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Certain multipliers of univalent harmonic functions

Om P. Ahuja, Jay M. Jahangiri*

Mathematical Sciences, Kent State University, 14111 Clairdon Troy Road, Burton, OH 544201-9500, United States

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Dedicated to UC Davis Late Professor Henry Ludwig Alder 03/26/1922–11/06/2002

Abstract

Inequalities involving multipliers using the sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers are introduced for complex-valued harmonic univalent functions. By specializing $\{c_n\}$ and $\{d_n\}$, we determine representation theorems, distortion bounds, convolutions, convex combinations, and neighbourhoods for such functions. The theorems presented, in many cases, confirm or generalize various well-known results for corresponding classes of harmonic functions.

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1. Introduction

Let H denote the class of complex-valued harmonic functions which are univalent, orientation preserving, and normalized in the open unit disk $\Delta = \{z : |z| < 1\}$. Functions in H can be written (e.g. see [1]) in the form $f = h + \bar{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1}$$

* Corresponding author.
E-mail addresses: ahuja@geauga.kent.edu (O.P. Ahuja), jay@geauga.kent.edu (J.M. Jahangiri).

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We call h the analytic part and g the co-analytic part of f. If the co-analytic part of f is zero, then H reduces to the class S of normalized analytic univalent functions in Δ.

The subclass $\overline{H}$ consists of functions $f = h + \bar{g} \in H$ where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \qquad |b_1| < 1. \tag{2}$$

The family SH consists of functions in H which are starlike in Δ. Furthermore, $S\overline{H} \equiv SH \cap \overline{H}$.

It is known that, for example see Jahangiri and Silverman [2], if

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1 - |b_1| \tag{3}$$

then $f \in SH$. The condition (3) is also necessary if $f \in S\overline{H}$.

A function $f = h + \bar{g}$, where h and g are given by (2), is said to be in the multiplier family $F_{\overline{H}}(\{c_n\}, \{d_n\})$ if there exist sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers such that

$$\sum_{n=2}^{\infty} c_n |a_n| + \sum_{n=1}^{\infty} d_n |b_n| \le 1, \qquad d_1 |b_1| < 1. \tag{4}$$

The multipliers $\{c_n\}$ and $\{d_n\}$ provide a transition from harmonic convex to harmonic starlike functions as well as including many more generalized classes of harmonic functions. For example, see Jahangiri and Silverman [2]. In this paper we determine representation theorem, distortion bounds, convolutions, convex combinations, and neighborhoods of functions in $F_{\overline{H}}(\{c_n\}, \{d_n\})$. Information is also obtained when the arguments of the coefficients are unrestricted.

2. Main results

If $n \le c_n$ and $n \le d_n$ then, by (3) we have $F_{\overline{H}}(\{c_n\}, \{d_n\}) \subset S\overline{H}$. Consequently, the functions in $F_{\overline{H}}(\{c_n\}, \{d_n\})$ are sense-preserving, harmonic, and univalent in Δ. We first observe that if $f_1(z) = z - \sum_{n=2}^{\infty} |a_{1_n}| z^n + \sum_{n=1}^{\infty} |b_{1_n}| \bar{z}^n$ and $f_2(z) = z - \sum_{n=2}^{\infty} |a_{2_n}| z^n + \sum_{n=1}^{\infty} |b_{2_n}| \bar{z}^n$ are in $F_{\overline{H}}(\{c_n\}, \{d_n\})$ and $0 \le \lambda \le 1$, then so is the linear combination $\lambda f_1 + (1-\lambda) f_2$, by (4). Therefore, $F_{\overline{H}}(\{c_n\}, \{d_n\})$ is a convex family.

For any compact family F, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull, $clcoF$. We determine the extreme points of the closed convex hull of the family $F_{\overline{H}}(\{c_n\}, \{d_n\})$.

Theorem 1. *$f \in clcoF_{\overline{H}}(\{c_n\}, \{d_n\})$ if and only if f has the representation $f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n(z) + \mu_n g_n(z))$, where $n \le c_n$, $n \le d_n$, $\lambda_n \ge 0$, $\mu_n \ge 0$, $\sum_{n=1}^{\infty} (\lambda_n + \mu_n) = 1$, $h_1(z) = z$, $h_n(z) = z - \frac{z^n}{c_n}$ $(n = 2, 3, \ldots)$ and $g_n(z) = z + \frac{\bar{z}^n}{d_n}$ $(n = 1, 2, \ldots)$. In particular, the extreme points of $F_{\overline{H}}(\{c_n\}, \{d_n\})$ are $\{h_n\}$ and $\{g_n\}$.*

Proof. For functions f of the form $f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n(z) + \mu_n g_n(z))$, we have $f(z) = \lambda_1 h_1(z) + \sum_{n=2}^{\infty} \lambda_n (z - \frac{z^n}{c_n}) + \sum_{n=1}^{\infty} \mu_n (z + \frac{\bar{z}^n}{d_n}) = z - \sum_{n=2}^{\infty} \frac{\lambda_n}{c_n} z^n + \sum_{n=1}^{\infty} \frac{\mu_n}{d_n} \bar{z}^n$. Then

$$\sum_{n=2}^{\infty} \frac{\lambda_n}{c_n} c_n + \sum_{n=1}^{\infty} \frac{\mu_n}{d_n} d_n = \sum_{n=2}^{\infty} \lambda_n + \sum_{n=1}^{\infty} \mu_n \le 1$$

and so $f \in clcoF_{\overline{H}}(\{c_n\},\{d_n\})$. Conversely, suppose that $f \in clcoF_{\overline{H}}(\{c_n\},\{d_n\})$. We set $\lambda_n = c_n|a_n|$ $(n = 2, 3, \ldots)$, $\mu_n = d_n|b_n|$ $(n = 1, 2, \ldots)$, and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n - \sum_{n=1}^{\infty}\mu_n$. Therefore, f can be written as

$$\begin{aligned}
f(z) &= z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\bar{z}^n \\
&= z - \sum_{n=2}^{\infty}\frac{\lambda_n}{c_n}z^n + \sum_{n=1}^{\infty}\frac{\mu_n}{d_n}\bar{z}^n \\
&= z + \sum_{n=2}^{\infty}(h_n(z) - z)\lambda_n + \sum_{n=1}^{\infty}(g_n(z) - z)\mu_n \\
&= \sum_{n=2}^{\infty}h_n(z)\lambda_n + \sum_{n=1}^{\infty}g_n(z)\mu_n + z\left(1 - \sum_{n=2}^{\infty}\lambda_n - \sum_{n=1}^{\infty}\mu_n\right) \\
&= \sum_{n=1}^{\infty}(h_n(z)\lambda_n + g_n(z)\mu_n).
\end{aligned}$$

Next we obtain distortion bounds for functions in $F_{\overline{H}}(\{c_n\},\{d_n\})$.

Theorem 2. *Consider the increasing sequences of positive numbers $\{c_n\}$ and $\{d_n\}$ so that $c_2 \le d_2$, $n \le c_n$, and $n \le d_n$ for all $n \ge 2$. If $f \in F_{\overline{H}}(\{c_n\},\{d_n\})$, then*

$$(1 - |b_1|)r - \left(\frac{1 - d_1|b_1|}{c_2}\right)r^2 \le |f(z)| \le (1 + |b_1|)r + \left(\frac{1 - d_1|b_1|}{c_2}\right)r^2.$$

The bounds given above are sharp for functions

$$f(z) = z \pm |b_1|\bar{z} + \left(\frac{1 - d_1|b_1|}{c_2}\right)\bar{z}^2, \qquad d_1|b_1| < 1.$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f \in F_{\overline{H}}(\{c_n\},\{d_n\})$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &= \left|z - \sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\bar{z}^n\right| \le (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n \\
&= (1 + |b_1|)r + \frac{1}{c_2}\sum_{n=2}^{\infty}c_2(|a_n| + |b_n|)r^n \le (1 + |b_1|)r + \sum_{n=2}^{\infty}\frac{1}{c_2}(c_2|a_n| + d_2|b_n|)r^n \\
&\le (1 + |b_1|)r + \sum_{n=2}^{\infty}\frac{1}{c_2}(c_n|a_n| + d_n|b_n|)r^2 \le (1 + |b_1|)r + \frac{1}{c_2}(1 - d_1|b_1|)r^2.
\end{aligned}$$

Corollary 1. *Let f be as in Theorem 2. Then*

$$\left\{w : |w| < \frac{1}{c_2}(c_2 - 1 - (c_2 - d_1)|b_1|)\right\} \subset f(\Delta).$$

Theorem 3. *If $n \le c_n$ and $n \le d_n$ for $n \ge 2$, then the family $F_{\overline{H}}(\{c_n\}, \{d_n\})$ is closed under convex combinations.*

Proof. Consider $f_i(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n + \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n$, where $\sum_{n=1}^{\infty} t_i = 1$ and $0 \le t_i \le 1$. If $f_i \in F_{\overline{H}}(\{c_n\}, \{d_n\})$ then

$$\sum_{n=2}^{\infty} c_n |a_{i_n}| + \sum_{n=1}^{\infty} d_n |b_{i_n}| \le 1, \qquad i = 1, 2, \ldots. \tag{5}$$

On the other hand, we have

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

From this and (5) we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} c_n \left| \sum_{i=1}^{\infty} t_i \right| a_{i_n} \left\| + \sum_{i=1}^{\infty} d_n \left| \sum_{i=1}^{\infty} t_i \right| b_{i_n} \right\| &= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} c_n |a_{i_n}| + \sum_{n=1}^{\infty} d_n |b_{i_n}| \right\} \\ &\le \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Hence $\sum_{i=1}^{\infty} t_i f_i \in F_{\overline{H}}(\{c_n\}, \{d_n\})$, by an application of (4).

In [3], Ruscheweyh showed that the analytic counterparts of $F_{\overline{H}}(\{n\}, \{n\})$ are useful in studying questions of neighbourhood. The δ-neighbourhood of the functions $f = h + \bar{g}$ in $F_{\overline{H}}(\{nc_n\}, \{nd_n\})$ is defined as the set $N_\delta(f) = F(z) = z + B_1\bar{z} + \sum_{n=2}^{\infty}(A_n z^n + B_n \bar{z}^n)$ so that $\sum_{n=2}^{\infty} n|a_n - A_n| + \sum_{n=1}^{\infty} n|b_n - B_n| \le \delta, \delta > 0$. Our next result guarantees that the functions in a neighbourhood of $F_{\overline{H}}(\{nc_n\}, \{nd_n\})$ are starlike functions.

Theorem 4. *Consider the increasing sequences $\{c_n\}$ and $\{d_n\}$ so that $n \le c_n$ and $n \le d_n$ for $n \ge 2$. If $\delta = [c_2 - 1 - (c_2 - d_1)|b_1|]/c_2$, then $N_\delta(F_{\overline{H}}(\{nc_n\}, \{nd_n\})) \subset S\overline{H}$.*

Proof. Suppose $f = h + \bar{g} \in F_{\overline{H}}(\{nc_n\}, \{nd_n\})$. Let $F = H + \overline{G} \in N_\delta(f)$ where $H(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $G(z) = \sum_{n=1}^{\infty} B_n z^n$. We need to show that $F \in S\overline{H}$. In other words, it suffices to show that F satisfies the condition $M(F) = \sum_{n=2}^{\infty} n(|A_n| + |B_n|) + |B_1| \le 1$. We observe that

$$\begin{aligned} M(F) &= \sum_{n=2}^{\infty} n(|A_n| + |B_n|) + |B_1| \\ &= \sum_{n=2}^{\infty} n(|A_n - a_n + a_n| + |B_n - b_n + b_n|) + |B_1 - b_1 + b_1| \\ &\le \sum_{n=2}^{\infty} n(|A_n - a_n| + |B_n - b_n|) + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |B_1 - b_1| + |b_1| \\ &= \left(\sum_{n=2}^{\infty} n[|A_n - a_n| + |B_n - b_n|] + |B_1 - b_1| \right) + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |b_1| \end{aligned}$$

$$\leq \delta + |b_1| + \sum_{n=2}^{\infty}(n|a_n| + n|b_n|) = \delta + |b_1| + \frac{1}{c_2}\sum_{n=2}^{\infty}(c_2 n|a_n| + c_2 n|b_n|)$$
$$\leq \delta + |b_1| + \frac{1}{c_2}\sum_{n=2}^{\infty}(c_n n|a_n| + c_n n|b_n|) \leq \delta + |b_1| + \frac{1}{c_2}\sum_{n=2}^{\infty}(nc_n|a_n| + nd_n|b_n|)$$
$$\leq \delta + |b_1| + \frac{1}{c_2}(1 - d_1|b_1|).$$

Now this last expression is never greater than one provided that

$$\delta \leq 1 - |b_1| - \frac{1}{c_2}(1 - d_1|b_1|) = \frac{c_2 - 1 - (c_2 - d_1)|b_1|}{c_2}.$$

We say that a function $f = h + \bar{g}$, where h and g are given by (1), is in $V_H(\alpha_n, \beta_n)$ if $f \in H$ and $\alpha_n = \arg a_n$ and $\beta_n = \arg b_n$. If, further, there exists a real number θ such that $\alpha_n + (n-1)\theta \equiv \pi \pmod{2\pi}$ and $\beta_n + (n+1)\theta \equiv 0 \pmod{2\pi}$ for all n, then f is said to be in $V_H(\alpha_n, \beta_n; \theta)$. Also see [2]. The union of $V_H(\alpha_n, \beta_n; \theta)$ taken over all sequences $\{\alpha_n\}$, $\{\beta_n\}$, and all possible real θ, is denoted by V_H. Also, if $a_n = 0$ or $b_n = 0$ for some n, we define the argument as zero.

We define $G_H(\{c_n\}, \{d_n\})$ consisting of functions $f = h + \bar{g}$ of the form (1) if there exist sequences $\{c_n\}$ and $\{d_n\}$ of positive real numbers such that $\sum_{n=2}^{\infty} c_n|a_n| + \sum_{n=1}^{\infty} d_n|b_n| \leq 1$. We observe that $F_{\overline{H}}(\{c_n\}, \{d_n\}) \subset G_H(\{c_n\}, \{d_n\})$ for $c_n \geq n$ and $d_n \geq n$ and that many of the results given in this paper also hold for $G_H(\{c_n\}, \{d_n\})$. However, this generalization comes at a cost that the family $G_H(\{c_n\}, \{d_n\})$ is not convex. Therefore, the major difference would be in Theorem 1, which may be replaced with

Theorem 5. *If $c_n \geq n$ and $d_n \geq n$ for $n \geq 2$, then $clcoG_H(\{c_n\}, \{d_n\}) = \{f \in V_H\}$ where $\sum_{n=2}^{\infty}(c_n|a_n| + d_n|b_n|) \leq 1 - d_1|b_1|$. Also, for b_1 fixed, $0 < |b_1| < 1/d_1$, $|x| = 1 - |b_1|$, and $n \geq 2$, we have*

$$E(clcoG_H(\{c_n\}, \{d_n\})) = \left\{z + \left(\frac{1}{c_n}\right)xz^n + \overline{b_1 z}\right\} \cup \left\{z + \overline{b_1 z} + \left(\frac{1}{d_n}\right)x\bar{z}^n\right\}.$$

Proof. In view of the definition of $G_H(\{c_n\}, \{d_n\})$, the functions in $clcoG_H(\{c_n\}, \{d_n\})$ satisfy the required condition

$$\sum_{n=2}^{\infty} c_n|a_n| + \sum_{n=2}^{\infty} d_n|b_n| \leq 1 - d_1|b_1|. \tag{6}$$

Conversely, suppose that the function $f(z) = z + \sum_{n=2}^{\infty} |a_n|e^{i\alpha_n}z^n + \overline{b_1 z + \sum_{n=1}^{\infty} |b_n|e^{i\beta_n}z^n}$ satisfies the condition (6). We need to show that $f \in clcoG_H(\{c_n\}, \{d_n\})$. Set $h_1(z) = z$, $h_n(z) = z + \frac{1}{c_n}e^{i\alpha_n}z^n$, $g_1(z) = b_1 z$, $g_n(z) = b_1 z + \frac{1}{d_n}e^{i\beta_n}z^n$, $(n = 2, 3, \ldots)$. Letting $x_n = c_n|a_n|$ where $x_1 = 1 - \sum_{n=2}^{\infty} x_n$ and $y_n = d_n|b_n|$ where $y_1 = 1 - \sum_{n=2}^{\infty} y_n$, we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{x_n}{c_n}e^{i\alpha_n}z^n + \overline{b_1 z} + \sum_{n=2}^{\infty} \frac{y_n}{d_n}e^{-i\beta_n}\bar{z}^n$$
$$= z + \sum_{n=2}^{\infty} x_n(h_n(z) - z) + \overline{b_1 z} + \sum_{n=2}^{\infty}(\overline{g_n(z) - b_1 z})y_n$$

$$= \sum_{n=2}^{\infty}(x_n h_n(z) + y_n \overline{g_n(z)}) + z\left(1 - \sum_{n=2}^{\infty} x_n\right) + \overline{b_1 z}\left(1 - \sum_{n=2}^{\infty} y_n\right)$$
$$= \sum_{n=1}^{\infty}(x_n h_n(z) + y_n \overline{g_n(z)}).$$

In particular, setting $f_1(z) = z + \overline{b_1 z}$ and $f_n(z) = z + \frac{1}{c_n} x z^n + \overline{b_1 z} + \frac{1}{d_n} y \bar{z}^n$ where $n \geq 2$ and $|x| + |y| = 1 - d_1|b_1|$ we observe that $E(clcoG_H(\{c_n\}, \{d_n\})) \subset \{f_1, f_n\}$. Note that $f_1 \notin E(clcoG_H(\{c_n\}, \{d_n\}))$ because f_1 can be expressed as a linear combination $f_1(z) = \frac{1}{2}\left[f_1(z) + \frac{1-d_1|b_1|}{c_2} z^2\right] + \frac{1}{2}\left[f_1(z) - \frac{1-d_1|b_1|}{c_2} \bar{z}^2\right]$. In order to show that $\{f_n\} \subset E(clcoG_H(\{c_n\}, \{d_n\}))$, we only need to show that f_n are extremal coefficient bounds. This is the case because if both $|x| > 0$ and $|y| > 0$, we choose $\epsilon > 0$ small enough so that both

$$t_1(z) = z + \frac{1+\epsilon}{c_n} x z^n + \overline{b_1 z} + \frac{1-\epsilon}{d_n} y \bar{z}^n$$

and

$$t_1(z) = z + \frac{1-\epsilon}{c_n} x z^n + \overline{b_1 z} + \frac{1+\epsilon}{d_n} y \bar{z}^n$$

are in $clcoG_H(\{c_n\}, \{d_n\})$. Then $f_n(z) = \frac{1}{2} t_1(z) + \frac{1}{2} t_2(z)$. This completes the proof.

Remark. For $c_n = n = d_n$, the corresponding result was determined by Jahangiri and Silverman [2].

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