JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 173, 542–549 (1993)

Perturbations of Classes of Functions and Their Related Dual Spaces*

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Submitted by Steven G. Krantz

Received July 16, 1990

1. PRELIMINARY RESULTS

Let $\mathcal{A}$ denote the class of functions $f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ that are analytic in the unit disk $U=\{z: |z|<1\}$. For $\delta \geqslant 0$, St. Ruscheweyh [7] defined a δ-neighborhood of $f \in \mathcal{A}$ as $N_\delta(f) := \{g(z)=z+\sum_{k=2}^{\infty} b_k z^k \in \mathcal{A}: \sum_{k=2}^{\infty} k|a_k-b_k| \leqslant \delta\}$. With this notation, we can state a well-known [4] and classical result as $N_1(z) \subset St$, the class of starlike functions. A criterion for such a result with other starlike functions is given by

LEMMA A [7]. *If $f \in \mathcal{A}$ is such that $(f(z)+\varepsilon z)/(1+\varepsilon)$ is starlike for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon|<\delta$, then $N_\delta(f) \subset St$.*

* This work was completed while the first and third authors were on sabbatical leave and the second author was a Visiting Lecturer at UC Davis.

0022-247X/93 $5.00

Further, it was shown that for every f in the class of convex functions, $N_{1/4}(f) \subset St$.

Analogous results for δ-neighborhoods involving other subclasses of $\mathscr{A}$ can be found in [2, 3, 1].

For $\delta \geqslant 0$, $T = \{T_K\}_{k=2}^{\infty}$ a sequence of nonnegative reals, and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathscr{A}$, a T-δ-neighborhood of f is defined by

$$TN_{\delta}(f) = \left\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{A} : \sum_{k=2}^{\infty} T_k |a_k - b_k| \leqslant \delta \right\}.$$

The notion of a T-δ-neighborhood was introducted in [8] where many of the results known for δ-neighborhoods were generalized. Also given was a general criterion for determining appropriate T-δ-neighborhood information that related various subclasses of analytic functions.

A different coefficient variation of $f \in \mathscr{A}$ is obtained by considering $f(z) + \varepsilon z^n$, for fixed n, $n = 2, 3, 4, \ldots$. Q. I. Rahman and J. Stankiewicz [5] showed

LEMMA B. *If for some fixed n, $n \geqslant 2$, $f(z) + \varepsilon z^n \in St$ for all $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| < \delta$, then the same is true for $(f(z) + \varepsilon z)/(1 + \varepsilon)$.*

From Lemma A and Lemma B, it follows that if $f(z) + \varepsilon z^n \in St$ for some fixed $n \geqslant 2$ and for all ε with $|\varepsilon| < \delta$, then $N_{\delta}(f) \subset St$.

Remark. Note that the collection $f(z) + \varepsilon z^n$, $|\varepsilon| < \delta$, of perturbations of f is a set of linear variations in range over one coefficient of f. It is natural to ask about extending this to more than one coefficient. For instance, if $f(z) = z$, we see that both sets $f(z) + \varepsilon z^2$, $|\varepsilon| < 1/9$, and $f(z) + \varepsilon z^3$, $|\varepsilon| < 1/9$, consist of univalent functions. Extending this, we see that $\{z + \varepsilon z^n : n = 2, 3, \ldots, k, |\varepsilon| < 1/k\}$ consists of univalent functions. On the other hand, there is a limit on the extent to which this is possible. For example, there is no univalent function $f \in \mathscr{A}$ such that $f(z) + \varepsilon z^n$ is univalent for all n and for all ε, $|\varepsilon| < \delta$. To see this, let $g_{\varepsilon,n}(z) = f(z) + \varepsilon z^n$, $|\varepsilon| < \delta$, and choose an arc Γ from 0 to the boundary of U. To each $z_{\gamma} = r_{\gamma} e^{i\theta_{\gamma}} \in \Gamma$ and n fixed, set $f'(z_{\gamma}) = A_{\gamma} e^{i\gamma}$, $A_{\gamma} > 0$, and $\varepsilon = |\varepsilon| e^{i\beta}$, $\beta = \gamma - (n-1)\theta_{\gamma} + \pi$. Then $g'(z_{\gamma}) = e^{i\gamma}[A_{\gamma} - n|\varepsilon| r_{\gamma}^{n-1}]$. Since $nr_{\gamma}^{n-1} \to \infty$ as $n \to \infty$ for $r_{\gamma} = 1 - 1/n$, we see that $A_{\gamma} - n|\varepsilon| r_{\gamma}^{n-1} < 0$ for $r_{\gamma} = 1 - 1/n$, n (fixed) sufficiently large. We now let $|\varepsilon| \to 0$ and observe that $A_{\gamma} - n|\varepsilon| r_{\gamma}^{n-1} > 0$ for $|\varepsilon|$ sufficiently small. Thus, g' vanishes somewhere in the unit disk and g cannot be univalent.

In this note, we determine more general conditions under which $f(z) + \varepsilon z^n$, $|\varepsilon| < \delta$, in a class implies that $(f(z) + \varepsilon z)/(1 + \varepsilon)$, $|\varepsilon| < \delta$, is also. We apply our results to the following three classes.

For $-1 \leqslant B < A \leqslant 1$,

$$S[A, B] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1 + Az)/(1 + Bz) \text{ for } z \in U\}$$
$$K[A, B] = \{f \in \mathcal{A} : zf' \in S[A, B]\}.$$

For λ real, $|\lambda| < \pi/2$,

$$Sp(\lambda) = \left\{f \in \mathcal{A} : \operatorname{Re}\left\{\frac{e^{i\lambda} zf'(z)}{f(z)}\right\} > 0\right\}.$$

The classes $S[A, B]$ and $K[A, B]$ are subclasses of starlike and convex functions, respectively, while $Sp(\lambda)$ is the class of λ-spiral-like functions.

For $f \in \mathcal{A}$, let

$$f_{n,\varepsilon}(z) = \begin{cases} \dfrac{f(z) + \varepsilon z}{1 + \varepsilon}, & n = 1 \\ f(z) + \varepsilon z^n, & n = 2, 3, \ldots. \end{cases}$$

Note that if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then $f_{1,\varepsilon}(z) = z + \sum_{k=2}^{\infty} a_k (1 + \varepsilon)^{-1} z^k$. Hence, for fixed $\varepsilon \in \mathbb{C}$, we see that the expansion of $f_{1,\varepsilon}$ can be obtained from that of f by multiplying each coefficient—after the first one—by a constant, while $f_{n,\varepsilon}$ is obtained by adding a constant to the nth coefficient of f. Lemma B states that, over the class of starlike functions, expecting $f_{n,\varepsilon}$ to be in the class for a fixed n and $|\varepsilon| < \delta$ is at least as strong a requirement as expecting $f_{1,\varepsilon}$ to be in the class. The general conditions we will obtain are more conveniently stated in terms of dual spaces [6].

For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{A}$, the convolution or Hadamard product of f and g is $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Given a normal family $\mathcal{T} \subset \mathcal{A}$, the dual of $\mathcal{T}$, denoted by $\mathcal{T}^*$, is $\{f \in \mathcal{A} : f * g \neq 0 \text{ for all } g \in \mathcal{T},\ 0 < |z| < 1\}$. The classes $S[A, B]$ and $K[A, B]$ are known [10] to be the duals of

$$\mathcal{T}_1[A, B] = \left\{\frac{z - ((A - \zeta)/(A - B))z^2}{(1 - z)^2} : |\zeta| = 1\right\}$$

and

$$\mathcal{T}_2 = \left\{\frac{z + ((2\zeta - A - B)/(A - B))z^2}{(1 - z)^3} : |\zeta| = 1\right\}, \qquad \text{respectively,}$$

while $Sp(\lambda)$ was shown in [9] to be the dual of

$$\mathcal{T}_3[\lambda] = \left\{\frac{z + ((\zeta - e^{-2i\lambda})/(1 + e^{-2i\lambda}))z^2}{(1 - z)^2} : |\zeta| = 1\right\}.$$

For other dual spaces, see [7, 6, 11, 9].

Dual spaces offer a useful characterization for membership in classes. The characterizations can afford an alternative and simpler way to prove various extremal properties. For example, in [9] a convolution condition is used to obtain the radius of convexity for totally monotone functions. The convolution characterizations allow us to benefit from the distributive property of the Hadamard product. Consequently, they are easier to work with than the characterizations that involve quotients of linear functionals.

In the following application of a dual space characterization, we will use $\mathscr{T}_1[1, -1]$ to prove

THEOREM 1. *For fixed n,* $n \geq 2$, $f_{n,\varepsilon} \in St$ *for every* $f \in K = K[1, -1]$ *if and only if* $|\varepsilon| < 1/4n$.

Proof. As noted earlier, $N_{1/4}(K) \subset St$. Hence, $f(z) + \varepsilon z^n \in N_{1/4}(f)$ whenever $|\varepsilon| < 1/4n$. On the other hand, since $\mathscr{T}_1[1, -1]^* = St$, $z/(1-z) + \varepsilon z^n \in St$ if and only if

$$\left(\frac{z - ((1-\zeta)/2)z^2}{(1-z)^2}\right) * \left(\frac{z}{1-z} + \varepsilon z^n\right) \neq 0, \qquad |\zeta| = 1.$$

Substituting $\zeta = 1$ into the left-hand side yields

$$\left(\frac{z}{(1-z)^2}\right) * \left(\frac{z}{1-z} + \varepsilon z^n\right) = \frac{z}{(1-z)^2} + n\varepsilon z^n$$

which vanishes at $z = -1$ when $\varepsilon = (-1)^n/4n$. Consequently, a continuity argument justifies that $(z/(1-z) + \varepsilon z^n) \notin St$ for $|\varepsilon| > 1/4n$. ∎

Unless otherwise specified, $\mathscr{T} \subset \mathscr{A}$ denotes a normal family such that

$$\tau_k = \sup\left\{|t_k| : t(z) = z + \sum_{n=2}^{\infty} t_n z^n \in \mathscr{T}\right\} > 0.$$

It is easy to determine τ_n associated with $\mathscr{T}_1[A, B]$ and $\mathscr{T}_2[A, B]$. To find $\tau_{n,1} := \tau_n$ for $\mathscr{T}_1[A, B]$, observe that for $z + \sum_{n=2}^{\infty} t_n(\zeta) z^n := (z - ((A-\zeta)/(A-B))\, z^2)/(1-z^2)$, we have $t_n(\zeta) = n - (n-1)(A-\zeta)/(A-B)$. Hence,

$$|t_n(e^{i\theta})|^2 = \frac{(A-nB)^2 + 2(n-1)(A-nB)\cos\theta + (n-1)^2}{(A-B)^2}$$

$$\leq \frac{(A-nB)^2 + 2(n-1)\,|A-nB| + (n-1)^2}{(A-B)^2}.$$

Therefore,

$$\tau_{n,1}=\begin{cases}\dfrac{n-1+A-nB}{A-B}, & A-nB\geqslant 0\\[2mm] \dfrac{n-1-A+nB}{A-B}, & A-nB<0.\end{cases}$$

Note that $\tau_{n,1}\geqslant 1$. This follows from a direct computation when $A-nB\geqslant 0$. For the case $A-nB<0$, we see that $(n-1-A+nB)/(A-B)\geqslant 1$ if and only if $n-1-A+B\geqslant A-nB$. For $n\geqslant 3$, the latter follows because $n-1-A+B\geqslant 2-A+B=(1-A)+(1+B)\geqslant 0\geqslant A-nB$. For $n=2$, it suffices to show that $1\geqslant 2A-3B$. But $A-2B<0$ implies that $2A-3B<4B-3B=B<1$ as needed. It follows immediately that, for $\mathscr{T}_2[A,B]$, $\tau_n:=\tau_{n,2}=n\tau_{n,1}\geqslant n$.

For $\mathscr{T}_3[\lambda]$ we can show that $n\leqslant\tau_n:=\tau_{n,3}\leqslant n/\cos\lambda$. For $t(z)=z+\sum_{n=2}^{\infty}t_nz^n\in\mathscr{T}_3[\lambda]$, we have

$$t_n(\zeta)=\frac{n+e^{-2i\lambda}+(n-1)\zeta}{1+e^{-2i\lambda}}=\frac{e^{i\lambda}(n+e^{-2i\lambda}+(n-1)\zeta)}{2\cos\lambda}.$$

Then $|t_n(\zeta)|\leqslant n/\cos\lambda$ and $t_n(e^{-2i\lambda})=n$. Hence, $n/\cos\lambda\geqslant\tau_{n,3}\geqslant n$.

We conclude this section with a convenient containment result that is nicely stated in terms of duals.

THEOREM 2. *A function* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *is in* $\mathscr{T}^*$ *whenever* $\sum_{n=2}^{\infty}\tau_n|a_n|\leqslant 1$.

Proof. We know that $f\in\mathscr{T}^*$ if and only if $(1/z)[f(z)*t(z)]\neq 0$ for $z\in U$, for all $t(z)=z+\sum_{n=2}^{\infty}t_nz^n\in\mathscr{T}$. That is, $1+\sum_{n=2}^{\infty}a_nt_nz^{n-1}\neq 0$, for all $z\in U$. The last inequality certainly holds whenever $\sum_{n=2}^{\infty}\tau_n|a_n|\leqslant 1$. ∎

Note that for $\mathscr{T}=\mathscr{T}_1[1,-1]$, $\tau_n=n$ so that Theorem 2 generalizes the classical result: $\sum_{n=2}^{\infty}n|a_n|\leqslant 1$ implies that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in St$.

2. CONTAINMENT PROPERTIES FOR $\mathscr{T}^*$

We begin this section by showing that the starlikeness criterion of Lemma B is a special case of a property of dual spaces satisfying certain conditions. To see this, we will make use of

LEMMA C [8]. *Suppose* $\mathscr{T}\subset\mathscr{A}$ *is a normal family such that* $\mathscr{T}^*$ *is compact normal and* $\lambda(z)=z+\sum_{k=2}^{\infty}\lambda_kz^k\in\mathscr{A}$. *A necessary and sufficient condition for*

$$\left|\frac{(\lambda*t)(z)}{z}\right|\geqslant\delta>0\qquad(t\in\mathscr{T}),$$

is that

$$\frac{\lambda(z)+\delta xz}{1+\delta x} \in \mathscr{T}^* \qquad (|x| \leqslant 1).$$

THEOREM 3. *Suppose $\mathscr{T} \subset \mathscr{A}$ is a normal family such that $\mathscr{T}^*$ is a compact normal family. If for some fixed integer $n \geqslant 2$, $\tau_n \geqslant 1$ and $f_{n,\varepsilon} \in \mathscr{T}^*$ for all ε, $|\varepsilon| < \delta$, then $f_{1,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon| < \delta$.*

Proof. For $t(z) = z + \sum_{k=2}^{\infty} t_k z^k \in \mathscr{T}$, let $\varphi_n(z) = z^n/(f^*t)(z)$. Since $f(z) + \varepsilon z^n \in \mathscr{T}^*$, we know that $(f(z)+\varepsilon z^n)^*\, t(z) \neq 0$ for $0 < |z| < 1$. Hence, $\varphi_n(z) \neq -1/\varepsilon t_n$ for $|\varepsilon| < \delta$. Since $\varphi_n(0) = 0$ and $\tau_n \geqslant 1$, we conclude that $|\varphi_n(z)| \leqslant 1/\delta\tau_n \leqslant 1/\delta$. But $\varphi_n(z)$ is analytic in U and has an $(n-1)$-fold zero at the origin. Consequently, we have

$$|\varphi_n(z)| \leqslant \frac{|z|^{n-1}}{\delta} \text{ for } z \in U; \qquad \text{that is,} \quad \left|\frac{(f^*t)(z)}{z}\right| \geqslant \delta.$$

By Lemma C, it follows that $(f(z)+\delta xz)/(1+\delta x) \in \mathscr{T}^*$, $|x| \leqslant 1$. ∎

Remark. As noted earlier, we have $\tau_n \geqslant 1$ for each of $\mathscr{T}_1[A, B]$, $\mathscr{T}_2[A, B]$, and $\mathscr{T}_3[\lambda]$. Consequently, Theorem 3 applies to $\mathscr{T}^*$ where $\mathscr{T}$ is any one of these classes. That is, for fixed n, $n \geqslant 2$, $f_{n,\varepsilon}$ in $S[A, B]$, $K[A, B]$, or $Sp(\lambda)$ for $|\varepsilon| < \delta$ implies that $f_{1,\varepsilon}$ is in $S[A, B]$, $K[A, B]$, or $Sp(\lambda)$, respectively, for $|\varepsilon| < \delta$. Consequently, Theorem 3 generalizes the results obtained in [5].

Due to the following lemma, information about $(f(z)+\varepsilon z)/(1+\varepsilon)$, $|\varepsilon| < \delta < 1$, leads directly to results concerning T-δ-neighborhoods of f.

LEMMA D [8]. *Let $f \in \mathscr{A}$. If there exists a δ, $0 < \delta \leqslant 1$, such that $(f(z)+\varepsilon z)/(1+\varepsilon) \in \mathscr{T}^*$, for $|\varepsilon| < \delta$, then $TN_\delta(f) \subset \mathscr{T}^*$ where $T = \{\tau_k\}_{k=2}^{\infty}$.*

Combining Lemma D and Theorem 3 yields

COROLLARY 1. *Let $f \in \mathscr{A}$* and let $\mathscr{T}$ be such that $\tau_n \geqslant 1$. *If for some integer $n \geqslant 2$ there exists a δ, $0 < \delta < 1$, such that $f(z) + \varepsilon z^n \in \mathscr{T}^*$, then $TN_\delta(f) \subset \mathscr{T}^*$ where $T = \{\tau_k\}_{k=2}^{\infty}$.*

Since Theorem 3 applies to each of $\mathscr{T}_1[A, B]$, $\mathscr{T}_2[A, B]$, and $\mathscr{T}_3[\lambda]$, we immediately claim

COROLLARY 2. *If for a fixed integer n, $n \geqslant 2$, $f_{n,\varepsilon}$ is in $S[A, B]$ ($K[A, B]$) for ε, $|\varepsilon| < \delta \leqslant 1$, then $TN_\delta(f)$ is contained in $S[A, B]$ ($K[A, B]$), for $T = \{\tau_k\}_{k=2}^{\infty}$.*

COROLLARY 3. *Let n be a fixed integer n, $n \geqslant 2$. If $f_{n,\varepsilon} \in Sp(\lambda)$ for $|\varepsilon| < \delta \leqslant 1$, then $TN_\delta(f) \subset Sp(\lambda)$ for $T = \{\tau_k\}_{k=2}^{\infty}$.*

Remark. Whenever the class $\mathscr{T}$ is such that $\tau_n \leqslant \rho n$ for all n and for some $\rho \geqslant 1$, we have that $N_{\delta/\rho}(f) \subset TN_\delta(f)$ for $T=\{\tau_k\}_{k=2}^{\infty}$. When $\mathscr{T}=\mathscr{T}_1[A, B]$, it is easy to see that $\tau_{k,1} \leqslant k[(1+|B|)/(A-B)]$ with equality holding only for $A=-B=1$. Thus, by Corollary 2, we have that $f_{n,\varepsilon} \in S[A, B]$ for fixed n, $n \geqslant 2$, and $|\varepsilon|<\delta$ implies that $N_{(A-B)\delta/(1+|B|)}(f) \subset S[A, B]$. The case $B=-1$ and $A=1-2\alpha$, $0 \leqslant \alpha < 1$, was obtained by Q. I. Rahman and J. Stankiewicz [5]. For $\mathscr{T}=\mathscr{T}_3[\lambda]$, $\tau_{n,3} \leqslant n/\cos\lambda$ leads to $N_{\delta \cos\lambda}(f) \subset Sp(\lambda)$ whenever $f_{n,\varepsilon} \in Sp(\lambda)$, $|\varepsilon|<\delta$, as obtained in [5].

Let $\mathscr{T}$ be a normal family such that $\tau_n \geqslant 1$ and $\mathscr{T}^*$ is compact normal. By Theorem 3, we know that $f_{n,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta$, implies that $f_{1,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta$. On the other hand, $f_{n,\varepsilon} \in TN_\delta(f)$, $T=\{\tau_k\}_{k=2}^{\infty}$, whenever $|\varepsilon|<\delta/\tau_n$; $f_{1,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta$ implies that $f_{n,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta/\tau_n$. It is natural to ask if we can draw the conclusion that $f_{n,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta$, from the weakened hypothesis that $f_{n,\varepsilon} \in \mathscr{T}^*$ for $|\varepsilon|<\delta/\tau_n$. While we do not get this result, a slight modification of the proof of Theorem 3 enables us to make use of the infimum of the $|t_k|$ or any positive lower bound on the $|t_k|$ to obtain an alternative hypothesis that will be an improvement whenever the lower bound is greater than one.

THEOREM 4. *Let $f \in \mathscr{A}$, $n \geqslant 2$ a fixed integer, and $\mu_n = \inf\{|t_n|: z+\sum_{j=2}^{\infty} t_j z^j \in \mathscr{T}\} > 0$. If $f_{n,\varepsilon} \in \mathscr{T}^*$ for all ε such that $|\varepsilon|<\delta_n=\delta/\mu_n$, then $f_{1,\varepsilon} \in \mathscr{T}^*$ for $|\varepsilon|<\delta$.*

Proof. Proceeding as in the proof of Theorem 3, from $f_{n,\varepsilon} \in \mathscr{T}^*$ we conclude that $\varphi_n(z)=z^n/(f^*t)(z) \neq -1/\varepsilon t_n$, for all ε, $|\varepsilon|<\delta_n$. Hence, $|\varphi_n(z)| \leqslant 1/\delta_n|t_n| \leqslant 1/\delta_n\mu_n = 1/\delta$ which leads to $|(f^*t)(z)/z| \geqslant \delta$ or $(f(z)+\delta xz)/(1+\delta x) \in \mathscr{T}^*$ for $|x| \leqslant 1$. ∎

COROLLARY 4. *Let $f \in \mathscr{A}$ and $n \geqslant 2$ a fixed positive integer. If $f_{n,\varepsilon} \in \mathscr{T}^*$, $|\varepsilon|<\delta$, then $TN_{\delta\mu_n}(f) \subset \mathscr{T}^*$ where $T=\{\tau_k\}_{k=2}^{\infty}$.*

Proof. We know from the proof of Theorem 4 that $|f^*t(z)/z| \geqslant \delta\mu_n$. Now, for $g(z)=z+\sum_{k=2}^{\infty} b_k z^k \in TN_{\delta\mu_n}(f)$, with $f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ and $t(z)=z+\sum_{k=2}^{\infty} t_k z^k \in \mathscr{T}$, we have

$$\left|\frac{(g^*t)(z)}{z}\right| = \left|\frac{(f^*t)(z)}{z}+\frac{(t^*(g-f))(z)}{z}\right|$$

$$\geqslant \delta\mu_n - \sum_{k=2}^{\infty} |t_k|\,|b_k-a_k|\,|z|^{k-1}$$

$$\geqslant \delta\mu_n - |z| \sum_{k=2}^{\infty} \tau_k |b_k-a_k| \geqslant \delta\mu_n - |z|\,\delta\mu_n = \delta\mu_n(1-|z|) > 0.$$

Thus, $g \in \mathscr{T}^*$ as claimed. ∎

Remark. While Theorem 4 is stated with the infimum, any lower bound on the $|t_k|$ will do. For $\mathscr{T}_1[A, B]$, we may use $||A-Bn|-(n-1)|/(A-B)$. When $A=1$ and $B=0$, this yields that $|t_n| \geqslant (n-2)$. Thus, for $n>2$, we have that

$$f_{n,\varepsilon} \in S[1,0],\ |\varepsilon| < \frac{\delta}{n-2} \text{ implies that } f_{1,\varepsilon} \in S[1,0],\ |\varepsilon| < \delta.$$

This agrees with the result obtained in [5].

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