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PLANAR HARMONIC UNIVALENT AND RELATED MAPPINGS

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ABSTRACT. The theory of harmonic univalent mappings has become a very popular research topic in recent years. The aim of this expository article is to present a guided tour of the planar harmonic univalent and related mappings with emphasis on recent results and open problems and, in particular, to look at the harmonic analogues of the theory of analytic univalent functions in the unit disc.

Key words and phrases: Planar harmonic mappings, Orientation preserving harmonic functions, Univalent functions, Conformal mappings, Geometric function theory, Harmonic Koebe mapping, Harmonic starlike functions, Harmonic convex functions, Multivalent harmonic functions, Meromorphic harmonic functions, Sakaguchi-type harmonic functions.

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1. INTRODUCTION

Planar harmonic univalent mappings have long been used in the representation of minimal surfaces. For example, E. Heinz [34] in 1952 used such mappings in the study of the Gaussian curvature of nonparametric minimal surfaces over the unit disc. For more recent results and references, one may see [70]. Such mappings and related functions have applications in the seemingly diverse fields of Engineering, Physics, Electronics, Medicine, Operations Research, Aerodynamics, and other branches of applied mathematical sciences. For example, harmonic and meromorphic functions are critical components in the solutions of numerous physical problems, such as the flow of water through an underground aquifer, steady-state temperature distribution, electrostatic field intensity, the diffusion of, say, salt through a channel.

Harmonic univalent mappings can be considered as close relatives of conformal mappings. But, in contrast to conformal mappings, harmonic univalent mappings are not at all determined (up to normalizations) by their image domains. Another major difference is that a harmonic univalent mapping can be constructed on an interval of the boundary of the open unit disc. On

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the other hand, because of the natural analogy to Fourier series, harmonic mappings have a two-folded series structure consisting of an 'analytic part' which is a power series in the complex variable z, and a 'co-analytic part' which is a power series in the complex conjugate of z. In view of such fascinating properties, a study of harmonic univalent mappings is promising and important.

Harmonic univalent mappings have attracted the serious attention of complex analysts only recently after the appearance of a basic paper by Clunie and Sheil-Small [22] in 1984. Hengartner and Schober ([35], [37]) in 1986 made efforts to find an appropriate form of the Riemann Mapping Theorem for harmonic mappings. Their theory is based on the model provided by the theory of quasiconformal mappings. The works of these researchers and several others (e.g. see [36], [51], [52], [63], [64], [67]) gave rise to several fascinating problems, conjectures, and many tantalizing but perplexing questions. Though several researchers solved some of these problems and conjectures, yet many perplexing questions are still unanswered and need to be investigated.

The purpose of this expository article is to provide a guided tour of planar harmonic univalent mappings with emphasis on recent results and open problems and, in particular, to look at the harmonic analogues of the theory of analytic univalent functions in the unit disc. Since there are several survey articles and books ([21], [23], [24], [27], [49]) on harmonic mappings and related areas, we present only a selection of the results relevant to our precise objective. We begin the next section with a quick review of the theory of analytic univalent functions.

2. THEORY OF ANALYTIC UNIVALENT FUNCTIONS (1851 – 1985)

Let $D_1 \neq \mathbb{C}$ be any given simply connected domain in the *z-plane* $\mathbb{C}$. Let D_2 be any given simply connected domain in the *w-plane*. In 1851, G. Bernard Riemann showed that there always exists an analytic function f that maps D_1 onto D_2. This original version of the *Riemann mapping theorem* gave rise to the birth of *geometric function theory*. But, this theorem was incomplete and so it could not find many applications until the beginning of the 20th century. It was Koebe [48] who, in 1907, discovered that the functions which are both analytic and univalent in a simply connected domain $D = D_1 \neq \mathbb{C}$ have a nice property stated in Theorem 2.1. Here *univalent function* or *univalent mapping* is the complex analyst's term for "one-to-one": $f(z_1) \neq f(z_2)$ unless $z_1 \neq z_2$.

Theorem 2.1. *If $z_0 \in D$, then there exists a unique function f, analytic and univalent in D which maps D onto the open unit disc $\Delta := \{z : |z| < 1\}$ in such a way that $f(z_0) = 0$ and $f'(z_0) > 0$.*

This powerful version of the Riemann mapping theorem allows pure and applied mathematicians and engineers to reduce problems about simply connected domains to the special case of the open unit disc Δ or half-plane. An analytic univalent function is also called a *conformal mapping* because it preserves angles between curves.

The theory of univalent functions is so vast and complicated that certain simplifying assumptions are necessary. In view of the modified version of the Riemann Mapping Theorem 2.1, we can replace the arbitrary simply connected domain D with Δ. We further assume the normalization conditions: $f(0) = 0$, $f'(0) = 1$. It is easy to show that these normalization conditions are harmless. We let S denote the *family of analytic, univalent and normalized functions defined in* Δ. Thus a function f in S has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta. \tag{2.1}$$

The theory of univalent functions is largely related to S. It is well-known that S is a compact subset of the locally convex linear topological space of all analytic normalized functions defined on Δ with respect to the topology of uniform convergence on compact subsets of Δ. The Koebe function

$$k(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} nz^n \tag{2.2}$$

and its rotations are extremal for many problems in S. Note that $k(\Delta)$ is the entire complex plane minus the slit along the negative real axis from $-\infty$ to $-1/4$. For the family S, we have the following powerful and fascinating result which was discovered in 1907 by Koebe [48]:

Theorem 2.2. *There exists a positive constant c such that*

$$\bigcap_{f\in S} f(\Delta) \supset \{w : |w| \le c\}.$$

But, this interesting result did not find many applications until Bieberbach [19] in 1916 proved that $c = 1/4$. More precisely, he proved that that the open disc $|w| < 1/4$ is always covered by the map of Δ of any function $f \in S$. Interestingly, the one-quarter disc is the largest disc that is contained in $k(\Delta)$, where k is the Koebe function given by (2.2). In the same paper, Bieberbach also observed the following.

Conjecture 2.3 (Bieberbach [19]**).** *If $f \in S$ is any function given by (2.1), then $|a_n| \le n$, $n \ge 2$. Furthermore,$|a_n| = n$ for all n for the Koebe function k defined by (2.2) and its rotations.*

Failure to settle the Bieberbach conjecture until 1984 led to the introduction and investigation of several subclasses of S. An important subclass of S, denoted by S^*, consists of the functions that map Δ onto a domain *starshaped with respect to the origin*. Another important subclass of S is the family K which maps Δ onto a convex domain. Note that the Koebe function and its rotations do not belong to K. Furthermore, a function f, analytic in Δ, is said to be *close-to-convex* in Δ, $f \in C$, if $f(\Delta)$ is a close-to-convex domain; that is, if the complement of $f(\Delta)$ can be written as a union of non-crossing half-lines. It is well-known that $K \subset S^* \subset C \subset S$. We remark that various subclasses of these classes have been studied by many researchers including the author in ([3], [17], [31], [32]).

Various attempts to prove or disprove the Bieberbach conjecture gave rise to eight major conjectures which are related to each other by a chain of implications; see for example, [3]. Many powerful new methods were developed and a large number of related problems were generated in attempts to prove these conjectures, which were finally settled in mid 1984 by Louis de Branges [20]. For a historical development of the Bieberbach Conjecture and its implications on univalent function theory, one may refer to the survey by the author [3].

3. HARMONIC UNIVALENT MAPPINGS: BACKGROUND AND DEFINITIONS

A complex-valued continuous function $w = f(z) = u(z) + iv(z)$ defined on a domain D is *harmonic* if u and v are real-valued harmonic functions on D, that is u, v satisfy, respectively, the Laplace equations $\Delta u = u_{xx} + v_{yy} = 0$ and $\Delta v = v_{xx} + v_{yy} = 0$. A one-to-one mapping $u = u(z)$, $v = v(z)$ from a region D_1 in the $xy-$plane to a region D_2 in the $uv-$plane is a *harmonic mapping* if u and v are harmonic. It is well-known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. It follows that every analytic function is a complex-valued harmonic function. However, not every complex-valued harmonic function is analytic, since no two solutions of the

Laplace equation can be taken as the components u and v of an analytic function in D, they must be related by the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$.

An analytic function of a harmonic function may not be harmonic. For example, x is harmonic but x^2 is not. But, a product of any pair of analytic functions is analytic. On the other hand, the harmonic function of an analytic function can be shown to be harmonic, but the composition of two harmonic functions may not be harmonic. Moreover, the inverse of a harmonic function need not be harmonic. The simplest example of a harmonic univalent function which need not be conformal is the linear mapping $w = \alpha z + \beta \bar{z}$ with $|\alpha| \neq |\beta|$. Another simple example is $w = z + \bar{z}^2/2$ which maps Δ harmonically onto a region inside a hypocycloid of three cusps.

Theorem 3.1 ([22]). *Most general harmonic mappings of the whole complex plane* $\mathbb{C}$ *onto itself are the affine mappings* $w = \alpha z + \beta \bar{z} + \gamma \ (|\alpha| \neq |\beta|)$.

Let $f = u + iv$ be a harmonic function in a simply connected domain D with $f(0) = 0$. Let F and G be analytic in D so that $F(0) = G(0) = 0$, $\operatorname{Re} F = \operatorname{Re} f = u$, $\operatorname{Re} G = \operatorname{Im} f = v$. Write $h = (F + iG)/2$, $g = (F - iG)/2$. It is now a routine exercise to show that $f = h + \bar{g}$, where h and g are analytic functions in D. We call h the *analytic part* and $\bar{g}$ the *co-analytic part* of f. Moreover,

$$h' = f_z = \frac{\partial f/\partial x - i\partial f/\partial y}{2}, \quad \overline{g'} = f_{\bar{z}} = \frac{\partial f/\partial x + i\partial f/\partial y}{2}$$

are always (globally) analytic functions on D. For example, $f(z) = z - 1/\bar{z} + 2\ln|z|$ is a harmonic univalent function from the exterior of the unit disc Δ onto $\mathbb{C}\setminus\{0\}$, where $h(z) = z + \log z$ and $g(z) = \log z - 1/z$.

A subject of considerable importance in harmonic mappings is the *Jacobian* J_f of a function $f = u + iv$, defined by $J_f(z) = u_x(z)v_y(z) - u_y(z)v_x(z)$. Or, in terms of f_z and $f_{\bar{z}}$, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

where $f = h + \bar{g}$ is the harmonic function Δ. When J_f is positive in D, the harmonic function f is called *orientation–preserving* or *sense-preserving* in D. An analytic univalent function is a special case of an orientation-preserving harmonic univalent function. For analytic functions f, it is well-known that $J_f(z) \neq 0$ if and only if f is locally univalent at z. For harmonic functions we have the following useful result due to Lewy.

Theorem 3.2 ([50]). *A harmonic mapping is locally univalent in a neighborhood of a point* z_0 *if and only if the Jacobian* $J_f(z) \neq 0$ *at* z_0.

The first key insight into harmonic univalent mappings came from Clunie and S. Small [22], who observed that $f = h + \bar{g}$ is locally univalent and orientation-preserving if and only if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ $(z \in \Delta)$. This is equivalent to

$$|g'(z)| < |h'(z)| \quad (z \in \Delta). \tag{3.1}$$

The function $w = g'/h'$ is called the *second dilation* of f. Note that $|w(z)| < 1$. More generally, we have

Theorem 3.3 ([22]). *A non-constant complex-valued function* f *is a harmonic and orientation-preserving mapping on* D *if and only if* f *is a solution of the elliptic partial differential equation* $\overline{f_{\bar{z}}(z)} = w(z)f_z(z)$.

A function $f = h + \bar{g}$ harmonic in the open unit disc Δ can be expanded in a series

$$f(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \quad (0 \leq r < 1),$$

where $h(z) = \sum_0^\infty a_n z^n$, $g(z) = \sum_1^\infty \bar{a}_{-n} z^n$. We may normalize f so that $h(0) = 0 = h'(0) - 1$. For the sake of simplicity, we may write $b_n = \bar{a}_{-n}$. We denote by S_H the *family of all harmonic, complex-valued, orientation-preserving, normalized and univalent mappings* defined on Δ. Thus a function f in S_H admits the representation $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{3.2}$$

are analytic functions in Δ. It follows from the orientation-preserving property that $|b_1| < 1$. Therefore, $\left(f - \overline{b_1 f}\right) / \left(1 - |b_1|^2\right) \in S_H$ whenever $f \in S_H$. Thus we may restrict our attention to the subclass S_H^0 defined by $S_H^0 = \{f \in S_H : g'(0) = b_1 = 0\}$.

We observe that $S \subset S_H^0 \subset S_H$. Both families S_H and S_H^0 are normal families. That is every sequence of functions in S_H (or S_H^0) has a subsequence that converges locally uniformly in Δ. Note that S_H^0 is a compact family (with respect to the topology of locally uniform convergence) [22]. However, in contrast to the families S and S_H^0, the family S_H is not compact because the sequence of affine functions $f_n(z) = (n/(n+1))\,\bar{z} + z$ is in S_H but as $n \to \infty$ it is apparent that $f_n(z) \to f(z) = 2x$ (where $z = x + iy$) uniformly in Δ and the limit function f is not univalent (nor is it constant).

Analogous to well-known subclasses of the family S, one can define various subclasses of the families S_H and S_H^0. A sense-preserving harmonic mapping $f \in S_H$ ($f \in S_H^0$) is in the class S_H^* (S_H^{*0} respectively) if the range $f(\Delta)$ is starlike with respect to the origin. A function $f \in S_H^*$ (or $f \in S_H^{*0}$) is called a *harmonic starlike mapping* in Δ. Likewise a function f defined in Δ belongs to the class K_H (K_H^0) if $f \in S_H$ (or $f \in S_H^0$ respectively) and if $f(\Delta)$ is a *convex domain*. A function $f \in K_H$ (or $f \in K_H^0$) is called *harmonic convex* in Δ. Analytically, we have

$$f \in S_H^* \Leftrightarrow \frac{\partial}{\partial \theta}\left(\arg f(re^{i\theta})\right) > 0, \quad (z \in \Delta) \tag{3.3}$$

$$f \in K_H \Leftrightarrow \frac{\partial}{\partial \theta}\left\{\arg\left(\frac{\partial}{\partial \theta}\arg f(re^{i\theta})\right)\right\} > 0, \tag{3.4}$$
$$(z \in re^{i\theta},\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1).$$

Similar to the subclass C of S, let C_H and C_H^0 denote the subsets, respectively, of S_H and S_H^0, such that for any $f \in C_H$ or C_H^0, $f(\Delta)$ is a *close-to-convex* domain. Recall that a domain D is close-to-convex if the complement of D can be written as a union of non-crossing half-lines.

Comparable to the positive order defined in the subclasses S^* and K of S, we can introduce the order α $(0 \le \alpha < 1)$ in S_H^* and K_H by replacing '0' on the right sides of inequalities (3.3) and (3.4) by α. Denote the corresponding subclasses of the functions which are *harmonic starlike of order* α and *harmonic convex of order* α, respectively, by $S_H^*(\alpha)$ and $K_H(\alpha)$. Note that $S_H^*(0) \equiv S_H^*$ and $K_H(0) \equiv K_H$. Also, note that whenever the co-analytic parts of each $f = h + \bar{g}$, that is g, is zero, then $S_H^*(\alpha) \equiv S^*(\alpha)$ and $K_H(\alpha) \equiv K(\alpha)$, where $S^*(\alpha)$ and $K(\alpha)$ are the subclasses of the family S which consist of functions, respectively, of *starlike of order* α and *convex of order* α.

The convolution of two complex-valued harmonic functions

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{i_n} z^n + \sum_{n=1}^{\infty} \bar{b}_{i_n} \bar{z}^n \quad (i = 1, 2)$$

is given by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1_n} a_{2_n} z^n + \overline{\sum_{n=1}^{\infty} b_{1_n} b_{2_n} z^n}. \tag{3.5}$$

The above convolution formula reduces to the famous Hadamard product if the co-analytic parts of f_1 and f_2 are zero.

4. BIRTH OF THE THEORY OF HARMONIC UNIVALENT FUNCTIONS

After the discovery of the proof of the 69-year old Bieberbach conjecture for the family S by Louis de Branges [20] in 1984, it was natural to ask whether the classical collection of results for the family S and its various subclasses could be extended in any way to the families S_H and S_H^0 of harmonic univalent functions. In 1984, Clunie and Sheil-Small [22] gave an affirmative answer. They discovered that though estimates for these families are not the same, yet with suitable interpretations there are analogous estimates for harmonic mappings in S_H and S_H^0. This gave rise to the theory of planar harmonic univalent functions. Since then, it has been growing faster than any one could even imagine. We first state the following interesting result:

Theorem 4.1 ([22]). *A harmonic function $h + \bar{g}$ is univalent and convex in the direction of the real axis (CRA) if and only if the analytic function $h - g$ is univalent and CRA. (Here a function f defined in D is CRA if the intersection of $f(D)$ with each horizontal is connected).*

Using Theorem 4.1, Clunie and Sheil-Small [22] discovered a result for the family S_H^0, analogous to the Koebe function $k \in S$ defined by (2.2). In fact, they constructed the *harmonic Koebe function* $k_0 = h + \bar{g} \in S_H^0$ defined by

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \quad g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}. \tag{4.1}$$

It can be shown that k_0 maps Δ univalently onto $\mathbb{C}$ minus the real slit $-\infty < t < -1/6$. Moreover, $k_0(z) = -1/6$ for every z on the unit circle except $z = 1$. Unlike for the family S, there is no overall positive lower bound for $|f(z)|$ depending on $|z|$, when $f \in S_H$. This is because, for example, $z + \varepsilon\bar{z} \in S_H$ for all ε with $|\varepsilon| < 1$. However, by using an extremal length method, Clunie and Sheil-Small discovered the following interesting result analogous to the distortion property for functions in the family S.

Theorem 4.2 ([22]). *If $f \in S_H^0$, then*

$$|f(z)| \geq \frac{1}{4} \frac{|z|}{(1+|z|)^2} \quad (z \in \Delta).$$

In particular, $\{w \in \mathbb{C} : |w| < 1/16\} \subset f(\Delta) \forall f \in S_H^0$.

The result in Theorem 4.2 is non-sharp. However, the harmonic Koebe function k_0 suggests that the 1/16 radius can be improved to 1/6.

Conjecture 4.3 ([22]). $\{w \in \mathbb{C} : |w| < 1/6\} \subset f(\Delta) \forall f \in S_H^0$.

This conjecture is true for close-to-convex functions in C_H^0([22], [64]). Clunie and Sheil-Small [22] posed the following harmonic analogues of the Bieberbach conjecture (*see* Conjecture 2.3) for the family S_H^0:

Conjecture 4.4. *If $f = h + \bar{g} \in S_H^0$ is given by (3.2), then*

$$
\begin{aligned}
||a_n| - |b_n|| &\leq n \quad (n = 2, 3, ...) \\
|a_n| &\leq \frac{(2n+1)(n+1)}{6}, \qquad (4.2)\\
|b_n| &\leq \frac{(2n-1)(n-1)}{6}, \quad (n = 2, 3, ...).
\end{aligned}
$$

Equality occurs for $f = k_0$.

For $f = h + \bar{g} \in S_H^0$, applying Schwarz's lemma to (3.1), we have $|g'(z)| \leq |h'(z)|$ $(z \in \Delta)$. In particular, it follows that $|b_2| \leq 1/2$. Conjecture 4.4 was proved for functions in the class S_H^{*0}, and when $f(\Delta)$is *convex in one direction* ([22], [64]). The results also hold if all the coefficients of f in S_H^{*0} are real ([22], [64]). It was proved in [69] that this conjecture is also true for $f \in C_H^0$.

Later Sheil-Small [64] developed Conjecture 4.4 and proposed the following generalization of the Bieberbach conjecture.

Conjecture 4.5. *If*

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} a_{-n} z^n} \in S_H,
$$

then

$$
|a_n| < \frac{2n^2 + 1}{3} \quad (|n| = 2, 3, ...).
$$

In [22], it was discovered that $|a_2(f)| < 12,173$ for all $f \in S_H$. This result was improved to $|a_2(f)| < 57.05$ for all $f \in S_H$in [64]. These bounds were further improved in [62]. On the other hand, Conjecture 4.5 was proved for the class $\tilde{C}_H$, where $\tilde{C}_H$denotes the closure of C_H [22]. Wang [69] established the conjecture for $f \in C_H$. He also proposed to rewrite the generalization of Bieberbach conjecture as follows:

Conjecture 4.6. *If*

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in S_H,
$$

then

(1) $||a_n| - |b_n|| \leq (1 + |b_1|)\, n, \quad (n = 2, 3, \dots),$

(2) $|a_n| \leq \dfrac{(n+1)(2n+1)}{6} + |b_1| \dfrac{(n-1)(2n-1)}{6} \quad (n = 2, 3, \dots),$

(3) $|b_n| \leq \dfrac{(n-1)(2n-1)}{6} + |b_1| \dfrac{(n+1)(2n+1)}{6} \quad (n = 2, 3, \dots).$

Since $|b_1| < 1$, the above conjecture may be rewritten as:

Conjecture 4.7. *If*

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in S_H,
$$

then

(1) $||a_n| - |b_n|| \leq 2n, \quad (n = 2, 3, \dots),$

(2) $|a_n| < \dfrac{2n^2 + 1}{3}, \quad (n = 2, 3, \dots),$

(3) $|b_n| < \dfrac{2n^2+1}{3}, \quad (n = 2, 3, \ldots)$.

Results of these types have been previously obtained only for functions in the special subclass C_H; see [69]. However, necessary coefficient conditions for functions in C_H were also found in [22]. The next result provides a sufficient condition for the function to be in C_H.

Theorem 4.8 ([44]). *If $f = h + \bar{g}$ with*

$$\sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| \le 1,$$

then $f \in C_H$. The result is sharp.

Next we construct an example of a function f_0 in the familyK_H^0. The function

$$\begin{aligned} f_0(z) &= h(z) + \overline{g(z)} \\ &= \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\left(\frac{\frac{1}{2}z^2}{(1-z)^2}\right)} \\ &= \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right) \end{aligned}$$

is in $K_H^0$and it maps Δ onto the half plane; see [22]. Moreover, parallel to a well-known coefficient bound theorem in the case of univalent analytic mappings in Δ, we have

Theorem 4.9 ([22]). *If $f \in K_H^0$, then for $n = 1, 2, \ldots$ we have*

$$||a_n| - |b_n|| \le 1, \quad |a_n| \le \frac{(n+1)}{2}, \quad |b_n| \le \frac{(n-1)}{2}.$$

The results are sharp for the function $f = f_0$ as given above.

In view of the sharp coefficient bounds given for functions in K_H^0 in Theorem 4.9, we may take $f_1, f_2 \in K_H^0$ and define $f_1 * f_2$ by (3.5). Clunie and Sheil-Small [22] showed that if $\varphi \in K$ and $f = h + \bar{g} \in K_H$, then $f * (\varphi + \alpha\bar{\varphi}) = h * \varphi + \alpha\overline{g * \varphi}$, $|\alpha| \le 1$, is a univalent mapping of Δ onto a close-to-convex domain. They raised the following problem.

Open Problem 1. Which complex-valued harmonic functions, φ have the property that $\varphi * f \in K_H$ for all $f \in K_H$?

A related open problem for the univalent analytic functions was proved by Ruscheweyh and Sheil-Small in the following.

Lemma 4.10 ([61]).

(a) $\phi, \psi \in K \Rightarrow \phi * \psi \in K$.
(b) $\phi \in K \Rightarrow (\phi * f)(z) \in C$ *if* $f \in C$.

Note that the first part of the above lemma is the famous *Polya-Schoenberg conjecture*. Analogous results for the harmonic mappings are the following:

Theorem 4.11 ([15]). *If $f \in K_H$ and $\phi \in K$, then $\left(\alpha\bar{\phi} + \phi\right) * f \in C_H \;(|\alpha| \le 1)$.*

Theorem 4.12 ([15]). *If h and g are analytic in Δ, then*

(1) *If $h, \phi \in K$ with $|g'(z)| < |h'(z)|$ for each $z \in \Delta$, then for each $|\varepsilon| \le 1$ we have $\left(\phi + \varepsilon\bar{\phi}\right) * (h + \bar{g}) \in C_H$.*
(2) *If $\phi \in K, |g'(0)| < |h'(0)|$ and $h + \varepsilon g \in C$ for each $\varepsilon\,(|\varepsilon| = 1)$, then $\left(\phi + \bar{\sigma}\bar{\phi}\right) * (h + \bar{g}) \in C_H, |\sigma| = 1$.*

In this direction, one may also refer to articles in ([41], [42], [44], [62]). In the next theorem we give necessary and sufficient convolution conditions for convex and starlike harmonic functions.

Theorem 4.13 ([15])**.** *Let* $f = h + \bar{g} \in S_H$. *Then*

(i) $$f \in S_H^* \Leftrightarrow h(z) * \frac{z + ((\varsigma - 1)/2) z^2}{(1-z)^2} - \overline{g(z)} * \frac{\varsigma \bar{z} - ((\varsigma - 1)/2) \bar{z}^2}{(1-\bar{z})^2} \neq 0, \quad |\varsigma| = 1,\ 0 < |z| < 1.$$

(ii) $$f \in K_H \Leftrightarrow h(z) * \frac{z + \varsigma z^2}{(1-z)^3} + \overline{g(z)} * \frac{\varsigma \bar{z} + \bar{z}^2}{(1-\bar{z})^3} \neq 0, \quad |\varsigma| = 1,\ 0 < |z| < 1.$$

The above theorem yields the following sufficient coefficient bounds for starlike and convex harmonic functions.

Theorem 4.14 ([15], [38], [39], [65])**.** *If* $f = h + \bar{g}$ *with* h *and* g *of the form (3.2), then*

(i) $$\sum_{n=2}^{\infty} n\,|a_n| + \sum_{n=1}^{\infty} n\,|b_n| \leq 1 \Rightarrow f \in S_H^*.$$

(ii) $$\sum_{n=2}^{\infty} n^2\,|a_n| + \sum_{n=1}^{\infty} n^2\,|b_n| \leq 1 \Rightarrow f \in K_H.$$

In [29], the researcher constructed some examples in which the property of convexity is preserved for convolution of certain convex harmonic mappings. On the other hand, the researchers in [33] obtained the integral means of extreme points of the closures of univalent harmonic mappings onto the right half plane $\{w : \operatorname{Re} w > -1/2\}$ and onto the one-slit plane $\mathbb{C} \setminus (-\infty, a]\,, a < 0$.

It is of interest to determine the largest disc $|z| < r$ in which all the members of one family possess properties of those in another. For example, all functions in K_H are convex in $|z| < \sqrt{2} - 1$ [62]. It is known that $\{w : |w| < 1/2\} \subset f(\Delta)$ for all $f \in K_H^0$[22]. It is also a known fact [64] that if $f \in C_H$, then f is convex for $|z| < 3 - \sqrt{8}$. However, analogous to the radius problem for the family S and its subclasses, nothing much is known for S_H, S_H^0 and their subclasses. For example

Open Problem 2. Find the radius of starlikeness for starlike mappings in S_H.

Another challenging area is the Riemann Mapping Theorem related to the harmonic univalent mappings. The best possible Riemann Mapping Theorem was obtained by Hengartner and Schober in [35]. But, the uniqueness problem of mappings in their theorem is still open.

The boundary behavior of a function $f \in S_H$ along a closed subarc of boundary $\partial\Delta$ of Δ was investigated in [2]. These authors gave a prime-end theory for univalent harmonic mappings. Also, see ([1], [25], [71]). Corresponding to the neighborhood problem and duality techniques for the family S, Nezhmetdinov [54] studied problems related to the family S_H^0.

5. SUBCLASSES OF HARMONIC UNIVALENT AND RELATED MAPPINGS

Since it is difficult to directly prove several results or obtain sharp estimates for the families S_H and S_H^0, one usually attempts to investigate them for various subclasses of these families. Denote by S_{RH}, S_{RH}^0, S_{RH}^*, and K_{RH}, respectively, the subclasses of S_H, S_H^0, S_H^* and K_H

consisting of functions $f = h + \overline{g}$ so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0,\ b_n \geq 0,\ b_1 < 1. \tag{5.1}$$

Our next result shows that the coefficient bounds in Theorem 4.14 cannot be improved.

Theorem 5.1 ([38], [65], [66])**.** *If $f = h + \overline{g}$ is given by (5.1), then*

(i)
$$f \in S^*_{RH} \Leftrightarrow \sum_{n=2}^{\infty} n a_n + \sum_{n=1}^{\infty} n b_n \leq 1,$$

(ii)
$$f \in K_{RH} \Leftrightarrow \sum_{n=2}^{\infty} n^2 a_n + \sum_{n=1}^{\infty} n^2 b_n \leq 1.$$

Jahangiri ([38], [39]) proved the following sufficient conditions, akin to Theorem 4.14, for functions in the classes $S^*_H(\alpha)$ and $K_H(\alpha)$.

Theorem 5.2. *If $f = h + \bar{g}$where h and g are given by (3.2), then*

(a)
$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \leq 1, \quad 0 \leq \alpha < 1 \Rightarrow f \in S^*_H(\alpha).$$

(b)
$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \leq 1, \quad 0 \leq \alpha < 1 \Rightarrow f \in K_H(\alpha).$$

Let $S^*_{RH}(\alpha)$ and $K_{RH}(\alpha)$denote, respectively, the subclasses of $S^*_H(\alpha)$ and $K_H(\alpha)$, consisting of functions $f = h + \overline{g}$ where h and g are given by (5.1). In ([38], [39]) it was discovered that the above-mentioned inequalities in (a) and (b) are the necessary as well as sufficient conditions, respectively, for the functions in $S^*_{RH}(\alpha)$ and $K_{RH}(\alpha)$. Using these characterizing conditions, he also found various extremal properties, extreme points, distortion bounds, covering theorems, convolution properties, and others for the families $S^*_{RH}(\alpha)$ and $K_{RH}(\alpha)$.

In several other papers, including ([11], [13], [18], [28], [42], [46], [47], [56], [57]), the researchers obtained the necessary and/or sufficient coefficient conditions for functions in various subclasses of S^*_H and K_H. In ([8], [43]), the researchers used an argument variation for the coefficients of h and g that contain several previously studied cases. Let V_H denote the class of functions $f = h + \bar{g}$ for which h and g are of the form (3.2) and there exists ϕ so that, $\operatorname{mod} 2\pi$,

$$\alpha_n + (n-1)\,\phi \equiv \pi, \quad \beta_n + (n-1)\,\phi \equiv 0, \quad n \geq 2, \tag{5.2}$$

where $\alpha_n = \arg(a_n)$ and $\beta_n = \arg(b_n)$. We also let $V_{H^*} = V_H \cap S^*_H$, $V_H P(\alpha) = V_H \cap P_H(\alpha)$, and $V_H R(\alpha) = V_H \cap R_H(\alpha)$, $0 \leq \alpha < 1$, where $P_H(\alpha)$ and $R_H(\alpha)$, $0 \leq \alpha < 1$, are the classes of functions $f = h + \bar{g} \in S_H$ which satisfy, respectively, the conditions

$$\operatorname{Re}\left(\frac{(\partial/\partial\theta) f(z)}{(\partial/\partial\theta) z}\right) \geq \alpha \quad \text{and} \quad \operatorname{Re}\left(\frac{(\partial^2/\partial\theta^2) f(z)}{(\partial^2/\partial\theta^2) z}\right) \geq \alpha, \quad z = re^{i\theta} \in \Delta.$$

Earlier the classes $P_H(\alpha)$ and $R_H(\alpha)$ were investigated, respectively, in [11] and [13]. We remark that if $g \equiv 0$ for $f = h + \bar{g}$, then $P_H(\alpha)$ and $R_H(\alpha)$ reduce, respectively, to the well-known classes

$$P(\alpha) = \{h : \operatorname{Re}(h'(z)) \geq \alpha\} \quad \text{and} \quad R(\alpha) = \{h : \operatorname{Re}(h'(z) + zh''(z)) \geq \alpha\}$$

of analytic univalent functions. While V_H and V_{H^*} were studied in [43], $V_H P(\alpha)$ and $V_H R(\alpha)$ were investigated in [8]. In both these papers, the authors determined necessary and sufficient conditions, distortion bounds, and extreme points.

A function F is said to be in $S^*_{H_c}(\alpha)$ for some c, $0 \le c < 1$, if F can be expressed by

$$F(z) = \frac{f(cz)}{c} = \frac{h(cz)}{c} + \overline{\frac{g(cz)}{c}} \tag{5.3}$$

for some $f = h + \bar{g}$, where h and g are functions of the form (3.2) and f satisfies the inequality (a) in Theorem 5.2. Analogous to $S^*_{H_c}(\alpha)$ is the family $K_{H_c}(\alpha)$ consisting of functions F that can be expressed as (5.2), where f satisfies the condition (b) in Theorem 5.2. Also, let $S^{*0}_{H_c}(\alpha)$ and $K^0_{H_c}(\alpha)$ be the corresponding classes where $b_1 = 0$. It is natural to ask whether there exists $c_0 = c_0(\alpha, \beta)$, $0 \le \alpha \le \beta < 1$, such that $S^{*0}_{H_c}(\alpha) \subset K^0_H(\beta)$ for $|c| \le c_0$. As it turns out, the answer is affirmative. The researchers in [14] extended several known results to the contractions of the mappings (5.3) in $S^{*0}_{H_c}(\alpha)$ and $K^0_{H_c}(\alpha)$.

There is a challenge in fixing the second coefficient in the power series representation of an analytic univalent function in the class S. This challenge is even greater when it comes to a family $\mathbb{F}^p_H(\{c_n\}, \{d_n\})$ of harmonic functions with fixed second coefficient. For $0 \le p \le 1$, a function $f = h + \bar{g}$ where

$$h(z) = z - \frac{p}{c_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{5.4}$$

is said to be in the family $\mathbb{F}^p_H(\{c_n\}, \{d_n\})$ if there exist sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers such that

$$p + \sum_{n=3}^{\infty} c_n |a_n| + \sum_{n=1}^{\infty} d_n |b_n| \le 1, \quad d_1 |b_1| < 1. \tag{5.5}$$

Also, let $\mathbb{F}^p_{H^0}(\{c_n\}, \{d_n\}) \equiv \mathbb{F}^p_H(\{c_n\}, \{d_n\}) \cap S^0_H$. The families $\mathbb{F}^p_H(\{c_n\}, \{d_n\})$ and $\mathbb{F}^p_{H^0}(\{c_n\}, \{d_n\})$ incorporate many subfamilies, respectively, of S_{RH} and S^0_{RH} consisting of functions with a fixed second coefficient. For example, for functions $f = h + \bar{g}$ of the form (5.4), we have $\mathbb{F}^p_{RH}(\{n\}, \{n\}) \equiv \{f : f \in S^*_{RH}\}$ and $\mathbb{F}^p_{RH}(\{n^2\}, \{n^2\}) \equiv \{f : f \in K_{RH}\}$. It is known [12] that if $c_n \ge n$ and $d_n \ge n$ for all n, then $\mathbb{F}^p_H(\{c_n\}, \{d_n\})$ consists of starlike sense-preserving harmonic mappings in Δ. Additionally, each function in $\mathbb{F}^p_{H^0}(\{c_n\}, \{d_n\})$ maps the disc $|z| = r < 1/2$ onto a convex domain [12]. In the same paper, they also determined extreme points, convolution conditions, and convex combinations for these types of functions.

6. MULTIVALENT HARMONIC FUNCTIONS

Passing from the harmonic univalent functions to the harmonic multivalent functions turns out to be quite non-trivial. We need the following argument principle for harmonic functions obtained by Duren, Hengartner, and Laugesen.

Theorem 6.1 ([26]). *Let f be a harmonic function in a Jordan domain D with boundary Γ. Suppose f is continuous in $\bar{D}$ and $f(z) \neq 0$ on Γ. Suppose f has no singular zeros in D, and let m be the sum of the orders of the zeros in D. Then $\Delta_\Gamma \arg(f(z)) = 2\pi m$, where $\Delta_\Gamma \arg(f(z))$ denotes the change of argument of $f(z)$ as z traverses Γ.*

The above theorem motivated the author and Jahangiri [5] to introduce and study certain subclasses of the family $H(m)$, $m \ge 1$, of all multivalent harmonic and orientation-preserving functions in Δ. A function f in $H(m)$ can be expressed as $f = h + \bar{g}$, where h and g are of the form

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1. \tag{6.1}$$

For $m \geq 1$, let $SH(m)$ denote the subclass of $H(m)$ consisting of harmonic starlike functions that map the unit disc Δ onto a closed curve that is starlike with respect to the origin. Observe that $m-$valent mappings need not be orientation-preserving. For example, $f(z) = z + \bar{z}^2$ is 4-valent on $D = \{z : |z| < 2\}$ and we have $|a(0)| = 0$ and $|a(1.5)| = 3$.

Theorem 6.2 ([5]). *If a function $f = h + \bar{g}$ given by (6.1) satisfies the condition*

$$\sum_{n=1}^{\infty} (n+m-1)\left(|a_{n+m-1}| + |b_{n+m-1}|\right) \leq 2m \tag{6.2}$$

where $a_m = 1$ and $m \geq 1$, then f is harmonic and sense preserving in Δ and $f \in SH(m)$.

Let $TH(m)$, $m \geq 1$, denote the class of functions $f = h + \bar{g}$ in $SH(m)$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad a_{n+m-1} \geq 0, \tag{6.3}$$

$$g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad b_{n+m-1} \geq 0.$$

It was proved in [5] that a function $f = h + \bar{g}$ given by (6.3) is in the class $TH(m)$ if and only if condition (6.2) is satisfied. They also determined the extreme points, distortion and covering theorems, convolution and convex combination conditions for the functions in $TH(m)$.

During the last five years, there have been several papers on multivalent harmonic functions in the open unit disc. For example, see ([4], [7], [53]).

7. MEROMORPHIC HARMONIC FUNCTIONS

To begin let us turn our attention to the special classes of harmonic functions which are defined on the exterior of the unit disc $\tilde{\Delta} = \{z : |z| > 1\}$ for which $f(\infty) = \lim_{z\to\infty} f(z) = \infty$. Such functions were recently studied by Hengartner and Schober who obtained the main idea in the following:

Theorem 7.1 ([36]). *Let f be a complex-valued, harmonic, orientation-preserving, univalent function defined on $\tilde{\Delta}$, satisfying $f(\infty) = \infty$. Then f must admit the representation*

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \tag{7.1}$$

where $A \in \mathbb{C}$ and

$$h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \tag{7.2}$$

are analytic in $\tilde{\Delta}$ and $0 \leq |\beta| < |\alpha|$. In addition, $w = \bar{f}_{\bar{z}}/f_z$ is analytic and satisfies $|w(z)| < 1$.

In view of the aforementioned result, the researchers in [45] found the following sufficient coefficient condition for which functions of the form (7.1) are univalent.

Theorem 7.2. *If f given by (7.1) together with (7.2) satisfies the inequality*

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) \leq |\alpha| - |\beta| - |A|,$$

then f is orientation-preserving and univalent in $\tilde{\Delta}$.

By applying an affine transformation $\left(\bar{\alpha}f - \overline{\beta f} - \bar{\alpha}a_0 + \overline{\beta a_0}\right) / \left(|\alpha|^2 - |\beta|^2\right)$, we may normalize f so that $\alpha = 1,\ \beta = 0,$ and $a_0 = 0$ in (7.2). In view of Theorem 7.1, $w = \bar{f}_{\bar{z}}/f_z$ is analytic and satisfies $|w(z)| < 1$. Therefore let Σ'_H be the set of all harmonic, orientation-preserving, univalent mappings f given by (7.1), where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \tag{7.3}$$

are analytic in $\tilde{\Delta}$. Also, let $\Sigma_H = \{f \in \Sigma'_H : A = 0\}$, that is, the subclass without logarithmic singularity. Note that in contrast to analytic univalent functions, there is no elementary isomorphism between S_H and Σ_H. Finally, let Σ_H^0 denote the non-vanishing class defined by

$$\Sigma_H^0 = \left\{ f - c : f \in \Sigma'_H \quad \text{and} \quad c \notin f(\tilde{\Delta}) \right\}.$$

Using Schwarz's lemma and Theorem 7.1, Hengartner and Schober proved the following estimates:

Theorem 7.3 ([36])**.**

(a) $f \in \Sigma'_H \Rightarrow |A| \le 2$ *and* $|b_1| \le 1$.

(b) $f \in \Sigma_H \Rightarrow |b_1| \le 1$ *and* $|b_2| \le \frac{1}{2}\left(1 - |b_1|^2\right) \le \frac{1}{2}$.

(c) $f \in \Sigma'_H$ *has expansion (7.1) together with (7.3)*$\Rightarrow \sum_{k=1}^{\infty} k\left(|a_k|^2 - |b_k|^2\right) \le 1 + 2\operatorname{Re} b_1$.

All the results are sharp.

The next result gives the distortion theorem:

Theorem 7.4 ([36])**.** *If* $f - c \in \Sigma_H^0$, *then* $|f(z)| \le 4\left(1 + |z|^2\right)/|z|$ *for all* $z \in \tilde{\Delta}$, $f(\tilde{\Delta})$*contains the set* $\{w : |w| > 16\}$, *and* $|c| \le 16$.

The bound for c in Theorem 7.4 is equivalent to the following

Corollary 7.5 ([36])**.** *If* $f \in \Sigma_H$, *then* $f\left(\tilde{\Delta}\right) \supseteq \{w : |w| > 16\}$.

The next result concerns the compactness of the families.

Theorem 7.6 ([36])**.** *The families* $\Sigma_H^0, \Sigma'_H,$ *and* Σ_H *are compact with respect to the topology of locally uniform convergence.*

Related to the famous classical area theorem (see [32]), we have the following result.

Theorem 7.7 ([36])**.** *If* $f \in \Sigma'_H$ *has expansion (7.1) along with (7.3), then*

$$\sum_{n=1}^{\infty} n\left(|a_n|^2 - |b_n|^2\right)^2 \le 1 + 2\operatorname{Re} b_1.$$

Equality occurs if and only if $\mathbb{C}\backslash f(\tilde{\Delta})$ *has area never zero.*

Comparable to the subclasses of S_H and S_H^0, it may be possible to define and study subclasses of the meromorphic harmonic functions. Denote by Σ_H^* the subfamily of Σ_H consisting of functions that are starlike with respect to the origin in $\tilde{\Delta}$. Also, let Σ_{RH}^* denote the subfamily of Σ_H^* consisting of functions f of the form $f = h + \bar{g}$ for which h and g are restricted by

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty} b_n z^{-n}, \quad a_n \ge 0,\ b_n \ge 0. \tag{7.4}$$

The classes Σ_H^*, Σ_{RH}^* and their subclasses were studied in [45]. In particular, it was found in [45] that $f \in \Sigma_{RH}^*$ if and only if $\sum_{n=1}^{\infty}(a_n + b_n) \leq 1$. Analogous results were also found for the convex case. These results were generalized in [10] for the class $\Sigma_H R(\alpha, \lambda)$ consisting of functions $f = h + \bar{g}$ which satisfy the condition

$$\operatorname{Re}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda\frac{\frac{\partial}{\partial\theta}f(z)}{\frac{\partial}{\partial\theta}z}\right\} > \alpha, \quad 0 \leq \alpha < 1, \quad \lambda \geq 0,\ z \in \bar{\Delta}.$$

These authors obtained sufficient conditions, coefficient characterizations, inclusion and convexity conditions, and extreme properties for $\Sigma_H R(\alpha, \lambda)$ and its subclasses [10]. In this area, one may also refer to the papers in ([9], [59], [68]).

8. Other Function Classes Related to Harmonic Mappings

In this section we discuss certain other function classes related to harmonic mappings. First let us look at a special subclass of the family S. Sakaguchi introduced a subclass of S consisting of functions which are starlike with respect to symmetrical points (e.g. see [32, p. 165]). Can such a strategy be implemented for the harmonic mappings? In [6], this concept was extended to include the harmonic functions. For $0 \leq \alpha < 1$, let $SH(\alpha)$ denote the class of complex-valued, orientation-preserving, harmonic univalent functions f of the form (3.2) which satisfy the condition

$$\operatorname{Re}\left(\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})}\right) \geq \alpha,$$

where $z = re^{i\theta}$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$. A function $f \in SH(\alpha)$ is called a *Sakaguchi-type harmonic function.* In [6], the authors obtained the following sufficient condition for functions in the family $SH(\alpha)$.

Theorem 8.1. *If a function $f = h + \bar{g}$ given by (3.2) satisfies the inequality*

$$\sum_{n=1}^{\infty}\left\{\frac{2(n-1)}{1-\alpha}(|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha}|a_{2n-1}| + \frac{2n-1+\alpha}{1-\alpha}|b_{2n-1}|\right\} \leq 2,$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then f is orientation-preserving harmonic univalent in Δ and $f \in SH(\alpha)$.

Theorem 8.1 is fundamental in the proof of the following characterization of functions in $SH(\alpha)$.

Theorem 8.2 ([6])**.** *A harmonic function $f = h + \bar{g}$ of the form (3.2) is in $SH(\alpha)$, $0 \leq \alpha < 1$, if and only if*

$$h(z) * \frac{(1-\alpha)z + (\alpha+\xi)z^2}{(1-z)^2(1+z)} - \overline{g(z)} * \frac{(\alpha+\xi)\bar{z} + (1-\alpha)\bar{z}^2}{(1-\bar{z})^2(1+\bar{z})} \neq 0,$$

where $|\xi| = 1, \xi \neq -1,$ and $0 < |z| < 1$.

Goodman in [30] introduced the geometrically defined class *UCV* of uniformly convex functions. Analogous to *UCV* is the class *UST* of uniformly starlike functions that was studied in [58]. It is a natural question to ask whether it is possible to extend the known results of the classes *UCV* and *UST* and their subclasses to include harmonic functions. Generalizing the class *UST* to include harmonic functions, let $G_H(\gamma)$ denote the subclass of S_H consisting of functions $f = h + \bar{g} \in S_H$ that satisfy the condition

$$\operatorname{Re}\left\{\left(1 + e^{i\alpha}\right)\frac{zf'(z)}{z'f(z)} - e^{i\alpha}\right\} \geq \gamma, \quad 0 \leq \gamma < 1,$$

where $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$, $f'(z) = \frac{\partial}{\partial\theta}\left(f(z) = f(re^{i\theta}\right)$, $0 \le r < 1,$ and α and θ are real. Also, let $G_{RH}(\gamma)$ denote the subclass of $G_H(\gamma)$ consisting of functions $f = h + \bar{g}$ such that h and g are of the form (5.1). The class $G_{RH}(\gamma)$ was studied in [60] where they found coefficient conditions, extreme points, convolution conditions, and convex combinations of the functions in $G_{RH}(\gamma)$. An analogous class of complex-valued harmonic convex univalent functions related to the class *UCV* was studied in [47].

Finally, although the connections between the theory of family S and hypergeometric functions have been investigated by several researchers, the corresponding connections between the family S_H (or its subclasses) and hypergeometric functions have not been explored. Recently, the author and Silverman [16] have discovered some of the inequalities associating hypergeometric functions with planar harmonic mappings.

9. CONCLUSION

In this article, we have made an attempt to present a survey of the newly emerging theory of harmonic mappings in the plane. We have been compelled to omit a number of related areas and interesting problems. However, we hope that this article may serve as a useful guide for new and old researchers in the theory of planar harmonic mappings and related areas. This article may also be useful to pure and applied mathematicians working in several diverse areas.

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