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Planar harmonic convolution operators generated by hypergeometric functions

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Let $S_{\hat{H}}$ be the family of all planar harmonic, univalent and sense-preserving mappings $f = h + \bar{g}$ where h and g are analytic funtions in the open unit disk. The purpose of this article is to investigate connections between the theory of harmonic mappings in the plane and hypergeometric functions by applying certain planar harmonic convolution operators on various subclasses of $S_{\hat{H}}$.

Keywords: Planar harmonic mappings; Hypergeometric functions; Harmonic convolution operators; Harmonic starlike; Harmonic convex; Harmonic close-to-convex; Harmonic typically real

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1. Introduction

Harmonic mappings in the plane are complex-valued harmonic functions $w = f(z)$ which map the open unit disk $\Delta := \{z \in \mathbb{C}: \ = |z| < 1\}$ onto a domain $D \subset \mathbb{C}$. The major difference of such mappings from analytic functions is the canonical representation $f = h + \bar{g}$ where h and g are analytic functions in Δ. Let $\hat{H}$ be the collection of all harmonic mappings of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1 \tag{1}$$

are analytic functions in Δ. A necessary and sufficient condition for $f = h + \bar{g}$ to be locally univalent and sense-preserving in Δ is that $|h'(z)| > |g'(z)|$ in Δ. See Clunie and Sheil-Small [1]. Denote by $S_{\hat{H}}$ the subclass of $\hat{H}$ that are univalent and sense-preserving in Δ. Note that $(f - \overline{B_1 f}))/(1 - |B_1|^2) \in S_{\hat{H}}$ whenever $f \in S_{\hat{H}}$. Thus we may restrict our

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attention to the subclass $S^0_{\hat{H}}$ of $S_{\hat{H}}$ defined by

$$S^0_{\hat{H}} := \{f = h + \bar{g} \ \in S_{\hat{H}} : g'(0) = B_1 = 0\}.$$

The classes $S^0_{\hat{H}}$ and $S_{\hat{H}}$ were first studied in [1].

Analogous to the well-known subclasses of the family S of all analytic, univalent and normalized functions in Δ, there are several subclasses of $S^0_{\hat{H}}$ and $S_{\hat{H}}$ Note that $S \subset S^0_{\hat{H}} \subset S_{\hat{H}}$. A function $f \in S_{\hat{H}} (f \in S^0_{\hat{H}})$ is said to be in $S^*_{\hat{H}}$ ($S^{*0}_{\hat{H}}$ respectively), the class of all harmonic starlike functions in Δ, if the range $f(\Delta)$ is starlike with respect to the origin. Likewise $f \in S_{\hat{H}}$ ($f \in S^0_{\hat{H}}$) is said to be in $K_{\hat{H}}$ ($K^0_{\hat{H}}$ respectively), the class of all harmonic convex functions in Δ, if $f(\Delta)$ is a convex domain. Analytically, for $z = \mathrm{re}^{i\theta}$, $0 \le \theta \le 2\pi$, and $0 \le r \le 1$ we have

$$f \in S^*_{\hat{H}} \Longleftrightarrow \frac{\partial}{\partial\theta}(\arg\left(f\left(\mathrm{re}^{i\theta}\right)\right) > 0,$$

$$f \in K_{\hat{H}} \Longleftrightarrow \frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta} f\left(\mathrm{re}^{i\theta}\right)\right)\right) > 0.$$

The order $\alpha (0 \le \alpha < 1)$ in $S^*_{\hat{H}}$ and $K_{\hat{H}}$ is defined by replacing '0' on the right side of the above inequalities by α. Denote the corresponding subclasses of the functions which are harmonic starlike of order α and harmonic convex of order α, respectively, by $S^*_{\hat{H}}(\alpha)$ and $K_{\hat{H}}(\alpha)$. Note that $S^*_{\hat{H}}(0) = S_{\hat{H}}$ and $K_{\hat{H}}(0) = K_{\hat{H}}$. Similar to the class C of all close-to-convex analytic functions in Δ, let $C_{\hat{H}}$ and $C^0_{\hat{H}}$ denote the subclasses, respectively, of $S_{\hat{H}}$ and $S^0_{\hat{H}}$ such that for any $f \in C_{\hat{H}}$ or $f \in C^0_{\hat{H}}$, $f(\Delta)$ is close-to-convex in Δ. Recall that a domain D is close-to-convex if the complement of D can be written as a union of non-increasing half-lines. For more information, see refs. [2–4].

We shall consider the functions $\phi_1$and ϕ_2 defined by

$$\phi_1(z) := zF(a_1, b_1; c_1; z), \quad \phi_2(z) := zF(a_2, b_2; c_2; z) \quad (z \in \Delta) \tag{2}$$

where $F(a, b, c; z)$ is a well-known Gaussian hypergeometric function; see, for example, [5] and [6]. A hypergeometric function $F(a, b; c; z)$ is analytic in Δ and plays an important role in the theory of univalent functions. See, for example, the works by de Branges [7], Dziok and Srivastava [8], Miller and Mocanu [5], Owa and Srivastava [9], Ruscheweyh and Singh [10], and Srivastava and Manocha [6].

Corresponding to $f = h + \bar{g} \in \hat{H}$ and ϕ_1, ϕ_2 given by (2), we define convolution operator $\Omega : \hat{H} \to \hat{H}$ by

$$\Omega(f) := \Omega(a_1, b_{1,} c_{1,} a_2, b_{2,} c_2; f) = f \tilde{\star} \left(\phi_1 + \overline{\phi_2}\right) = h \star \phi_1 + \overline{g \star \phi_2}, \tag{3}$$

where $\star$ denotes the usual Hadamard or convolution product of power series. For any function $f = h + \bar{g} \in \hat{H}$ given by (1), we have $\Omega(f) = H + \bar{G} \in \hat{H}$ where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n, \tag{4}$$

and where $|B_1| < 1$. Note that we may represent $f = h + \bar{g} \in \hat{H}$ as the Hadmard product according to

$$f(z) = f(z)\tilde{\star}\left[\phi_1(1, 1; 1; z) + \overline{\phi_2(1, 1; 1; z)}\right] = f(z)\tilde{\star}\left[\frac{z}{1-z} + \overline{\frac{z}{1-z}}\right].$$

We also define the convolution operator $L\colon \hat{H} \to \hat{H}$ by

$$L(f) := L(b_1, c_1, b_2, c_2; f) = f \tilde{\star} (\varphi_1 + \overline{\varphi_2}) = h \star \varphi_1 + \overline{g \star \varphi_2}, \tag{5}$$

where φ_1 and φ_2 are incomplete beta functions defined by

$$\varphi_1(z) := \varphi(b_1, c_1; z) = zF(1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(b_1)_{n-1}}{(c_1)_{n-1}} z^n, \tag{6}$$

$$\varphi_2(z) := \varphi(b_2, c_2; z) = zF(1, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(b_2)_{n-1}}{(c_2)_{n-1}} z^n, \quad |B_1| < 1, \tag{7}$$

and where $c_1, c_2 \neq 0, -1, -2, \ldots$. An incomplete beta function $\varphi(b, c; z)$ has an analytic continuation to the z-plane cut along the positive real line from 1 to ∞. Note that $\varphi(b, 1; z) = z/(1-z)^b$ and $\varphi(2, 1; z)$ is the Koebe function. We also observe that if $b, c \neq 0, -1, -2, \ldots$, then $L(b, b, b, b; f)$ is the identity and $L(b, c, b, c; f)$ has a continuous inverse of the operator $L(c, b, c, b; f)$. In particular, convolution of an analytic function with an incomplete beta function was first studied by Carlson and Shaffer [11].

The connections of hypergeometric functions with the theory of univalent functions are well-known in the literature, but the corresponding connections with harmonic mappings have not yet received much attention. The purpose of this article is to investigate the following:

1.1 *Main problem*

Under what restrictions on the parameters a_1, b_1, c_1, a_2, b_2, and c_2 are the convolution operators $\Omega\colon \hat{H} \to \hat{H}$ and $L\colon \hat{H} \to \hat{H}$ map various subclasses of $S_{\hat{H}}$ ($S^0_{\hat{H}}$) into various subclasses of $S_{\hat{H}}$ (respectively $S^0_{\hat{H}}$)? More precisely, this article investigates connections between various subclasses of $S_{\hat{H}}$ and hypergeometric functions by applying convolution operators Ω and L on certain subclasses of $S_{\hat{H}}$ and $S^0_{\hat{H}}$.

Throughout this article, we will frequently use the notations

$$\Omega(f) := \Omega(a_1, b_1, c_1, a_2, b_2, c_2; f), \quad L(f) := L(b_1, c_1, b_2, c_2; f)$$

and a well-known formula

$$F(a, b, c; 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \quad c \neq 0, -1, -2, \ldots \tag{8}$$

2. Preliminaries and key lemmas

First four Lemmas are due to Clunie and Sheil-Small [1].

LEMMA 1 *A harmonic function $f = h + \bar{g} \in \hat{H}$ is locally univalent and sense-preserving in Δ if only if the Jacobian*

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \Delta$$

or equivalently $|g'(z)| < |h'(z)| (z \in \Delta)$.

LEMMA 2 *Suppose H and G are analytic functions in Δ with $|G'(0)| < |H'(0)|$ and that $H + \varepsilon G \in C$ for each $\varepsilon(|\varepsilon| = 1)$. Then $H + \overline{G} \in C_{\hat{H}}$.*

LEMMA 3 *A function $f = h + \bar{g} \in K_{\hat{H}}$ with h and g given by* (1) *if and only if the analytic functions $h(z) - e^{it}g(z)$ $(0 \le t \le 2\pi)$ are convex in the direction $t/2$ and f is suitably normalized.*

LEMMA 4 *If $f = h + \bar{g} \in K^0_{\hat{H}}$ is given by* (1), *then $|A_n| \le (n+1)/2$, $|B_n| \le (n-1)/2$, $n = 2, 3, \ldots$*

The proof of next lemma is available in [12].

LEMMA 5 *If $q(z) = z + \sum_{n=2}^{\infty} t_n z^n$ is analytic in Δ, then*

(i) q maps onto a starlike domain if $\sum_{n=2}^{\infty} n|t_n| \le 1$,
(i) q maps onto a convex domain if $\sum_{n=2}^{\infty} n^2|t_n| \le 1$.

A complex-valued function f harmonic in Δ is said to be typically real provided that $f(z)$ is real if and only if z is real. Suppose the class $T_{\hat{H}}$ consists of all sense-preserving typically real harmonic functions $f = h + \bar{g} \in \hat{H}$. The subclass of $T_{\hat{H}}$ with $g'(0) = 0$ is denoted by $T^0_{\hat{H}}$. The results in the next two lemmas may be found in [1] and [13], respectively.

LEMMA 6 *If $f = h + \bar{g} \in C^0_{\hat{H}}$ ($S^{*0}_{\hat{H}}$ or $T^0_{\hat{H}}$) with h and g as given by* (1), *then*

$$|A_n| \le \frac{(2n+1)(n+1)}{6} \quad (n = 2, 3, \ldots),$$
$$|B_n| \le \frac{(2n-1)(n-1)}{6} \quad (n = 2, 3, \ldots).$$

LEMMA 7 *If $f = h + \bar{g} \in C_{\hat{H}}$ is given by* (1), *then*

$$|A_n| \le \frac{(2n+1)(n+1)}{6} + \frac{(2n-1)(n-1)}{6}|B_1| \quad (n = 2, 3, \ldots),$$
$$|B_n| \le \frac{(2n-1)(n-1)}{6} + \frac{(2n+1)(n+1)}{6}|B_1| \quad (n = 1, 2, \ldots).$$

The results in the next lemma were found in [14] and [15].

LEMMA 8 *If $f = h + \bar{g}$ with h and g of the form* (1), *then*

(i)

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}|A_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha}|B_n| \le 1 \Longrightarrow f \in S^*_{\hat{H}}(\alpha),$$

(ii)

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}|A_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha}|B_n| \le 1 \Longrightarrow f \in K_{\hat{H}}(\alpha).$$

For $\alpha = 0$, the corresponding results in Lemma 8 were found in [16] and [17]. Let $S^*_{R\hat{H}}(\alpha)$ and $K_{R\hat{H}}(\alpha)$ denote, respectively, the subclasses of $S^*_{\hat{H}}(\alpha)$ and $K_{\hat{H}}(\alpha)$ consisting of functions $f = h + \bar{g}$ where

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n, \quad |B_1| < 1. \tag{9}$$

Let $S^*_{R\hat{H}} := S^*_{RH}(0)$, $K_{R\hat{H}} := K_{R\hat{H}}(0)$. The results in the next Lemma, found in [14] and [15], provide the coefficient characterizations of the functions in $S^*_{R\hat{H}}(\alpha)$ and $K_{R\hat{H}}(\alpha)$.

LEMMA 9 *If $f = h + \bar{g}$ with h and g of the form* (9), *then*

(i)

$$f \in S^*_{R\hat{H}}(\alpha) \Longleftrightarrow \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |B_n| \le 1,$$

(ii)

$$f \in K_{R\hat{H}}(\alpha) \Longleftrightarrow \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |B_n| \le 1.$$

LEMMA 10 *If $a, b, c > 0$, then*

(i)

$$F(a+k, b+k; c+k; 1) = \frac{(c)_k}{(c-a-b-k)_k} F(a, b; c; 1),$$
$$\text{for } k = 0, 1, 2, \ldots, \quad \text{if } c > a+b+k$$

(ii)

$$\sum_{n=2}^{\infty} (n-1) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c-a-b-1} F(a, b; c; 1) \quad \text{if } c > a+b+1$$

(iii)

$$\sum_{n=2}^{\infty} (n-1)^2 \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$= \left[\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1} \right] F(a, b; c; 1) \quad \text{if } c > a+b+2$$

(iv)

$$\sum_{n=2}^{\infty} (n-1)^3 \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left[\frac{(a)_3(b)_3}{(c-a-b-3)_3} + \frac{3(a)_2(b)_2}{(c-a-b-2)_2} \right.$$
$$\left. + \frac{ab}{c-a-b-1} \right] F(a, b; c; 1) \text{ if } c > a+b+3.$$

Proof (i) The proof follows by using (8) and a well-known property $\Gamma(\mathrm{d}+k) = (\mathrm{d})_k \Gamma(\mathrm{d})$.

(ii) In view of (8) and by an application of Part (i) of this lemma, Part (ii) immediately follows.

(iii) Again applying (8), we have

$$\sum_{n=2}^{\infty}(n-1)^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c}\sum_{n=3}^{\infty}(n-2)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}$$

$$= \frac{(a)_2(b)_2}{(c)_2}F(a+2,b+2;c+2;1) + \frac{ab}{c}F(a+1,b+1;c+1;1).$$

The result follows by using Part (i) of this lemma.

(iv) The proof is similar to the proof of Part (iii) and so it is omitted. ■

3. Main results

We first determine the conditions which ensures that the operator Ω maps $\hat{H}$ into $S_{\hat{H}}$.

THEOREM 1 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form* (1), $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ *for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \le 1,\ \ |B_1| < 1,$$

(ii)

$$\sum_{j=1}^{2}\left[\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right]F(|a_j|,|b_j|;c_j;1) \le 2,$$

then $\Omega(f)$ is sense-preserving, harmonic, and univalent in Δ; and so $\Omega(\hat{H}) \subset S_{\hat{H}}$.

Proof Note that $|A_n| \le 1$ and $|B_n| \le 1$ for all $n \ge 2$ by the given condition (i). Therefore, in view of Lemma 1 we observe that $\Omega(f)$ defined by (3) is locally univalent and sense-preserving in Δ provided that the Jacobian

$$J_{\Omega(f)}(z) = \big(|(h \star \phi_1'(z)| + |(g \star \phi_2'(z)|)(|(h \star \phi_1'(z)| - |(g \star \phi_2'(z)|\big) \ge 0.$$

However, it suffices to prove that $|(h \star \phi_1'(z)| - |(g \star \phi_2'(z)| \ge 0$. In view of Lemma 10(i), it follows that

$$\big|(h \star \phi_1'(z)\big| - \big|(g \star \phi_2'(z)\big| \ge 1 - \sum_{n=2}^{\infty}n\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=1}^{\infty}n\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$$

$$= 2 - \left[\frac{|a_1b_1|}{c_1 - |a_1| - |b_1| - 1} + 1\right] F(|a_1|, |b_1|; c_1; 1)$$
$$- \left[\frac{|a_2b_2|}{c_2 - |a_2| - |b_2| - 1} + 1\right] F(|a_2|, |b_2|; c_2; 1)$$

is non-negative by the given condition (ii). To show that $\Omega(f) = H + \bar{G}$ given by (3) is univalent in Δ, we assume that $z_1, z_2 \in \Delta$, $z_1 \neq z_2$. Then by using (4) we have

$$\begin{aligned}\left|\frac{\Omega(f)(z_1) - \Omega(f)(z_2)}{H(z_1) - H(z_2)}\right| &\geq 1 - \left|\frac{G(z_1) - G(z_2)}{H(z_1) - H(z_2)}\right| \\ &\geq 1 - \frac{\sum_{n=1}^{\infty} n((|a_2|)_{n-1}(|b_2|)_{n-1})/((c_2)_{n-1}(1)_{n-1})}{1 - \sum_{n=2}^{\infty} n((|a_1|)_{n-1}(|b_1|)_{n-1})/((c_1)_{n-1}(1)_{n-1})} \\ &\geq 1 - \frac{|G'(z)|}{|H'(z)|}.\end{aligned}$$

Since by the first part of the theorem, $\Omega(f) = H + \bar{G}$ is locally univalent and sense-preserving in Δ, an application of Lemma 1 shows that the last inequality is strictly positive and, therefore, $\Omega(f)$ is univalent in Δ. ■

We next investigate the conditions which gaurantee that a harmonic convex function maps the unit disk Δ into a harmonic close-to-convex domain.

THEOREM 2 *If $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 2$ for $j = 1, 2$ are such that*

$$\begin{aligned}&\left[\frac{(|a_1|)_2(|b_1|)_2}{2(c_1 - |a_1| - |b_1| - 2)_2} + \frac{2|a_1b_1|}{c_1 - |a_1| - |b_1| - 1} + 1\right] F(|a_1|, |b_1|; c_1; 1) \\ &\quad + \left[\frac{(|a_2|)_2(|b_2|)_2}{2(c_2 - |a_2| - |b_2| - 2)_2} + \frac{|a_2b_2|}{c_2 - |a2| - |b_2| - 1}\right] F(|a_2|, |b_2|; c_2; 1) \leq 2,\end{aligned} \tag{10}$$

then $\Omega(K_{\hat{H}}^0) \subset C_{\hat{H}}^0$.

Proof Let $f = h + \bar{g} \in K_{\hat{H}}^0$ with h and g of the form (1) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{G} \in C_{\hat{H}}^0$, where H and G given by (4) with $B_1 = 0$ are analytic functions in Δ. Note that $|H'(0)| = 1 > |B_1| = |G'(0)|$. Therefore, in view of Lemma 2, it suffices to prove that $H + \varepsilon G = z + \sum_{n=2}^{\infty} t_n z^n \in C$ for every $\varepsilon(|\varepsilon| = 1)$, where

$$t_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n + \varepsilon \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n. \tag{11}$$

As an application of Lemma 5(i), we only need to prove that $\sum_{n=2}^{\infty} n|t_n| \leq 1$. Using Lemma 4 we have

$$\sum_{n=2}^{\infty} n|t_n| \leq \frac{1}{2} \sum_{n=2}^{\infty} \left[n(n+1)\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + n(n-1)\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right].$$

Writing

$$n(n+1) = (n-1)^2 + 3(n-1) + 2, \quad n(n-1) = (n-1)^2 + (n-1),$$

and using Lemma 10 (i–iii) and (10), it is a routine matter to show that $\sum_{n=2}^{\infty} n|t_n| \leq 1$. ■

THEOREM 3 *If $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 3$ for $j = 1, 2$ are such that*

$$\begin{aligned}&\left[\frac{(|a_1|)_3(|b_1|)_3}{3(c_1-|a_1|-|b_1|-3)_3} + \frac{5(|a_1|)_2(|b_1|)_2}{2(c_1-|a_1|-|b_1|-2)_2} + \frac{4|a_1 b_1|}{c_1-|a_1|-|b_1|-1} + 1\right]\\ &\times F(|a_1|, |b_1|; c_1; 1)\\ &+\left[\frac{(|a_2|)_3(|b_2|)_3}{3(c_2-|a_2|-|b_2|-3)_3} + \frac{3(|a_2|)_2(|b_2|)_2}{2(c_2-|a_2|-|b_2|-2)_2} + \frac{|a_2 b_2|}{c_2-|a_2|-|b_2|-1}\right]\\ &\times F(|a_2|, |b_2|; c_2; 1) \le 2,\end{aligned}$$

then $\Omega(C^0_{\hat{H}}) \subset C^0_{\hat{H}}$, $\Omega(S^{*0}_{\hat{H}}) \subset C^0_{\hat{H}}$ and $\Omega(T^0_{\hat{H}}) \subset C^0_{\hat{H}}$.

Proof Let $f = h + \bar{g} \in C^0_{\hat{H}}$ ($S^{*0}_{\hat{H}}$ or $T^0_{\hat{H}}$) with h and g of the form (1). We need to show that $\Omega(f) = H + \bar{G} \in C^0_{\hat{H}}$, where H and G given by (4) are analytic functions in Δ. Using Lemma 2, Lemma 5(i) and proceeding similar to the proof of Theorem 2, it suffices to prove that $\sum_{n=2}^{\infty} n|t_n| \le 1$, where t_n is given by (11). Using Lemma 6 and identities

$$\begin{aligned}n(2n+1)(n+1) &= 2(n-1)^3 + 9(n-1)^2 + 13(n-1) + 6,\\ n(2n-1)(n-1) &= 2(n-1)^3 + 3(n-1)^2 + (n-1),\end{aligned}$$

it follows by applications of Lemma 10(i–iv) that $\sum_{n=2}^{\infty} n|t_n| \le 1$, by the given condition. ■

Letting $a_1 = a_2 = 1$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$ in Theorem 3, we obtain the following result for the harmonic convolution operator L defined by (5).

COROLLARY 1 *Let $b \in \mathbb{C}\setminus\{0\}$, $c \in \mathbb{R}$, and $c > |b| + 4$. If*

$$\left[\frac{4(|b|)_3}{(c-|b|-4)_3} + \frac{8(|b|)_2}{(c-|b|-3)_2} + \frac{5|b|}{c-|b|-2} + 1\right]\varphi(b; c; 1) \le 2,$$

then $L(C^0_{\hat{H}}) \subset C^0_{\hat{H}}$, $L(S^{*0}_{\hat{H}}) \subset C^0_{\hat{H}}$, $L(T^0_{\hat{H}}) \subset C^0_{\hat{H}}$.

Using Lemma 7 we can extend the above theorem for the full class $C_{\hat{H}}$. However, the corresponding statement and proof for $C_{\hat{H}}$ becomes very lengthy and messy. We, therefore, look at the special case when $a_1 = a_2 = a$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$ so that $\phi_1 = \phi_2 = \phi$ and $\Omega(f) = f\tilde{\star}(\phi + \bar{\phi}) = h \star \phi + \overline{g \star \phi}$ for any $f = h + \bar{g} \in C_{\hat{H}}$. For the next result, we use the notation $\Omega(f) = \Omega(a, b, c; f)$.

THEOREM 4 *If $a,\ b \in \mathbb{C}\setminus\{0\}$, $c \in \mathbb{R}$, $c > |a| + |b| + 3$, then a sufficient condition for $\Omega(a, b, c; C_{\hat{H}}) \subset C_{\hat{H}}$ is such that*

$$\begin{aligned}&\left[\frac{2(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3} + \frac{12(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{15|ab|}{c-|a|-|b|-1} + 3\right]\\ &\times F(|a|, |b|; c; 1) \le \frac{6}{1-|B_1|}\end{aligned}$$

for all $f \in C_{\hat{H}}$, where $|B_1| = |f_{\bar{z}}(0)| < 1$.

Proof Let $f = h + \bar{g} \in C_{\hat{H}}$ with h and g of the form (1). Then

$$\Omega(a, b, c; f)(z) = H(z) + \overline{G(z)}$$
$$= z + \overline{B_1 z} + \sum_{n=2}^{\infty} \left[\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n z^n + \overline{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} B_n z^n} \right].$$

We need to show that $H + \bar{G} \in C_{\hat{H}}$, where H and G are analytic functions in Δ and $|H'(0)| = 1 > |B_1| = |G'(0)|$. In view of Lemma 2, it suffices to prove that

$$\frac{H + \varepsilon G}{1 + \varepsilon B_1} = z + \sum_{n=2}^{\infty} \ell_n z^n \in C$$

for every $\varepsilon (|\varepsilon| = 1)$, where

$$\ell_n := \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \left[\frac{A_n + \varepsilon B_n}{1 + \varepsilon B_1} \right] \tag{12}$$

As an application of Lemma 5(i), we only need to show that $\sum_{n=2}^{\infty} n|\ell_n| \leq 1$. Using Lemma 7, the identity

$$n(2n^2 + 1) = 2(n-1)^3 + 6(n-1)^2 + 7(n-1) + 3,$$

we have

$$\sum_{n=2}^{\infty} n|\ell_n| \leq \frac{2(1 + |B_1|)}{6(1 - |B_1|)} \sum_{n=2}^{\infty} n(2n^2 + 1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$= \frac{(1 + |B_1|)}{3(1 - |B_1|)} \left[\frac{2(|a|)_3(|b|)_3}{(c - |a| - |b| - 1)_3} + \frac{12(|a|)_2(|b|)_2}{(c - |a| - |b| - 1)_2} \right.$$
$$\left. + \frac{15|ab|}{c - |a| - |b| - 1} + 3 \right] F(|a|, |b|; c; 1) - \frac{(1 + |B_1|)}{(1 - |B_1|)}.$$

The last inequality is bounded above by one because of the given condition. ∎

We next determine sufficient conditions for a mapping f that maps $\hat{H}$ into a harmonic close-to-convex domain.

THEOREM 5 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1), $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j|$ for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| \leq 1, |B_1| < 1,$$

(ii)

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1| < 3, \quad \textit{then } \Omega(f) \in C_{\hat{H}}.$$

Proof Note that $\Omega(f) = H + \bar{G}$, where H and G are given by (4). Then

$$\omega(z) := \frac{H(z) + \varepsilon G(z)}{1 + \varepsilon B_1(z)} = z + \sum_{n=2}^{\infty} s_n$$

where

$$s_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{A_n}{1 + \varepsilon B_1} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{\varepsilon B_n}{1 + \varepsilon B_1}.$$

From the given condition (i), it follows that $|A_n| \le 1/n$ and $|B_n| \le 1/n \; \forall n \ge 2$. We, therefore, have

$$\sum_{n=2}^{\infty} n|s_n| \le \sum_{n=2}^{\infty} \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{1}{1 - |B_1|} + \sum_{n=2}^{\infty} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{1}{1 - |B_1|}$$
$$= [F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) - 2] \frac{1}{1 - |B_1|} \le 1,$$

by the given condition (ii). As an application of Lemma 5(i), it follows that ω is starlike in Δ and therefore $\omega \in C$. Hence by Lemma 2, $\Omega(f) = H + \bar{G}$ is in $C_{\hat{H}}$. ■

Lemma 9(i) together with the above theorem yield the following

Corollary 2 *If $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j|$ for $j = 1, 2$ are such that*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) < 3,$$

then $\phi_1 + \varepsilon\overline{\phi_2} \in C_{\hat{H}}$ for each $\varepsilon (|\varepsilon| < 1)$.

Proof Note that for each $\varepsilon (|\varepsilon| < 1)$, we have

$$f(z) := \frac{z}{1 - z} + \varepsilon\overline{\left(\frac{z}{1 - z}\right)} = z + \sum_{n=2}^{\infty} z^n + \sum_{n=1}^{\infty} \overline{\bar{\varepsilon} z^n} \in S_{\hat{H}},$$

and $\Omega(f) = f \tilde{*} (\phi_1 + \overline{\phi_2}) = h * \phi_1 + \overline{g * \phi_2} = \phi_1 + \varepsilon\overline{\phi_2}$. Also

$$\sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| = |\varepsilon| + (1 + |\varepsilon|) \sum_{n=2}^{\infty} n \le 1,$$

because $|\varepsilon| \le \left(1 - \sum_{n=2}^{\infty} n\right) / \left(1 + \sum_{n=2}^{\infty} n\right) < 1$. The result follows from the above theorem. ■

Analogous to Theorem 5, we now obtain sufficient conditions for a mapping f in $\hat{H}$ into a harmonic convex domain.

Theorem 6 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1), $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty} n^2|A_n| + \sum_{n=1}^{\infty} n^2|B_n| \le 1, \; |B_1| < 1,$$

(ii)

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1| < 3,$$

then $\Omega(f) \in K_{\hat{H}}$.

Proof If $\Omega(f) = H + \bar{G}$, where H and G are given by (4), then

$$\varsigma(z) := \frac{H(z) - e^{it}G(z)}{1 - B_1 e^{it}} = z + \sum_{n=2}^{\infty} \rho_n z^n,$$

where

$$\rho_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{A_n}{1 - e^{it}B_1} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{e^{it}B_n}{1 - e^{it}B_1}.$$

On the other hand, condition (i) yields $|A_n| \leq 1/n^2$ and $|B_n| \leq 1/n^2$, $\forall n \geq 2$ and so using the arguments similar to the proof of Theorem 5, it is easy to prove that $\sum_{n=2}^{\infty} n^2 |\rho_n| \leq 1$, by the condition (ii). It therefore follows from Lemma 5(ii) that ς is convex in Δ and therefore by Lemma 3, $\Omega(f) \in K_{\hat{H}}$. ■

Using Lemma 9(ii), the above theorem gives the following

Corollary 3 *If* $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ *for* $j = 1, 2$ *and* $|B_1| < 1$ *are such that*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1|,$$

then $\Omega(K_{R\hat{H}}) \subset K_{\hat{H}}$.

The next result is more general than the above corollary and the method of proof is also different.

Theorem 7 *If* $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ *for* $j = 1, 2,$ *and* $0 \leq \alpha < 1$ *are such that*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 2,$$

then $\Omega(K_{R\hat{H}}(\alpha)) \subset K_{\hat{H}}(\alpha)$.

Proof Let $f = h + \bar{g} \in K_{R\hat{H}}(\alpha)$ with h and g are of the form (9). It follows from Lemma 9(ii) that

$$|A_n| \leq \frac{1-\alpha}{n(n-\alpha)} \ \forall n \geq 2, |B_n| \leq \frac{1-\alpha}{n(n+\alpha)} \forall n \geq 1, |B_1| < 1.$$

Note that

$$(\Omega(f))(z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}. \tag{13}$$

In view of Lemma 8(ii), $\Omega(f) \in K_{\hat{H}}(\alpha)$ because

$$\begin{aligned} P_1 &:= \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| \right| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| \right| \\ &\leq \sum_{n=2}^{\infty} \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \\ &= F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) - 1 \leq 1, \end{aligned}$$

by the given condition. ■

We now dilute the restrictions on complex coefficients and obtain a different sufficient condition for harmonic convolution operator Ω applied on $K_{R\hat{H}}(\alpha)$.

THEOREM 8 *If* $a_1, b_1 > -1, a_1b_1 < 0, a_2, b_2 \in \mathbb{C}\setminus\{0\}, c_1 > a_1 + b_1, c_2 > |a_2| + |b_2|$, *then a sufficient condition for* $\Omega(K_{R\hat{H}}(\alpha)) \subset K_{\hat{H}}(\alpha)$ *is that*

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0.$$

Proof If $f = h + \bar{g} \in K_{R\hat{H}}(\alpha)$ with h and g in (9), then $\Omega(f)$ in (13) can be rewritten as

$$z + \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}}|A_n|z^n + \overline{\sum_{n=1}^{\infty}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|z^n}.$$

In view of Lemma 8(ii), $\Omega(f) \in K_{\hat{H}}(\alpha)$ if $P_2 \leq 1$ where

$$P_2 := \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{n(n-\alpha)}{1-\alpha}\left|\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}}|A_n|\right| + \sum_{n=1}^{\infty}\frac{n(n+\alpha)}{1-\alpha}\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right|.$$

Following the steps of the preceeding theorem and using Lemma 10(i) and the given condition, it can easily be proved that $P_2 \leq 1$. ■

Our next result will parallel the above theorem for the case of harmonic starlike functions of order α and so its proof is ommitted. However, note that the proof of the next theorem uses Lemma 9(i), Lemma 8(i) and Lemma 10(i).

THEOREM 9 *If* $a_1, b_1 > -1, a_1b_1 < 0, a_2, b_2 \in \mathbb{C}\setminus\{0\}, c_1 > a_1 + b_1, c_2 > |a_2| + |b_2|$, *then a sufficient condition for* $\Omega(S^*_{R\hat{H}}(\alpha)) \subset S^*_{\hat{H}}(\alpha)$ *is that*

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0.$$

Finally, by imposing stronger conditions on a_1,a_2, b_1,b_2 c_1, and c_2 we obtain a characterization for the operator Ω which maps $S^*_{RH}(\alpha)$ into itself.

THEOREM 10 *If* $a_j, b_j > 0, c_j > a_j + b_j$ *for* $j = 1, 2,$ *then a necessary and sufficient condition for* $\Omega(S^*_{R\hat{H}}(\alpha)) \subset S^*_{R\hat{H}}(\alpha)$ *is that*

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 2. \tag{14}$$

Proof If $f = h + \bar{g} \in S^*_{R\hat{H}}(\alpha)$ with h and g given in (9), then $\Omega(f)$ has the form (13). As an application of Lemma 9(i) it follows that $\Omega(f) \in S^*_{R\hat{H}}(\alpha)$ if and only if $P_3 \leq 1$ where

$$P_3 := \sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|\right| + \sum_{n=1}^{\infty}\frac{n+\alpha}{1-\alpha}\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right|.$$

Again using Lemma 9(i) and following the procedure as in Theorem 8, it is straightforward to show that $P_3 \leq 1$ if and only if the expression (14) holds. ■

Using similar arguments and Lemma 9(ii) in place of Lemma 9(i) in the previous theorem, we obtain the following characterization for the operator Ω which maps $K_{R\hat{H}}(\alpha)$ onto itself.

THEOREM 11 *If* $a_j, b_j > 0, c_j > a_j + b_j$ *for* $j = 1, 2,$ *then* $\Omega(K_{RH}(\alpha)) \subset K_{RH}(\alpha)$ *if and only if* (14) *is satisfied* .

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