1)-tm(k+\beta)]|a_{i_{n+m-1}}| \\
&\qquad\qquad + \sum_{n=1}^{\infty}[(n+m-1)(k+1)+tm(k+\beta)]|b_{i_{n+m-1}}|\Bigg\} \\
&\qquad\qquad \le \begin{cases} m(1-\beta)\displaystyle\sum_{i=1}^{\infty}t_i = m(1-\beta), & \text{if} \quad m(1-\beta)\le 1, \\ \displaystyle\sum_{i=1}^{\infty}t_i = 1, & \text{if} \quad m(1-\beta)\ge 1, \end{cases}
\end{aligned}$$

and so $\sum_{i=1}^{\infty}t_if_i(z)\in G_{\overline{H}}(k,m,\beta,t)$. □

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