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Convolutions for Special Classes of Harmonic Univalent Functions

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Abstract—Ruscheweyh and Sheil-Small proved the Pólya-Schoenberg conjecture that the class of convex analytic functions is closed under convolution or Hadamard product. They also showed that close-to-convexity is preserved under convolution with convex analytic functions. In this note, we investigate harmonic analogs. Beginning with convex analytic functions, we form certain harmonic functions which preserve close-to-convexity under convolution. An auxiliary function enables us to obtain necessary and sufficient convolution conditions for convex and starlike harmonic functions, which lead to sufficient coefficient bounds for inclusion in these classes. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [1], Kneser [2], Lewy [3], and Rado [4]. Recent interest in harmonic complex functions has been triggered by geometric function theorists Clunie and Sheil-Small [5]. We let H denote the family of continuous complex-valued functions which are harmonic in the open unit disk $\Delta = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in Δ . Clunie and Sheil-Small in [5] developed the basic theory of harmonic functions $f \in H$ which are univalent in Δ and have the normalization $f(0) = 0$ and $f_z(0) = 1$. Such functions may be written as $f = h + \bar{g}$, where h and g are members of A . In this case, f is sense-preserving if $|g'| < |h'|$ in Δ , or equivalently, if the dilatation function $w = g'/h'$ satisfies $|w(z)| < 1$ for $z \in \Delta$. To this end, without loss of generality, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad (1.1.1)$$

and let S_H denote the family of functions $f = h + \bar{g}$ which are harmonic, univalent, and sense-preserving in Δ , where h and g are in A and are of form (1.1.1).

For the harmonic function $f = h + \bar{g}$, we call h the analytic part and g the coanalytic part of f . Note that the family S_H reduces to the class S of normalized analytic univalent functions in Δ if the coanalytic part of f is identically zero.

We let $K_H, S_H^*,$ and C_H denote the subclasses of S_H consisting of harmonic functions which are, respectively, convex, starlike, and close-to-convex in Δ . A function is said to be convex, starlike, or close-to-convex in Δ if (e.g., see [5,6]) it maps each $|z| = r < 1$ onto a convex, starlike, or close-to-convex domain, respectively.

Finally, we define the convolution of two complex-valued harmonic functions $f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \bar{b}_{1n} \bar{z}^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \bar{b}_{2n} \bar{z}^n$ by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n + \sum_{n=1}^{\infty} \bar{b}_{1n} \bar{b}_{2n} \bar{z}^n.$$

The above convolution formula reduces to the famous Hadamard product if the coanalytic parts of f_1 and f_2 are zero. In 1973, Ruscheweyh and Sheil-Small [6] proved the following.

LEMMA 1.1. *Let ϕ and ψ be convex analytic in Δ . Then we have the following.*

- i. $(\phi * \psi)(z)$ is convex analytic in Δ .
- ii. $(\phi * f)(z)$ is close-to-convex analytic in Δ if f is close-to-convex analytic in Δ .
- iii. $(\phi * z f') / (\phi * z \psi')$ takes all its values in a convex domain D if f' / ψ' takes all its values in D .

The first part of the above lemma is the famous Pólya-Schoenberg conjecture (e.g., see [7]). Clunie and Sheil-Small [5, Theorem 5.14] proved that if ϕ is convex analytic in Δ and $f \in K_H$ then $(\phi + \epsilon \bar{\phi}) * f \in C_H$ for $|\epsilon| \leq 1$. We can show that the required convexity condition for ϕ cannot be replaced by starlikeness. For example, set

$$f(z) = h(z) + \overline{g(z)} = \frac{z - (1/2)z^2}{(1 - z)^2} + \frac{-(1/2)\bar{z}^2}{(1 - \bar{z}^2)^2}$$

and consider the starlike analytic function $\phi(z) = z + z^n/n$ in Δ . Then, for $\epsilon = 0$, we obtain the convolution function

$$\left(\phi(z) + (0) \cdot \overline{\phi(z)}\right) * \left(h(z) + \overline{g(z)}\right) = (\phi * h)(z) = z + \frac{n+1}{2n} z^n, \quad n \geq 2,$$

which is not even univalent in Δ .

It is the purpose of this paper to investigate various functions and their convolutions which lead to close-to-convex, starlike, and convex harmonic univalent functions.

2. MAIN RESULTS

The first theorem extends Lemma 1.1.ii to the close-to-convex harmonic case.

THEOREM 2.1. *Let h and g be analytic in Δ so that $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex in Δ for each ϵ ($|\epsilon| = 1$). If ϕ is convex analytic in Δ , then*

$$(\phi + \overline{\sigma \phi}) * (h + \bar{g}) \in C_H, \quad |\sigma| = 1.$$

We shall need the following lemma, which is due to Clunie and Sheil-Small [5].

LEMMA 2.2. Let $h \in A$ and $g \in A$.

- i. If $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex analytic in Δ for each ϵ ($|\epsilon| = 1$), then $f = h + \bar{g} \in C_H$.
- ii. If $f = h + \bar{g}$ is harmonic and locally univalent in Δ and if $h + \epsilon g$ is convex analytic in Δ for some ϵ ($|\epsilon| \leq 1$), then $f = h + \bar{g} \in C_H$.

PROOF OF THEOREM 2.1. Write $(\phi + \overline{\sigma\phi}) * (h + \bar{g}) = \phi * h + \overline{\sigma\phi} * g = H + \bar{G}$. The condition $|G'(0)| < |H'(0)|$ is established, by observing that

$$|G'(0)| = |(\sigma\phi * g)'|_{z=0} = \left| \frac{1}{z} \phi * \sigma g' \right|_{z=0} < \left| \frac{1}{z} \phi * h' \right|_{z=0} = |(\phi * \sigma h)'|_{z=0} = |H'(0)|.$$

Using Lemma 1.1, we obtain for each $\beta = \epsilon/\sigma$ that the convolution function

$$H + \beta G = \phi * h + \beta \sigma \phi * g = \phi * (h + \epsilon g)$$

is close-to-convex analytic in Δ . Therefore, by Lemma 2.2.i, we conclude that

$$H + \bar{G} = (\phi + \overline{\sigma\phi}) * (h + \bar{g}) \in C_H.$$

Now we construct a family of close-to-convex harmonic functions.

COROLLARY 2.3. Let the analytic functions ϕ and h be convex in Δ . Also let ω be a Schwarz function. Then

$$(\phi + \epsilon\bar{\phi}) * \left[h(z) + \int_0^z \omega(t)h'(t) dt \right] \in C_H, \quad |\epsilon| = 1.$$

PROOF. Letting $g'(z) = \omega(z)h'(z)$, the proof follows from Lemma 2.2.i.

Next we obtain a modification of Lemma 1.1.i for close-to-convex harmonic case.

THEOREM 2.4. Suppose h and ϕ are convex analytic in Δ , and g is analytic in Δ with $|g'(z)| < |h'(z)|$ for each $z \in \Delta$. Then for each $|\epsilon| \leq 1$ we have

$$(\phi + \epsilon\bar{\phi}) * (h + \bar{g}) \in C_H.$$

PROOF. Write $(\phi + \epsilon\bar{\phi}) * (h + \bar{g}) = \phi * h + \overline{\epsilon(\phi * g)} = H + \bar{G}$. By Lemma 1.1.i, the analytic function $H = \phi * h$ is convex in Δ . Hence, by Lemma 2.2.ii, it suffices to show that $H + \bar{G}$ is locally univalent in Δ . For H and G as above we can write

$$\left| \frac{G'}{H'} \right| \leq \left| \frac{(\phi * g)'}{(\phi * h)'} \right| = \left| \frac{\phi * zg'}{\phi * zh'} \right|.$$

Note that ϕ is convex analytic in Δ and $|g'/h'| < 1$ by hypothesis. Since $|g'/h'| < 1$, we conclude from Lemma 1.1.iii that $|\phi * zg'|/|\phi * zh'| < 1$ and so the proof is complete.

The following corollary is a consequence of Theorem 2.4 and Lemma 2.2.ii.

COROLLARY 2.5. For analytic functions b and ϕ let $|b(z)| < 1/|1 - z|^2$ and ϕ be convex in Δ . Then

$$(\phi + \epsilon\bar{\phi}) * \left[\frac{z}{1 - z} + \int_0^z b(t) dt \right] \in C_H, \quad |\epsilon| \leq 1.$$

PROOF. Letting $h(z) = z/(1 - z)$ and $g(z) = \int_0^z b(t) dt$ in Theorem 2.4, the proof follows from Lemma 2.2.ii.

In the next two theorems we give necessary and sufficient convolution conditions for convex and starlike harmonic functions.

THEOREM 2.6. *Let $f = h + \bar{g} \in S_H$. Then $f \in S_H^*$ if and only if*

$$h(z) * \frac{z + ((\zeta - 1)/2) z^2}{(1 - z)^2} - \overline{g(z)} * \frac{\zeta \bar{z} - ((\zeta - 1)/2) \bar{z}^2}{(1 - \bar{z})^2} \neq 0, \quad |\zeta| = 1, \quad 0 < |z| < 1.$$

PROOF. A necessary and sufficient condition for the function f to be starlike (see [6]) is that $\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) > 0$. Therefore, for h and g of form (1.1.1), we must have

$$\Re \left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) > 0. \tag{2.6.1}$$

Since $(zh'(z) - \overline{zg'(z)})/(h(z) + \overline{g(z)}) = 1$ at $z = 0$, the required condition (2.6.1) is equivalent to

$$\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \neq \frac{\zeta - 1}{\zeta + 1}, \quad |\zeta| = 1, \quad \zeta \neq -1, \quad 0 < |z| < 1. \tag{2.6.2}$$

By a simple algebraic manipulation, inequality (2.6.2) yields

$$\begin{aligned} & 0 \neq (\zeta + 1) [zh'(z) - \overline{zg'(z)}] - (\zeta - 1) [h(z) + \overline{g(z)}] \\ &= h(z) * \left[\frac{(\zeta + 1)z}{(1 - z)^2} - \frac{(\zeta - 1)z}{1 - z} \right] - \overline{g(z)} * \left[\frac{(\bar{\zeta} + 1)z}{(1 - z)^2} + \frac{(\bar{\zeta} - 1)z}{1 - z} \right] \\ &= h(z) * \left[\frac{2z + (\zeta - 1)z^2}{(1 - z)^2} \right] - \overline{g(z)} * \left[\frac{2\bar{\zeta}z - (\bar{\zeta} - 1)z^2}{(1 - z)^2} \right], \end{aligned}$$

which is the condition required by Theorem 2.6.

The above theorem yields a sufficient coefficient bound for harmonic starlike functions.

COROLLARY 2.7. *Let $f = h + \bar{g} \in S_H$. If $\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1$, then $f \in S_H^*$.*

PROOF. Note that for h and g as in (1.1.1), we have

$$\begin{aligned} & \left| h(z) * \left(\frac{z + ((\zeta - 1)/2) z^2}{(1 - z)^2} \right) - \overline{g(z)} * \left(\frac{\bar{\zeta}z - ((\bar{\zeta} - 1)/2) z^2}{(1 - z)^2} \right) \right| \\ &= \left| z + \sum_{n=2}^{\infty} \left(n + \frac{(n - 1)(\zeta - 1)}{2} \right) a_n z^n + \sum_{n=1}^{\infty} \left(n\bar{\zeta} - \frac{(n - 1)(\bar{\zeta} - 1)}{2} \right) b_n z^n \right| \\ &\geq |z| \left[1 - \sum_{n=2}^{\infty} \left| \frac{2n + (n - 1)(\zeta - 1)}{2} \right| |a_n| |z|^{n-1} - \sum_{n=1}^{\infty} \left| \frac{2n\bar{\zeta} - (n - 1)(\bar{\zeta} - 1)}{2} \right| |b_n| |z|^{n-1} \right] \\ &> |z| \left[1 - \sum_{n=2}^{\infty} \left| \frac{n + 1 + (n - 1)\zeta}{2} \right| |a_n| - \sum_{n=1}^{\infty} \left| \frac{n - 1 + (n + 1)\bar{\zeta}}{2} \right| |b_n| \right] \\ &\geq |z| \left[1 - \sum_{n=2}^{\infty} \frac{n + 1 + (n - 1)|\zeta|}{2} |a_n| - \sum_{n=1}^{\infty} \frac{n - 1 + (n + 1)|\zeta|}{2} |b_n| \right] \\ &= |z| \left[1 - \sum_{n=2}^{\infty} n|a_n| - \sum_{n=1}^{\infty} n|b_n| \right]. \end{aligned}$$

This last expression is nonnegative by the hypothesis of the corollary and so $f \in S_H^*$.

THEOREM 2.8. *Let $f = h + \bar{g} \in S_H$. Then $f \in K_H$ if and only if*

$$h(z) * \frac{z + \zeta z^2}{(1 - z)^3} + \overline{g(z)} * \frac{\zeta \bar{z} + \bar{z}^2}{(1 - \bar{z})^3} \neq 0, \quad |\zeta| = 1, \quad 0 < |z| < 1.$$

PROOF. A harmonic function f is said to be convex (see [6]) if and only if for $z = re^{i\theta}$ in Δ we have

$$\frac{\partial}{\partial \theta} \left[\arg \left(\frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right] = \frac{\partial}{\partial \theta} \left[\arg \left(\frac{\partial}{\partial \theta} \left[h(re^{i\theta}) + \overline{g(re^{i\theta})} \right] \right) \right] > 0.$$

Therefore, for h and g of form (1.1.1), we must have

$$\Re \left(\frac{z(zh'(z))' + z(zg'(z))'}{zh'(z) - \overline{zg'(z)}} \right) > 0.$$

Since $(z(zh'(z))' + \overline{z(zg'(z))'}) / (zh'(z) - \overline{zg'(z)}) = 1$ at $z = 0$, the above required condition is equivalent to

$$\frac{z(zh'(z))' + \overline{z(zg'(z))'}}{zh'(z) - \overline{zg'(z)}} \neq \frac{\zeta - 1}{\zeta + 1}, \quad |\zeta| = 1, \quad \zeta \neq -1, \quad 0 < |z| < 1. \tag{2.8.1}$$

By a simple algebraic manipulation, inequality (2.8.1) yields

$$\begin{aligned} 0 &\neq (\zeta + 1) \left[(zh'(z))' + \overline{(zg'(z))'} \right] - (\zeta - 1) \left[zh'(z) - \overline{zg'(z)} \right] \\ &= zh'(z) * \left[\frac{(\zeta + 1)z}{(1 - z)^2} - \frac{(\zeta - 1)z}{1 - z} \right] + \overline{zg'(z)} * \left[\frac{(\bar{\zeta} + 1)z}{(1 - z)^2} + \frac{(\bar{\zeta} - 1)z}{1 - z} \right] \\ &= zh'(z) * \left[\frac{2z + (\zeta - 1)z^2}{(1 - z)^2} \right] + \overline{zg'(z)} * \left[\frac{2\bar{\zeta}z - (\bar{\zeta} - 1)z^2}{(1 - z)^2} \right] \\ &= h(z) * \left[\frac{2z + (\zeta - 1)z^2}{(1 - z)^2} \right]' + \overline{g(z)} * \left[\frac{2\bar{\zeta}z - (\bar{\zeta} - 1)z^2}{(1 - z)^2} \right]' \\ &= 2 \left[h(z) * \left(\frac{z + \zeta z^2}{(1 - z)^3} \right) + \overline{g(z)} * \left(\frac{\bar{\zeta}z + z^2}{(1 - z)^3} \right) \right], \end{aligned}$$

which is the condition required by Theorem 2.8.

Consequently, we obtain a sufficient coefficient bound for harmonic convex functions.

COROLLARY 2.9. *Let $f = h + \bar{g} \in S_H$. If $\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq 1$, then $f \in K_H$.*

The proof is similar to that given for Corollary 2.7 and we omit it.

REMARK 2.10. For the coanalytic part of $f = h + \bar{g}$ being identically zero, i.e., $g \equiv 0$, Theorems 2.6 and 2.8 yield the results presented by Silverman, Silvia and Telage [8] for the analytic case.

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