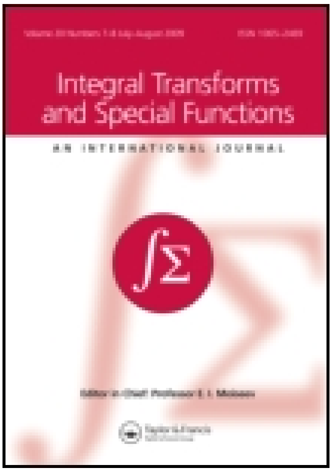

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gitr20

Harmonic starlikeness and convexity of integral operators generated by hypergeometric series

Om. P. Ahuja [a]
[a] Department of Mathematical Sciences, Kent State University, 14111 Claridon-Troy Road, Burton, Ohio, 44021-9500, USA
Published online: 06 Jul 2009.

To cite this article: Om. P. Ahuja (2009) Harmonic starlikeness and convexity of integral operators generated by hypergeometric series, Integral Transforms and Special Functions, 20:8, 629-641, DOI: 10.1080/10652460902734124

To link to this article: http://dx.doi.org/10.1080/10652460902734124

Integral Transforms and Special Functions
Vol. 20, No. 8, August 2009, 629–641

Harmonic starlikeness and convexity of integral operators generated by hypergeometric series

Om. P. Ahuja*

Department of Mathematical Sciences, Kent State University, 14111 Claridon-Troy Road, Burton, Ohio 44021-9500, USA

(*Received 4 August 2008*)

The object of this article is to study harmonic starlikeness and harmonic convexity of integral operators generated by hypergeometeric series. We also consider special cases of harmonic starlikeness and convexity of integral operators generated by incomplete beta functions.

Keywords: harmonic functions; harmonic starlike; harmonic convex; convolution; integral operators; hypergeometric functions; incomplete beta functions

AMS Classification: Primary: 30C45, 30C50; Secondary: 30C55

1. Introduction

A continuous complex-valued mapping $f = u + \mathrm{i}v$ is defined as *harmonic* in a simply connected domain $\mathbb{D}$ in the complex plane if it is satisfies $f_{z\bar{z}} \equiv 0$ on $\mathbb{D}$, i.e., u and v are real harmonic functions in $\mathbb{D}$. Such a harmonic function f can be expressed as the canonical representation $f = h + \bar{g}, \quad g(0) = 0$, where h and g are analytic and g denotes the function $z \to \overline{g(z)}$. Let $\hat{H}$ denote the class of all harmonic mappings of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1 \tag{1}$$

are analytic functions in the open unit disk Δ. A necessary and sufficient condition for f in the class $\hat{H}$ to be locally univalent and sense-preserving is that $|h'(z)| > |g'(z)|$ in Δ. Denote by HS the subclass of $\hat{H}$ consisting of functions that are univalent and sense-preserving in Δ.

*Email: oahuja@kent.edu

ISSN 1065-2469 print/ISSN 1476-8291 online

DOI: 10.1080/10652460902734124
http://www.informaworld.com

For $0 \le \alpha < 1$, define

$$HS^*(\alpha) := \left\{ f \in \hat{H} : \frac{\partial}{\partial\theta}(\arg(f(re^{i\theta})) > \alpha \right\}$$
$$HK(\alpha) := \left\{ f \in \hat{H} : \frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}(f(re^{i\theta})\right)\right) > \alpha \right\}$$

for $z = re^{i\theta}$, $0 \le \theta \le 2\pi$ and $0 \le r < 1$. Note that $HS^* \equiv HS^*(0)$ is the class of all harmonic starlike functions with respect to origin in Δ. Likewise $HK \equiv HK(0)$ is the class of all harmonic convex functions in Δ. Thus the parameter α in $HS^*(\alpha)$ and $HK(\alpha)$ defines order α, respectively, in the classes $HS^*$and HK. Also, let K_H^0 be the subclass of HK defined by

$$K_H^0 := \{f = h + \overline{g} \in HK : g'(0) = B_1 = 0\}.$$

For additional information about these classes, one may refer to [1,4,5].

Clunie and Sheil-Small [4] in 1984 observed that if h given by Equation (1) is analytic and univalent in Δ, then $h + \sigma\overline{h}$ for $|\sigma| < 1$ is harmonic univalent and sense-preserving in Δ, that is $h + \sigma\overline{h} \in HS$. This observation motivates us to consider the integral operator G_1 defined by

$$G_1(z) := \phi_1(a_1, b_{1;}c_1; z) + \sigma\overline{\phi_2(a_2, b_{2;}c_2; z)}, \quad |\sigma| < 1 \tag{2}$$

where

$$\phi_1(a_1, b_{1;}c_1; z) := \int_0^z F(a_1, b_{1;}c_1; t)\,\mathrm{d}t, \quad \phi_2(a_2, b_{2;}c_2; z) := \int_0^z F(a_2, b_{2;}c_2; t)\,\mathrm{d}t$$

are integral operators associated with the Gauss hypergeometric series (function)

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n.$$

Here a, b, c are complex parameters, and c is neither zero nor a negative integer. Note that $F(a, b; c; z)$ is analytic function in Δ. Univalence, starlikeness, and convexity properties of normalized hypergeometric function $zF(a, b; c; z)$ and related convolution operators have been studied extensively by several researchers; for example, in [3,6,9,10,12,13]. We next define the integral operator $\Lambda : \hat{H} \to \hat{H}$ by

$$\Lambda(f)(z) := \Lambda\begin{bmatrix} a_1, b_{1,}c_1 \\ a_2, b_{2,}c_2 \end{bmatrix}(f)(z) := f(z)\widetilde{\star}G_1(z)$$
$$= h(z) \star \int_0^z F(a_1, b_{1;}c_1; t)\,\mathrm{d}t + \sigma\overline{g(z) \star \int_0^z F(a_2, b_{2;}c_2; t)\,\mathrm{d}t}, \quad |\sigma| < 1,$$

where $\star$ denotes the usual Hadamard or convolution product of two harmonic functions and $f = h + \overline{g}$ is in $\hat{H}$. We observe that

$$\Lambda\begin{bmatrix} a_1, 2, a_1 \\ a_2, 2, a_2 \end{bmatrix} f(z) = f(z) = f(z)\widetilde{\star}\left(\frac{z}{1-z} + \overline{\frac{z}{1-z}}\right)$$

is the identity mapping. Setting $a_1 = a_2 = b_1 = b_2 = -m$, $m \in \mathbb{N}$, $c_1 = c_2 = c > 0$, the integral operator Λ reduces to the polynomial

$$\Lambda \begin{bmatrix} -m, -m, c \\ -m, -m, c \end{bmatrix} f(z) = f(z) \widetilde{\star} \left(z + \sum_{n=2}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c)_{n-1}(1)_n} z^n + \sum_{n=1}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c)_{n-1}(1)_n} \overline{z^n} \right).$$

We also define the integral operator $L : \hat{H} \to \hat{H}$ by

$$L(f) \equiv L \begin{bmatrix} b_{1,} c_1 \\ b_{2,} c_2 \end{bmatrix} (f)(z) := f(z) \widetilde{\star} (\varphi_1(b_{1;} c_1; z) + \sigma \overline{\varphi_2(b_{2;} c_2; z)}), \quad |\sigma| < 1$$

where

$$\varphi_1(b_{1;} c_1; z) := \int_0^z F(1, b_{1;} c_1; t)\, \mathrm{d}t, \quad \varphi_2(b_{2;} c_2; z) := \int_0^z F(1, b_{2;} c_2; t), \mathrm{d}t$$

are integral operators associated with incomplete beta functions.

The purpose of the this article is to study harmonic starlikeness and harmonic convexity of the integral operators

$$G_1(z) = \phi_1(a_1, b_{1;} c_1; z) + \sigma \overline{\phi_2(a_2, b_{2;} c_2; z)}, \quad G_2(z) = \varphi_1(b_{1;} c_1; z) + \sigma \overline{\varphi_2(b_{2;} c_2; z)})$$

$$\Lambda \begin{bmatrix} a_1, b_{1,} c_1 \\ a_2, b_{2,} c_2 \end{bmatrix} \quad \text{and} \quad L \begin{bmatrix} b_{1,} c_1 \\ b_{2,} c_2 \end{bmatrix}, \quad |\sigma| < 1.$$

In particular, we determine certain hypergeometric inequalities that ensure membership of functions in various subclasses of harmonic mappings in Δ.

Throughout this article, we will frequently use the well-known Gauss summation formula

$$F(a, b, c; 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0. \tag{3}$$

2. Preliminaries and Lemmas

LEMMA 1 ([2]) *If* $a, b, c > 0$, *then*

(i) $F(a+k, b+k; c+k; 1) = (c)_k/(c-a-b-k)_k F(a, b; c; 1)$ *for* $k = 0, 1, 2 \ldots,$ *if* $c > a+b+k$
(ii) $\sum_{n=2}^{\infty}(n-1)((a)_{n-1}(b)_{n-1})/((c)_{n-1}(1)_{n-1}) = ab/(c-a-b-1)F(a, b; c; 1)$, *if* $c > a+b+1$
(iii) $\sum_{n=2}^{\infty}(n-1)^2((a)_{n-1}(b)_{n-1})/((c)_{n-1}(1)_{n-1}) = [((a)_2(b)_2)/ ((c-a-b-2)_2) + ab/(c-a-b-1)]F(a, b; c; 1)$, *if* $c > a+b+2$.

LEMMA 2 ([11]) *Let* $a, b \in \mathbb{C} \setminus \{0\}$, $|a| \neq 1$, $|b| \neq 1$, $c \in (0, 1) \cup (1, \infty)$, *and* $c > \max\{0, |a|+|b|-1\}$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a|)_{n-1}(|b|))_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{c-|a|-|b|}{(|a|-1)(|b|-1)} F(|a|, |b|; c; 1) - \frac{c-1}{(|a|-1)(|b|-1)}.$$

LEMMA 3 ([8]) *Let* $f = h + \overline{g} \in \hat{H}$ *with* h *and* g *of the form* (1). *If for some* $\alpha (0 \leq \alpha < 1)$, *the inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)|A_n| + \sum_{n=1}^{\infty}(n+\alpha)|B_n| \leq 1-\alpha \tag{4}$$

is satisfied, then f *is harmonic, sense-preserving, univalent in* Δ, *and* $f \in HS^*(\alpha)$.

Define $THS^*(\alpha) := HS^*(\alpha) \cap \check{T}$ and $THK(\alpha) := HK(\alpha) \cap \check{T}$, where $\check{T}$ consists of the functions $f = h + \overline{g}$ in $\hat{H}$ so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty}|A_n|z^n, \quad g(z) = \sum_{n=1}^{\infty}|B_n|z^n, \quad |B_1| < 1. \tag{5}$$

Remark 1 In [8], it is also shown that the $f = h + \overline{g}$ given by (5) is in the family $THS^*(\alpha)$ if and only if the coefficient condition (4) holds. Moreover, if $f \in THS^*(\alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{n-\alpha}, \ \ n \geq 2, \quad |B_n| \leq \frac{1-\alpha}{n+\alpha}, \ \ n \geq 1. \tag{6}$$

LEMMA 4 ([13]) *Let* $f = h + \overline{g}$ *where* h *and* g *are given by* (1). *If* $0 \leq \alpha < 1$ *and*

$$\sum_{n=2}^{\infty}n(n-\alpha)|A_n| + \sum_{n=1}^{\infty}n(n+\alpha)|B_n| \leq 1-\alpha, \tag{7}$$

then f *is harmonic, sense-preserving, univalent in* Δ, *and* $f \in HK(\alpha)$.

Remark 2 In [7], it is also shown that $f = h + \overline{g}$ given by Equation (5) is in the family $THK(\alpha)$ if and only if the coefficient condition (7) holds. Moreover, if $f \in THK(\alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{n(n-\alpha)}, \ \ n \geq 2, \quad |B_n| \leq \frac{1-\alpha}{n(n+\alpha)}, \ \ n \geq 1. \tag{8}$$

LEMMA 5 ([11]) *Let* $b, c > 0$, $b \neq 1$, *and* $c > 1 + b$. *Then*

$$\sum_{n=0}^{\infty}\frac{(b)_n}{(n+1)(c)_n} = \frac{c-1}{b-1}[\varsigma(c-1) - \varsigma(c-b)],$$

where the function ς *is defined by* $\varsigma(x) := \Gamma'(x)/\Gamma(x)$.

LEMMA 6 *If* $f = h + \overline{g} \in K^0_{\hat{H}}$ *where* h *and* g *are given by* (1) *with* $B_1 = 0$, *then*

$$|A_n| \leq \frac{n+1}{2}, \quad |B_n| \leq \frac{n-1}{2}. \tag{9}$$

3. Harmonic starlikeness

In our first result, we determine conditions which guarantee that the integral operator Λ is harmonic starlike in Δ.

THEOREM 1 *Let* $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j > \max\{0, |a_j| + |b_j|\}$ *for* $j = 1, 2$ *and* $|\sigma| < 1$. *Also, suppose* $f = h + \overline{g} \in \hat{H}$ *is given by* (1). *If the hypergeometric inequalities*

(i) $\sum_{n=2}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n| \leq 1, |B_1| < 1$

(ii) $\dfrac{\Gamma(c_1 - |a_1| - |b_1|)\Gamma(c_1)}{\Gamma(c_1 - |a_1|)\Gamma(c_1 - |b_1|)} + |\sigma| \dfrac{\Gamma(c_2 - |a_2| - |b_2|)\Gamma(c_2)}{\Gamma(c_2 - |a_2|)\Gamma(c_2 - |b_2|)} \leq 2$

are satisfied, then the integral operator Λ *is harmonic, sense-preserving, univalent and maps* $\hat{H}$ *into* HS^*.

Proof Note that

$$\Lambda(f)(z) = \Lambda \begin{bmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \end{bmatrix} (f)(z) := H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \sigma \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} B_n z^n. \tag{10}$$

In order to show that Λ is locally univalent and sense-preserving it suffices to prove that $|H'(z)| - |G'(z)| > 0$ in Δ. Using the condition (i) and Lemma 1, we obtain

$$\begin{aligned} |H'(z)| - |G'(z)| &> 1 - \sum_{n=2}^{\infty} n \frac{(|a_1|)_{n-1}(|b_1|))_{n-1}}{(c_1)_{n-1}(1)_n} - |\sigma| \sum_{n=1}^{\infty} n \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_n} \\ &= 2 - F(|a_1|, |b_1|; c_1; 1) - |\sigma| F(|a_2|, |b_2|; c_2; 1). \end{aligned}$$

The last inequality is non-negative because of the Gauss summation formula (3) and given condition (ii). To show that $\Lambda(f)$ is univalent in Δ, we follow the method of Theorem 1 in [8]. That is, for $z_1 \neq z_2$ in Δ, it suffices to prove that

$$\mathrm{Re} \frac{f(z_2) - f(z_1)}{z_2 - z_1} > \int_0^1 (\mathrm{Re}\, H'(z(t)) - |G'(z(t))|)\, \mathrm{d}t. \tag{11}$$

Since from the given condition (i) and $|\sigma| < 1$, we have

$$\mathrm{Re}\, H'(z) - |G'(z)| > 1 - \sum_{n=2}^{\infty} n \frac{(|a_1|)_{n-1}(|b_1|))_{n-1}}{(c_1)_{n-1}(1)_n} - |\sigma| \sum_{n=1}^{\infty} n \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_n},$$

it follows from the given hypothesis that the last inequality is positive. Therefore, from Inequality (11) we have

$$\mathrm{Re} \frac{f(z_2) - f(z_1)}{z_2 - z_1} > 0.$$

This proves the univalence of $\Lambda(f)$. In order to prove that $\Lambda(f) \in HS^* \equiv HS^*(0)$, it sufices to show that $S_1 \leq 1$ because of Lemma 3, where

$$S_1 := \sum_{n=2}^{\infty} n \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} \right| |A_n| + |\sigma| \sum_{n=1}^{\infty} n \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \right| |B_n|. \tag{12}$$

Since

$$|A_n| \leq 1, \quad |B_n| \leq 1, \quad \forall n \geq 2.$$

because of given condition (i), we obtain

$$S_1 \leq \sum_{n=2}^{\infty} \frac{(|a_1|)_{n-1}(|b_1|))_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + |\sigma| \sum_{n=1}^{\infty} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$$
$$= F(|a_1|, |b_1|; c_1; 1) + |\sigma| F(|a_2|, |b_2|; c_2; 1) - 1.$$

The last expression is bounded above by one because of Equation (3) and condition (ii). This completes the proof. ■

COROLLARY 1 *Under the parametric constraints and hypergeometric condition* (ii) *of Theorem* 1, *the integral operator G_1 defined by* (2) *is in HS^*.*

COROLLARY 2 *If $b_j \in \mathbb{C} - \{0\}, c_j \in \mathbb{R}$, $c_j > 1 + |b_j|$ for $j = 1, 2$ and $|\sigma| < 1$ satisfy the hypergeometric inequality*

$$\frac{c_1 - 1}{c_1 - |b_1| - 1} + |\sigma| \frac{c_2 - 1}{c_2 - |b_2| - 1} \leq 2,$$

then the integral operator G_2 is harmonic starlike in Δ.

THEOREM 2 *If $b_j \in \mathbb{C} - \{0\}, |b_j| \neq 1, c_j \in \mathbb{R}, c_j > |b_j| + 1$ for $j = 1, 2$ and $|\sigma| < 1$ satisfy the condition $(c_1 - 1)[1/(c_1 - |b_1| - 1) - (\alpha(\varsigma(c_1 - 1) - \varsigma(c_1 - |b_1|)))/(|b_1| - 1)] + (c_2 - 1)|\sigma|[1/(c_2 - |b_2| - 1) + (\alpha(\varsigma(c_2 - 1) - \varsigma(c_2 - |b_2|)))/(|b_2| - 1)] \leq 2(1 - \alpha)$ then the integral operator $G_2(z) = \varphi_1(b_1; c_1; z) + \sigma\overline{\varphi_2(b_2; c_2; z)}$ is in $HS^*(\alpha)$, where $\varsigma(x) = \Gamma'(x)/\Gamma(x)$.*

Proof Note that

$$G_2(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} \frac{(b_1)_{n-1}}{(c_1)_{n-1}} z^n + \sigma \overline{\sum_{n=1}^{\infty} \frac{1}{n} \frac{(b_2)_{n-1}}{(c_2)_{n-1}} z^n}.$$

In view of Lemma 2, we need to prove that

$$S_2 := \sum_{n=2}^{\infty} (n - \alpha) \left| \frac{1}{n} \frac{(b_1)_{n-1}}{(c_1)_{n-1}} \right| + |\sigma| \sum_{n=1}^{\infty} (n + \alpha) \left| \frac{1}{n} \frac{(b_2)_{n-1}}{(c_2)_{n-1}} \right| \leq 1 - \alpha.$$

However,

$$S_2 \leq \sum_{n=1}^{\infty} \frac{(|b_1|)_n}{(c_1)_n} - \alpha \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{(|b_1|)_n}{(c_1)_n} + |\sigma| \sum_{n=0}^{\infty} \frac{(|b_2|)_n}{(c_2)_n} + \alpha|\sigma| \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{(|b_2|)_n}{(c_2)_n}$$
$$= F(1, |b_1|, c_1; 1) - 1 - \frac{\alpha(c_1 - 1)}{|b_1| - 1}[\varsigma(c_1 - 1) - \varsigma(c_1 - |b_1|)] + \alpha$$
$$+ |\sigma| F(1, |b_2|, c_2; 1) + \frac{\alpha|\sigma|(c_2 - 1)}{|b_2| - 1}[\varsigma(c_2 - 1) - \varsigma(c_2 - |b_2|)]$$

by application of Lemma 5. In view of formula (3) and the hypothesis, the last expression is bounded above by $1 - \alpha$. ■

Remark 3 For $\alpha = 0$, previous theorem reduces to Corollary 2.

We next find a hypergeometric condition for which the integral operator Λ maps $K_{\hat{H}}^0$ into $HS^*(\alpha)$.

THEOREM 3 *Let $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $|a_j| \neq 1$, $|b_j| \neq 1$, $c_j \in \mathbb{R}$, and $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$. If for some $\sigma (|\sigma| < 1)$ and $\alpha (0 \leq \alpha < 1)$, the hypergeometric inequality*

$$P_1 \frac{\Gamma(c_1 - |a_1| - |b_1|)\Gamma(c_1)}{\Gamma(c_1 - |a_1|)\Gamma(c_1 - |b_1|)} + Q_1|\sigma| \frac{\Gamma(c_2 - |a_2| - |b_2|)\Gamma(c_1)}{\Gamma(c_2 - |a_2|)\Gamma(c_2 - |b_2|)} + \alpha R_1 \leq 4(1 - \alpha)$$

is satisfied, then $\Lambda(K^0_{\hat{H}}) \subset HS^*(\alpha)$, where

$$\begin{aligned}
P_1 &:= (2 - \alpha) + \frac{|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} - \frac{\alpha(c_1 - |a_1| - |b_1|)}{(|a_1| - 1)(|b_1| - 1)}, \\
Q_1 &:= \alpha + \frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1} - \frac{\alpha(c_2 - |a_2| - |b_2|)}{(|a_2| - 1)(|b_2| - 1)}, \\
R_1 &:= \frac{c_1 - 1}{(|a_1| - 1)(|b_1| - 1)} + \frac{c_2 - 1}{(|a_2| - 1)(|b_2| - 1)}.
\end{aligned}$$

Proof Let $f = h + \overline{g} \in K^0_{\hat{H}}$ where h and g are given by (1) with $B_1 = 0$. We need to prove that $\Lambda(f) = H + \overline{G} \in HS^*(\alpha)$ where H and G given by (10) with $B_1 = 0$ are analytic functions in Δ. In view of Lemma 3, we need to prove that $S_3 \leq 1 - \alpha$, where

$$S_3 := \sum_{n=2}^{\infty} (n - \alpha) \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} A_n \right| + \sum_{n=2}^{\infty} (n + \alpha) |\sigma| \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} B_n \right|. \tag{13}$$

Applying Lemma 6, we have

$$S_3 \leq \frac{1}{2} \sum_{n=2}^{\infty} (n + 1)(n - \alpha) \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_n} + \frac{1}{2}|\sigma| \sum_{n=2}^{\infty} (n - 1)(n + \alpha) \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_n}. \tag{14}$$

Using the identities

$$(n + 1)(n - \alpha) \equiv n(n - 1) + (2 - \alpha)n, \quad (n - 1)(n + \alpha) \equiv n(n - 1) + n\alpha - \alpha \tag{15}$$

and by Lemmas 1 and 2, we obtain

$$\begin{aligned}
S_3 \leq{} & \frac{1}{2} \left[(2 - \alpha) + \frac{|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} - \frac{\alpha(c_1 - |a_1| - |b_1|)}{(|a_1| - 1)(|b_1| - 1)} \right] F(|a_1|, |b_1|; c_1; 1) \\
& + \frac{1}{2}|\sigma| \left[\alpha + \frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1} - \frac{\alpha(c_2 - |a_2| - |b_2|)}{(|a_2| - 1)(|b_2| - 1)} \right] F(|a_2|, |b_2|; c_2; 1) \\
& - \frac{2(1 - \alpha)}{2} - \frac{\alpha}{2} \left[\frac{c_1 - 1}{(|a_1| - 1)(|b_1| - 1)} + \frac{|\sigma|(c_2 - 1)}{(|a_2| - 1)(|b_2| - 1)} \right].
\end{aligned}$$

The last expression is bounded above by $1 - \alpha$ because of Equation (3) and the given hypothesis. ■

For the special case when $a_1 = 1$ and $a_2 = 1$, next theorem provides a hypergeometric condition for which the integral operator L maps $K^0_{\hat{H}}$ into $HS^*(\alpha)$.

THEOREM 4 *If* $b_j \in \mathbb{C}\setminus\{0\}$, $|b_2| \neq 1, c_j > |b_j| + 2$ *for* $j = 1, 2$ *and* $|\sigma| < 1$ *satisfy the hypergeometric inequality*

$$\frac{(c_1-1)P_2}{c_1-|b_1|-1} + |\sigma|\frac{(c_2-1)Q_2}{c_2-|b_2|-1} - \alpha R_2 \leq 4-3\alpha$$

is satisfied, then $L(K^0_{\hat{H}}) \subset HS^*(\alpha)$, *where*

$$P_2 := (2-\alpha) + \frac{|b_1|}{c_1-|b_1|-2},\ Q_2 := \alpha + \frac{|b_2|}{c_2-|b_2|-2},\ R_2 := \frac{\varsigma(c_2-1) - \varsigma(c_2-|b_2|)}{|b_2|-1}.$$

Proof Letting $a_1 = 1$ and $a_2 = 1$ in Equation (13) and using (15), we obtain

$$\begin{aligned} S_3 \leq{}& \frac{1}{2}\sum_{n=2}^{\infty}(n-1)\frac{(|b_1|)_{n-1}}{(c_1)_{n-1}} + \frac{2-\alpha}{2}\sum_{n=1}^{\infty}\frac{(|b_1|)_n}{(c_1)_n} \\ &+ \frac{|\sigma|}{2}\left[\sum_{n=2}^{\infty}(n-1)\frac{(|b_2|)_{n-1}}{(c_2)_{n-1}} + \alpha\sum_{n=1}^{\infty}\frac{(|b_2|)_n}{(c_2)_n} - \alpha\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{(|b_2|)_n}{(c_2)_n}\right]. \end{aligned}$$

Applying Lemmas 1, 5, formula (3) and the given hypergeometric condition, it is now a routine to show that $S_3 \leq 1-\alpha$. In view of Lemma 3, it follows that $L(f) \in HS^*(\alpha)$ when $f \in K^0_{\hat{H}}$. ■

In our next result, we impose stronger conditions on the parameters a_1,a_2, b_1, b_2 c_1, c_2 and obtain a hypergeometric characterization for the integral operator G_1.

THEOREM 5 *If* $a_1, b_1 \in (-1,1) \cup (1,\infty), a_1b_1 < 0, a_2, b_2 \in (0,1) \cup (1,\infty), c_j \in \mathbb{R}$, $c_j > \max\{0, a_j + b_j - 1\}$ *for* $j = 1, 2$, *and* $|\sigma| < 1$, *then the integral operator*

$$G_1(z) = \int_0^z F(a_1, b_1; c_1; t)\,\mathrm{d}t + \sigma\overline{\int_0^z F(a_2, b_2; c_2; t)\,\mathrm{d}t}$$

is in the class $THS^*(\alpha)$ *if and only if the hypergeometric inequality*

$$\begin{aligned} &\left[\frac{\alpha\,(c_1-a_1-b_1)}{(a_1-1)\,(b_1-1)} - 1\right]\frac{\Gamma\,(c_1-a_1-b_1)\,\Gamma(c_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)} + |\sigma|\left[\frac{\alpha\,(c_2-a_2-b_2)}{(a_2-1)\,(b_2-1)} + 1\right] \\ &\times\frac{\Gamma\,(c_2-a_2-b_2)\,\Gamma(c_2)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} \leq \alpha\left[\frac{c_1-1}{(a_1-1)\,(b_1-1)} + \frac{c_2-1}{(a_2-1)\,(b_2-1)}\right] \end{aligned}$$

is satisfied.

Proof We observe that the integral operator G_1defined by Equation (2) may be expressed as

$$G_1(z) = z + \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n}z^n + \sigma\sum_{n=1}^{\infty}\overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}z^n}, \tag{16}$$

which in view of the given parametric constraints can be rewritten as

$$\Omega(z) = z - \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n}z^n + \sigma\sum_{n=1}^{\infty}\overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}z^n}.$$

In view of Remarks 1 and 4, G_1 is in $THS^*(\alpha)$ if and only if $S_4 \le 1-\alpha$, where

$$S_4 := \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}(n-\alpha)|\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n}| \tag{17}$$

$$+\sum_{n=1}^{\infty}(n+\alpha)|\sigma||\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}|. \tag{18}$$

Using Lemma 2, it is easy to show that

$$\begin{aligned} S_4 &\le \left[\frac{|a_1b_1|}{c_1}+\frac{\alpha(c_1-a_1-b_1)}{(a_1-1)(b_1-1)}\right]F(a_1,b_1;c_1;1) \\ &+|\sigma|\left[1+\frac{\alpha(c_2-a_2-b_2)}{(a_2-1)(b_2-1)}\right]F(a_2,b_2;c_2;1) \\ &+1-\alpha-\alpha\left[\frac{c_1-1}{(a_1-1)(b_1-1)}+|\sigma|\frac{c_2-1}{(a_2-1)(b_2-1)}\right]. \end{aligned}$$

The last inequality is bounded above by $1-\alpha$ by the formula (3) and the given hypergeometric condition. This completes the proof. ■

COROLLARY 3 *Let* $b_1 \in (-1,0)$, $b_2 \in (0,1)\cup(1,\infty)$, $c_j \in \mathbb{R}$, $c_j > |b_j|+1$ *for* $j=1,2$, $|\sigma|<1$. *Then the integral operator* $G_2(z)=\varphi_1(b_1;c_1;z)+\sigma\overline{\varphi_2(b_2;c_2;z)} \in THS^*(\alpha)$ *if and only if the hypergeometric inequality*

$$\begin{aligned} &\frac{c_1-1}{c_1-b_1-1}-|\sigma|\frac{c_2-1}{c_2-b_2-1} \\ &\ge \alpha\left[\frac{(c_1-1)(\varsigma(c_1-1)-\varsigma(c_1-b_1))}{b_1-1}+\frac{|\sigma|(c_2-1)(\varsigma(c_2-1)-\varsigma(c_2-b_2))}{b_2-1}\right] \end{aligned}$$

is satisfied.

Proof Letting $a_1=1$ and $a_2=1$ in the definition of S_4 in the previous Theorem, it reduces to

$$S_4 \le \frac{|b_1|}{b_1}\left[\sum_{n=1}^{\infty}\frac{(b_1)_n}{(c_1)_n}-\alpha\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{(b_1)_n}{(c_1)_n}\right]+|\sigma|\left[\sum_{n=0}^{\infty}\frac{(b_2)_n}{(c_2)_n}+\alpha\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{(b_2)_n}{(c_2)_n}\right].$$

Using Lemma 5, formula (3), and given hypothesis, it follows that $S_4 \le 1-\alpha$. This completes the proof. ■

Under some stronger parametric conditions, we next determine a hypergeometric inequality when $\sigma = 1$, which guarantees that Λ maps $THS^*(\alpha)$ into $THS^*(\alpha)$ and conversely. Taking $\sigma = 1$ in next theorem and its Corollary does not hurt because $|B_1| < 1$ by equations given by (5).

THEOREM 6 *Let* $a_j, b_j \in (0,1)\cup(1,\infty)$, $c_j > \max\{0, a_j+b_j-1\}$ *for* $j=1,2$, $0\le\alpha<1$, *and* $\sigma=1$. *Then* $\Lambda(THS^*(\alpha)) \subset THS^*(\alpha)$ *if and only if the hypergeometric inequality*

$$\begin{aligned} &\frac{c_1-a_1-b_1}{(a_1-1)(b_1-1)}\frac{\Gamma(c_1-a_1-b_1)\Gamma(c_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}+\frac{(c_2-a_2-b_2)}{(a_2-1)(b_2-1)}\frac{\Gamma(c_2-a_2-b_2)\Gamma(c_2)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} \\ &\le 2+\frac{c_1-1}{(a_1-1)(b_1-1)}+\frac{c_2-1}{(a_2-1)(b_2-1)} \end{aligned}$$

is satisfied.

Proof Let $f = h + \overline{g} \in THS^*(\alpha)$ with h and g of the form (5). In view of Remarks 1 and 4, the integral operator

$$\Lambda(f)(z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} |A_n| z^n + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} |B_n| z^n}$$

is in $THS^*(\alpha)$ if and only if $S_5 \le 1 - \alpha$ where

$$S_5 := \sum_{n=2}^{\infty} (n-\alpha) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} |A_n| + \sum_{n=1}^{\infty} (n+\alpha) \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} |B_n|.$$

Using Inequalities given by (6), we get

$$S_5 \le (1-\alpha) \left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - 1 + \sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \right]. \tag{19}$$

Applying Lemma 2, formula (3), and given hypergeometric inequality, it follows that S_5 is bounded above by $1 - \alpha$. ∎

For the special case when $a_1 = 1$ and $a_2 = 1$, next theorem provides a hypergeometric characterization for which the integral operator L maps $THS^*(\alpha)$ into $THS^*(\alpha)$.

COROLLARY 4 *If $b_j \in (0,1) \cup (1,\infty)$, $c_j > b_j$ for $j = 1, 2$, $0 \le \alpha < 1$, then $L(THS^*(\alpha)) \subset THS^*(\alpha)$ for $\sigma = 1$ if and only if the hypergeometric inequality*

$$\frac{(c_1-1)\left[\varsigma(c_1-1) - \varsigma(c_1-b_1)\right]}{b_1-1} + \frac{(c_2-1)\left[\varsigma(c_2-1) - \varsigma(c_2-b_2)\right]}{b_2-1} \le 2$$

is satisfied.

Proof Letting $a_1 = 1$ and $a_2 = 1$ in Inequality (19) and using Lemma 5, it is now a routine process to prove that $L(f) \in THS^*(\alpha)$ when $f \in THS^*(\alpha)$. ∎

4. Harmonic convexity

We next explore a sufficient hypergeometric condition, which ensures that Λ maps $K^0_{\hat{H}}$ into $HK(\alpha)$.

THEOREM 7 *Let $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 2$ for $j = 1, 2$ and $|\sigma| < 1$. If the inequality*

$$P_3 \frac{\Gamma(c_1 - |a_1| - |b_1|)\Gamma(c_1)}{\Gamma(c_1 - |a_1|)\Gamma(c_1 - |b_1|)} + Q_3|\sigma| \frac{\Gamma(c_2 - |a_2| - |b_2|)\Gamma(c_2)}{\Gamma(c_2 - |a_2|)\Gamma(c_2 - |b_2|)} \le 4(1-\alpha)$$

holds true, then $\Lambda(K^0_{\hat{H}}) \subset HK(\alpha)$*, where*

$$P_3 := \frac{(|a_1|)_2(|b_1|)_2}{(c_1-|a_1|-|b_1|-2)_2} + \frac{(4-\alpha)|a_1b_1|}{c_1-|a_1|-|b_1|-1} + 2(1-\alpha),$$
$$Q_3 := \frac{(|a_2|)_2(|b_2|)_2}{(c_2-|a_2|-|b_2|-2)_2} + \frac{(2+\alpha)|a_2b_2|}{c_2-|a_2|-|b_2|-1}.$$

Proof Let $f = h + \overline{g} \in K^0_{\hat{H}}$ with h and g of the form (1) with $B_1 = 0$. We need to show that $\Lambda(f) = H + \overline{G} \in HK(\alpha)$, where H and G are defined by (10) with $B_1 = 0$. In view of Lemma 4, it suffices to prove that $S_6 \le 1-\alpha$, where

$$S_6 := \sum_{n=2}^{\infty} n(n-\alpha)\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n}A_n\right| + \sum_{n=2}^{\infty} n(n+\alpha)|\sigma|\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}B_n\right|.$$

As an application of Lemma 6, we get

$$S_6 \le \frac{1}{2}\sum_{n=2}^{\infty}(n+1)(n-\alpha)\frac{(|a_1|)_{n-1}(|b_1|))_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{1}{2}|\sigma|\sum_{n=2}^{\infty}(n-1)(n+\alpha)\frac{(|a_2|)_n(|b_2|)_n}{(c_2)_n(1)_{n-1}}.$$

Using the identities

$$(n+1)(n-\alpha) \equiv (n-1)^2 + (n-1)(3-\alpha) + 2(1-\alpha)$$
$$(n-1)(n+\alpha) \equiv (n-1)^2 + (1+\alpha)(n-1),$$

and Lemma 1, we obtain

$$S_6 \le \frac{1}{2}\left[\frac{(|a_1|)_2(|b_1|)_2}{(c_1-|a_1|-|b_1|-2)_2} + \frac{(4-\alpha)|a_1b_1|}{c_1-|a_1|-|b_1|-1} + 2(1-\alpha)\right]F(|a_1|,|b_1|;c_1;1)$$
$$+ \frac{1}{2}|\sigma|\left[\frac{(|a_2|)_2(|b_2|)_2}{(c_2-|a_2|-|b_2|-2)_2} + \frac{(2+\alpha)|a_2b_2|}{c_2-|a_2|-|b_2|-1}\right]F(|a_2|,|b_2|;c_2;1) - (1-\alpha).$$

The result follows by applying formula (3) and given hypergeometric condition. ■

THEOREM 8 *Let* $a_j, b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ *for* $j = 1, 2$ *and* $|\sigma| < 1$. *If the hypergeometric inequality*

$$\left[\frac{|a_1||b_1|}{c_1-|a_1|-|b_1|-1} + (1-\alpha)\right]\frac{\Gamma(c_1-|a_1|-|b_1|)\Gamma(c_1)}{\Gamma(c_1-|a_1|)\Gamma(c_1-|b_1|)}$$
$$+ |\sigma|\left[\frac{|a_2||b_2|}{c_2-|a_2|-|b_2|-1} + (1+\alpha)\right]\frac{\Gamma(c_2-|a_2|-|b_2|)\Gamma(c_2)}{\Gamma(c_2-|a_2|)\Gamma(c_2-|b_2|)} \le 2(1-\alpha)$$

is satisfied, then the integral operator G_1 *given by Equation* (2) *is in* $HK(\alpha)$.

Proof It follows from Lemma 4 that G_1 is in $HK(\alpha)$ when

$$S_7 := \sum_{n=2}^{\infty} n(n-\alpha)|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n}| + |\sigma| \sum_{n=1}^{\infty} n(n+\alpha)|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}| \leq 1-\alpha.$$

Using Lemma 1, we obtain

$$\begin{aligned} S_7 &\leq \sum_{n=2}^{\infty}(n-1)\frac{(|a_1|)_{n-1}(|b_1|))_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(|a_1|)_n(|b_1|))_n}{(c_1)_n(1)_n} \\ &\quad + |\sigma|\sum_{n=2}^{\infty}(n-1)\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} + |\sigma|(1+\alpha)\sum_{n=0}^{\infty}\frac{(|a_2|)_n(|b_2|)_n}{(c_2)_n(1)_n} \\ &= \left[\frac{|a_1||b_1|}{c_1-|a_1|-|b_1|-1} + (1-\alpha)\right]F(|a_1|,|b_1|;c_1;1) - (1-\alpha) \\ &\quad + |\sigma|\left[\frac{|a_2||b_2|}{c_2-|a_2|-|b_2|-1} + (1+\alpha)\right]F(|a_2|,|b_2|;c_2;1). \end{aligned}$$

The result follows by an application of Equation (3) and the given hypergeometric inequality. ■

COROLLARY 5 *If* $b_j \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{R}$, $c_j > |b_j| + 2$ *for* $j = 1, 2$ *and* $|\sigma| < 1$, *then the integral operator* G_2 *is harmonic convex in* Δ *if the hypergeometric inequality*

$$\left[\frac{|b_1|}{c_1-|b_1|-2}+1\right]\frac{c_1-1}{c_1-|b_1|-1} + |\sigma|\left[\frac{|b_2|}{c_2-|b_2|-21}+1\right]\frac{c_2-1}{c_2-|b_2|-1} \leq 2$$

holds true.

The proof of the next theorem is similar to that of Theorem 7 and hence it is omitted.

THEOREM 9 *If all the parametric restrictions in Theorem* 7 *are satisfied, then* $\Lambda(THK(\alpha)) \subset THK(\alpha)$ *when* $\sigma = 1$ *if and only if the hypergeometric inequality in Theorem* 7 *holds true.*

Acknowledgements

This research was supported by a research grant of the University Research Council of Kent State Unversity. The author thanks the referee for some useful suggestions for improvement of the article.

References

[1] O.P. Ahuja, *Planar harmonic univalent and related mappings*, J. Inequal. Pure Appl. Math. 6(4), Article 122, pp. 1–18.
[2] O.P. Ahuja, *Planar harmonic convolution operators generated by hypergeometric functions*, Integral Transform. Spec. Funct 18(3), pp. 165–177.
[3] B.C. Carleson and D.B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. 15 (1984), pp. 737–745.
[4] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 9 (1984), pp. 3–25.
[5] P.L. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics Vol. 156, Cambridge University Press, Cambridge, 2004, ISBN: 0-521-64121-7.
[6] Y.E. Hohlov, *Convolution operators preserving univalent functions*, Ukrainian Math. J. 37 (1985), pp. 220–226 (Russian).
[7] J.M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Marie Curie-Sklodowska Sect. A, 52(2) (1998), pp. 57–66.
[8] J.M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl., 235 (1999), pp. 470–477.

[9] S. Miller and P.T. Mocanu, *Univalence of Gaussian and confluent hypergeometric functions*, Proc. Amer. Math. Soc. 110(2) (1990), pp. 333–342.
[10] S. Owa and H.M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. 39 (1987), pp. 1057–1077.
[11] S. Punnusamy and F. Rønning, *Starlikeness properties for convolutions involving hypergeometric series*, Ann. Univ. Mariae Curie Sklodowska L. II 1(16) (1998), pp. 141–155.
[12] S. Ruscheweyh and V. Singh, *On the order of starlikeness of hypergeometric functions*, J. Math. Anal. Appl. 113 (1986), pp. 1–11.
[13] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Limited and John Wiley & Sons, New York, Chichester, Toronto, 1984.