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Connections between various subclasses of planar harmonic mappings involving hypergeometric functions

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Abstract

The main purpose of this paper is to establish connections between various subclasses of harmonic mappings in the plane by applying certain convolution operators involving hypergeometric functions. To be more precise, we investigate such connections with harmonic k-uniformly starlike and harmonic k-uniformly convex mappings in the plane.

Keywords: Planar harmonic mappings; Hypergeometric functions; Harmonic starlike; Harmonic convex; Harmonic close-to-convex; Harmonic typically real; Harmonic k-uniformly convex; Harmonic k-uniformly starlike

1. Introduction

Let A denote the class of analytic functions in the unit disk Δ with the normalizations $f(0) = 0$ and $f'(0) = 1$. Hohlov [12] introduced the convolution operator $H(a,b,c) : A \to A$ defined by $H(a,b,c)f(z) := zF(a,b;c;z) \star f(z)$, where the symbol $\star$ denotes the convolution of two functions and where $F(a,b,c;z)$ is a well-known Gaussian hypergeometric function. Here a,b,c are complex numbers such that $c \neq 0,-1,-2,\ldots$ A hypergeometric function $F(a,b;c;z)$ is analytic in Δ and plays an important role in Geometric Function Theory. See, for example, the works by de Branges [6], Carleson and Shaffer [7], Miller and Mocanu [14], Owa and Srivastava [15], Ruscheweyh and Singh [19], and Srivastava and Manocha [22].

Let $\hat{H}$ be the family of all harmonic functions of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1,\ z \in \Delta \tag{1}$$

are in the class A. For complex parameters $a_1, b_1, c_1, a_2, b_2, c_2$ ($c_1, c_2 \neq 0, -1, -2, \ldots$) we define the functions $\phi_1(z) := zF(a_1,b_1;c_1;z)$ and $\phi_2(z) := zF(a_2,b_2;c_2;z)$. Corresponding to these functions, we consider the convolution operator

$$\Omega \equiv \Omega\begin{pmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \end{pmatrix} : \hat{H} \to \hat{H}$$

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defined by

$$\Omega\begin{pmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \end{pmatrix} f := f \tilde{\star} (\phi_1 + \overline{\phi_2}) = h \star \phi_1 + \overline{g \star \phi_2}$$

for any function $f = h + \bar{g}$ in $\hat{H}$. Letting

$$\Omega\begin{pmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \end{pmatrix} f(z) := H(z) + \overline{G(z)},$$

we have

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n. \tag{2}$$

We observe that

$$\Omega\begin{pmatrix} a_1, 1, a_1 \\ a_2, 1, a_2 \end{pmatrix} f(z) = f(z) = f(z) \widetilde{\star} \left(\frac{z}{1-z} + \overline{\frac{z}{1-z}} \right)$$

is the identity mapping. Setting $a_1 = a_2 = b_1 = b_2 = -m, m \in \mathbb{N}, c_1 = c_2 = c > 0$, we may reduce the convolution operator to the polynomial

$$\Omega\begin{pmatrix} -m, -m, c \\ -m, -m, c \end{pmatrix} f(z) = f(z) \tilde{\star} \left(z + \sum_{n=2}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c_1)_{n-1}(1)_{n-1}} z^n + \sum_{n=1}^{m+1} \frac{\{(-m)_{n-1}\}^2}{(c_1)_{n-1}(1)_{n-1}} \overline{z^n} \right). \tag{3}$$

We also define the convolution operator $L : \hat{H} \to \hat{H}$ by

$$L(f) := L\begin{pmatrix} b_1, c_1 \\ b_2, c_2 \end{pmatrix} f := f \widetilde{\star} (\varphi_1 + \overline{\varphi_2}) = h \star \varphi_1 + \overline{g \star \varphi_2},$$

where φ_1 and φ_2 are incomplete beta functions given by

$$\varphi_1(z) := \varphi(b_1, c_1; z) = zF(1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(b_1)_{n-1}}{(c_1)_{n-1}} z^n,$$

$$\varphi_2(z) := \varphi(b_2, c_2; z) = zF(1, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(b_2)_{n-1}}{(c_2)_{n-1}} z^n,$$

and where b_1, c_1, b_2, c_2 are non-zero complex parameters $(c_1, c_2 \neq 0, -1, -2, \ldots)$. These convolution operators Ω and L were defined and studied by the author in [2].

Denote by $S_{\hat{H}}$ the subclass of $\hat{H}$ that are univalent and sense-preserving in Δ. Note that $(f - \overline{B_1 f})/(1 - |B_1|^2) \in S_{\hat{H}}$ whenever $f \in S_{\hat{H}}$. Thus we may restrict our attention to the subclass $S^0_{\hat{H}}$ of $S_{\hat{H}}$ defined by

$$S^0_{\hat{H}} := \{f = h + \bar{g} \in S_{\hat{H}} : g'(0) = B_1 = 0\}.$$

The classes $S^0_{\hat{H}}$ and $S_{\hat{H}}$ were first studied in [8]. We let $K^0_{\hat{H}}, S^{0*}_{\hat{H}}$, and $C^0_{\hat{H}}$, denote the subclasses of $S^0_{\hat{H}}$ of harmonic functions which are, respectively, convex, starlike, and close-to-convex in Δ. Also, let $T^0_{\hat{H}}$ be the class of sense-preserving, typically real, harmonic functions $f = h + \bar{g}$ in $\hat{H}$. For definitions and properties of these classes, one may refer to [1,8] or [9].

For $0 \leqslant \beta < 1$, let

$$N_{\hat{H}}(\beta) := \left\{ f \in \hat{H} : \mathrm{Re} \frac{f'(z)}{z'} \geqslant \beta, z = re^{i\theta} \in \Delta \right\},$$

$$R_{\hat{H}}(\beta) := \left\{ f \in \hat{H} : \mathrm{Re} \frac{f''(z)}{z''} \geqslant \beta, z = re^{i\theta} \in \Delta \right\},$$

where

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}), \quad z'' = \frac{\partial}{\partial\theta}(z'), \quad f'(z) = \frac{\partial}{\partial\theta}f(re^{i\theta}), \quad f''(z) = \frac{\partial}{\partial\theta}(f'(z)).$$

Define $TN_{\hat{H}}(\beta) := N_{\hat{H}}(\beta) \cap \check{T}$ and $TR_{\hat{H}}(\beta) := R_{\hat{H}}(\beta) \cap \check{T}$, where Ť consists of the functions $f = h + \bar{g}$ in $\hat{H}$ so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \tag{4}$$

The classes $\check{T}, N_{\hat{H}}(\beta), TN_{\hat{H}}(\beta), R_{\hat{H}}(\beta)$, and $TR_{\hat{H}}(\beta)$ were initially introduced and studied, respectively, in [20,4,4,5,5]. Various subclasses of T were also studied in [21].

Let $HUC(k,\alpha)$ be a subclass of the functions $f = h + \bar{g}$ in $\hat{H}$ which satisfy the condition

$$\mathrm{Re}\left(1 + (1 + ke^{i\eta})\frac{z^2h''(z) + \overline{2zg' + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right) \geqslant \alpha$$

for some k $(0 \leqslant k < \infty)$, α $(0 \leqslant \alpha < 1)$ and $z \in \Delta$. Define $THUC(k,\alpha) := HUC(k,\alpha) \cap$ Ť. A mapping in $HUC(k,\alpha)$ or $THUC(k,\alpha)$ is called harmonic k-uniformly convex in Δ. These classes were studied in [13]. For $g \equiv 0$, $k = 1$, and $\alpha = 0$, the class $HUC(k,\alpha)$ reduces to the class UC of analytic uniformly convex functions defined by Goodman [11].

Analogous to $HUC(k,\alpha)$ is the class $HUS^*(k,\alpha)$ consisting of harmonic functions $f = h + \bar{g}$ in $\hat{H}$ which satisfy the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{z'f(z)} - \alpha\right) \geqslant k\left|\frac{zf'(z)}{z'f(z)} - 1\right|$$

for some k $(0 \leqslant k < \infty)$, α $(0 \leqslant \alpha < 1)$ and $z \in \Delta$. Also define $THUS^*(k,\alpha) := HUS^*(k,\alpha) \cap$ Ť. The mappings in $HUS^*(k,\alpha)$ or $THUS^*(k,\alpha)$ are called harmonic k-uniformly starlike in Δ. For $\alpha = 0$, these classes were studied in [18]. For $g \equiv 0$, $k = 1$, and $\alpha = 0$, $HUS^*(k,\alpha)$ reduces to the family $US^*$of analytic uniformly starlike functions defined by Ronning [17].

Throughout this paper, we will frequently use the notations

$$\Omega(f) \equiv \Omega\begin{pmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \end{pmatrix} f, \quad L(f) \equiv L\begin{pmatrix} b_1, c_1 \\ b_2, c_2 \end{pmatrix} f,$$

$$D_{n-1} := \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}, \quad E_{n-1} := \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}},$$

and a well-known formula

$$F(a,b,c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0.$$

The main object of this paper is to establish some important connections between the classes $HUC(k,\alpha), HUS^*(k,\alpha), K^0_{\hat{H}}, S^{0*}_{\hat{H}}, C^0_{\hat{H}}, N_{\hat{H}}(\beta)$, and $R_{\hat{H}}(\beta)$ by applying the convolution operators Ω and L.

2. Connections with harmonic uniformly convex mappings

In order to establish connection between harmonic convex mappings and harmonic uniformly convex mappings we need following results in Lemma 1 [8], Lemma 2 [13], and Lemma 3 [2].

Lemma 1. *If* $f = h + \bar{g} \in K^0_{\hat{H}}$ *where* h *and* g *are given by* (1) *with* $B_1 = 0$, *then*

$$|A_n| \leqslant \frac{n+1}{2}, \quad |B_n| \leqslant \frac{n-1}{2}.$$

Lemma 2. *Let $f = h + \bar{g}$ where h and g are given by (1). If $0 \leqslant k < \infty$, $0 \leqslant \alpha < 1$ and*

$$\sum_{n=2}^{\infty} n(n(k+1)-(k+\alpha))|A_n| + \sum_{n=1}^{\infty} n(n(k+1)+(k+\alpha))|B_n| \leqslant 1-\alpha, \tag{5}$$

then f is harmonic, sense-preserving, univalent in Δ, and $f \in HUC(k,\alpha)$.

Remark 1. In [13], it is also shown that $f = h + \bar{g}$ given by (4) is in the family $THUC(k,\alpha)$ if and only if the coefficient condition (5) holds. Moreover, if $f \in THUC(k,\alpha)$, then

$$|A_n| \leqslant \frac{1-\alpha}{n(n(k+1)-(k+\alpha))}, \quad n \geqslant 2,$$
$$|B_n| \leqslant \frac{1-\alpha}{n(n(k+1)+(k+\alpha))}, \quad n \geqslant 1.$$

Lemma 3. *If $a,b,c > 0$, then*

(i) $F(a+k,b+k;c+k;1) = \frac{(c)_k}{(c-a-b-k)_k} F(a,b;c;1)$, *for* $k = 0,1,2\ldots$, *if* $c > a+b+k$.
(ii) $\sum_{n=2}^{\infty}(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c-a-b-1}F(a,b;c;1)$, *if* $c > a+b+1$.
(iii) $\sum_{n=2}^{\infty}(n-1)^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left[\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right]F(a,b;c;1)$, *if* $c > a+b+2$.
(iv) $\sum_{n=2}^{\infty}(n-1)^3\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left[\frac{(a)_3(b)_3}{(c-a-b-3)_3} + \frac{3(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right]F(a,b;c;1)$, *if* $c > a+b+3$.

Theorem 1. *Let $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}$, and $c_j > |a_j| + |b_j| + 3$ for $j = 1, 2$. If for some k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$, the inequality*

$$Q_1 F(|a_1|,|b_1|;c_1;1) + R_1 F(|a_2|,|b_2|;c_2;1) \leqslant 4(1-\alpha)$$

is satisfied, then

$$\Omega(K^0_{\hat{H}}) \subset HUC(k,\alpha),$$

where

$$Q_1 := (k+1)\frac{(|a_1|)_3(|b_1|)_3}{(c_1-|a_1|-|b_1|-3)_3} + (6k+7-\alpha)\frac{(|a_1|)_2(|b_1|)_2}{(c_1-|a_1|-|b_1|-2)_2}$$
$$+ 2(3k+5-2\alpha)\frac{|a_1b_1|}{c_1-|a_1|-|b_1|-1} + 2(1-\alpha),$$
$$R_1 := (k+1)\frac{(|a_2|)_3(|b_2|)_3}{(c_2-|a_2|-|b_2|-3)_3} + (6k+5+\alpha)\frac{(|a_2|)_2(|b_2|)_2}{(c_2-|a_2|-|b_2|-2)_2}$$
$$+ 2(3k+2+\alpha)\frac{|a_2b_2|}{c_2-|a_2|-|b_2|-1}.$$

Proof. Let $f = h + \bar{g} \in K^0_{\hat{H}}$ with h and g of the form (1) with $B_1 = 0$. We need to show that $\Omega(f) = H + \bar{G} \in HUC(k,\alpha)$, where H and G defined by (2) with $B_1 = 0$ are analytic functions in Δ. In view of Lemma 2, we need to prove that $P_1 \leqslant 1-\alpha$ where

$$P_1 := \sum_{n=2}^{\infty} n(n(k+1)-(k+\alpha))\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n\right| + \sum_{n=2}^{\infty} n(n(k+1)+(k+\alpha))\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n\right|.$$

In view of Lemmas 1 and 3, it follows that

$$P_1 \leqslant \frac{1}{2}\sum_{n=2}^{\infty} n(n+1)(n(k+1)-(k+\alpha))D_{n-1} + \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(n(k+1)+(k+\alpha))E_{n-1}$$
$$= \frac{1}{2}\left[\sum_{n=2}^{\infty}(k+1)(n-1)^3+(3k+4-\alpha)(n-1)^2+(2k+5-3\alpha)(n-1)+2(1-\alpha)\right]D_{n-1}$$
$$+\frac{1}{2}\left[\sum_{n=2}^{\infty}(k+1)(n-1)^3+(3k+2+\alpha)(n-1)^2+(2k+1+\alpha)(n-1)\right]E_{n-1}$$
$$= \frac{1}{2}Q_1F(|a_1|,|b_1|;c_1;1)+\frac{1}{2}R_1F(|a_2|,|b_2|;c_2;1)-(1-\alpha).$$

Now $P_1 \leqslant 1-\alpha$ follows from the given condition. □

The following result is a consequence of Theorem 1, for the special case when $a_1 = a_2 = a$, $b_1 = b_2 = \bar{a}$, and $c_1 = c_2 = c$. This result is useful in characterizing polynomials, when $a = b = -m, m \in \mathbb{N}$.

Corollary 1. *Let $a \in \mathbb{C}\setminus\{0\}$, and suppose $c > 2|a| + 3$. If the inequality*

$$\left[2(k+1)\frac{(((|a|)_3))^2}{(c-2|a|-3)_3}+12(k+1)\frac{(((|a|)_2))^2}{(c-2|a|-2)_2}+2(6k+7-\alpha)\frac{(|a|)^2}{c-2|a|-1}+2(1-\alpha)\right]F(|a|,|a|;c;1)$$
$$\leqslant 4(1-\alpha)$$

is satisfied for some k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$, then

$$\left(\Omega\begin{pmatrix} a,\bar{a},c \\ a,\bar{a},c\end{pmatrix}\right)(K_{\hat{H}}^0) \subset HUC(k,\alpha).$$

In another special case when $a_1 = 1 = a_2$, $b_1 = b_2 = b$, $c_1 = c_2 = c$, $\alpha = 0$, $k = 1$, and if we write the incomplete beta operator

$$\Omega\begin{pmatrix} 1,b,c \\ 1,b,c\end{pmatrix} \equiv L\begin{pmatrix} b,c \\ b,c\end{pmatrix} := \Lambda_c^b,$$

then Theorem 1 reduces to the following result.

Corollary 2. *Let $b \in \mathbb{C}\setminus\{0\}$, and suppose $c > |b| + 4$. If the inequality*

$$\frac{12((|b|)_3)}{(c-|b|-4)_3}+\frac{24((|b|)_2)}{(c-|b|-3)_2}+\frac{13|b|}{c-|b|-2} \leqslant \frac{c-2|b|-1}{c-1}$$

is satisfied, then $\Lambda_c^b(K_{\hat{H}}^0) \subset G_H^0$, where $G_H^0 \equiv HUC(1,0)$ where $B_1 = 0$ is the class of harmonic uniformly convex functions in Δ.

In order to determine connection between $TN_{\hat{H}}(\beta)$ and $HUC(k,\alpha)$, we need the following result obtained in [4].

Lemma 4. *Let $f = h + \bar{g}$ where h and g are given by (4), and suppose $0 \leqslant \beta < 1$. Then*

$$f \in TN_{\hat{H}}(\beta) \Longleftrightarrow \sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| \leqslant 1-\beta.$$

Corollary 3. *If $f = h + \bar{g} \in TN_{\hat{H}}(\beta)$ where h and g are given by (4), then*

$$|A_n| \leqslant \frac{(1-\beta)}{n}, \quad |B_n| \leqslant \frac{(1-\beta)}{n}.$$

Theorem 2. *Let* $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}, c_j > |a_j| + |b_j| + 1$ *for* $j = 1, 2$. *If for some* k $(0 \leqslant k < \infty)$, α $(0 \leqslant \alpha < 1)$, *and* β $(0 \leqslant \beta < 1)$ *the condition*

$$Q_2 F(|a_1|, |b_1|; c_1; 1) + R_2 F(|a_2|, |b_2|; c_2; 1) \leqslant \frac{(1-\alpha)(2-\beta)}{(1-\beta)}$$

is satisfied, then

$$\Omega(TN_{\hat{H}}(\beta)) \subset HUC(k, \alpha),$$

where

$$Q_2 := (k+1)\frac{|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} + (1-\alpha),$$

$$R_2 := (k+1)\frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1} + (2k+1+\alpha).$$

Proof. Let $f = h + \bar{g} \in TN_{\hat{H}}(\beta)$ where h and g are given by (4). In view of Lemma 2, it is enough to show that $P_2 \leqslant 1 - \alpha$, where

$$P_2 := \sum_{n=2}^{\infty} n(n(k+1) - (k+\alpha))\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n\right| + \sum_{n=1}^{\infty} n(n(k+1) + (k+\alpha))\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n\right|.$$

Using Corollary 3 and Lemma 3, it follows that

$$P_2 \leqslant (1-\beta)\left[\sum_{n=2}^{\infty}(n(k+1) - (k+\alpha))D_{n-1} + \sum_{n=1}^{\infty}(n(k+1) + (k+\alpha))E_{n-1}\right]$$
$$\leqslant (1-\beta)[Q_2 F(|a_1|, |b_1|; c_1; 1) - (1-\alpha) + Q_3 F(|a_2|, |b_2|; c_2; 1)] \leqslant 1 - \alpha,$$

by the given hypothesis. □

For the relationship between the classes $TR_{\hat{H}}(\beta)$ and $HUC(k, \alpha)$, we need the following result obtained in [5].

Lemma 5. *Let* $f = h + \bar{g}$ *where* h *and* g *are given by* (4), *and suppose* $0 \leqslant \beta < 1$. *Then*

$$f \in TR_{\hat{H}}(\beta) \iff \sum_{n=2}^{\infty} n^2 |A_n| + \sum_{n=1}^{\infty} n^2 |B_n| \leqslant 1 - \beta.$$

Corollary 4. *If* $f = h + \bar{g} \in TR_{\hat{H}}(\beta)$ *where* h *and* g *are given by* (4), *then*

$$|A_n| \leqslant \frac{(1-\beta)}{n^2}, \quad |B_n| \leqslant \frac{(1-\beta)}{n^2}.$$

Lemma 6. *Let* $a, b \in \mathbb{C} \setminus \{0\}, a \neq 1, b \neq 1, c \in (0,1) \cup (1, \infty)$, *and* $c > \max\{0, |a| + |b| - 1\}$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{c - |a| - |b|}{(|a|-1)(|b|-1)} F(|a|, |b|; c; 1) - \frac{c-1}{(|a|-1)(|b|-1)}.$$

Proof. We can write

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{c-1}{(|a|-1)(|b|-1)} \sum_{n=1}^{\infty} \frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n}$$
$$= \frac{c-1}{(|a|-1)(|b|-1)}[F(|a|-1, |b|-1; c-1; 1) - 1]$$

and the result immediately follows. □

Theorem 3. *Let* $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}, c_j > \max\{0, |a_j| + |b_j| - 1\}, (|a_j| \neq 1, |b_j| \neq 1)$ *for* $j = 1, 2$. *If for some* k $(0 \leqslant k < \infty)$, α $(0 \leqslant \alpha < 1)$, *and* β $(0 \leqslant \beta < 1)$ *the inequality*

$$Q_3 F(|a_1|, |b_1|; c_1; 1) + R_3 F(|a_2|, |b_2|; c_2; 1) \leqslant \frac{(1-\alpha)(2-\beta)}{(1-\beta)} - (k+\alpha)\left(\frac{c_1 - 1}{(|a_1|-1)(|b_1|-1)} - \frac{c_2 - 1}{(|a_2|-1)(|b_2|-1)}\right)$$

is satisfied, then

$$\Omega(TR_{\hat{H}}(\beta)) \subset HUC(k, \alpha),$$

where

$$Q_3 := (k+1) - (k+\alpha)\frac{c_1 - |a_1| - |b_1|}{(|a_1|-1)(|b_1|-1)},$$
$$R_3 := (k+1) + (k+\alpha)\frac{c_2 - |a_2| - |b_2|}{(|a_2|-1)(|b_2|-1)}.$$

Proof. Making use of Lemma 2 and the definition of P_2 in Theorem 2, we only need to prove that $P_2 \leqslant 1 - \alpha$. Using Corollary 4 and Lemma 3(iii), it follows that

$$P_2 \leqslant (1-\beta)\left[\sum_{n=2}^{\infty}\left(k+1-\frac{(k+\alpha)}{n}\right)D_{n-1} + \sum_{n=1}^{\infty}\left(k+1+\frac{(k+\alpha)}{n}\right)E_{n-1}\right]$$
$$= (1-\beta)\left[Q_3 F(|a_1|, |b_1|; c_1; 1) + R_3 F(|a_2|, |b_2|; c_2; 1) - (1-\alpha) + \frac{(k+\alpha)(c_1-1)}{(|a_1|-1)(|b_1|-1)} - \frac{(k+\alpha)(c_2-1)}{(|a_2|-1)(|b_2|-1)}\right] \leqslant 1-\alpha,$$

by Lemma 6 and the given hypothesis. □

In order to prove the corresponding result for incomplete beta operator L, we need the following result obtained in [16].

Lemma 7. *Let* $b, c > 0$, $b \neq 1$, *and* $c > 1 + b$. *Then*

$$\sum_{n=0}^{\infty}\frac{(b)_n}{(n+1)(c)_n} = \frac{c-1}{b-1}[\varsigma(c-1) - \varsigma(c-b)],$$

where $\varsigma(x) = \Gamma'(x)/\Gamma(x)$.

Theorem 4. *Let* $b_j \in \mathbb{C} \setminus \{0\}, |b_j| \neq 1, c_j > 0, c_j > 1 + |b_j|$ *for* $j = 1, 2$. *For some* k $(0 \leqslant k < \infty)$, α $(0 \leqslant \alpha < 1)$ *and* β $(0 \leqslant \beta < 1)$, *assume that*

$$(k+1)\left[\frac{|b_1|}{c_1 - |b_1| - 1} + \frac{c_2 - 1}{c_2 - |b_2| - 1}\right] - (k+\alpha)\vartheta \leqslant \frac{(1-\alpha)}{1-\beta},$$

where

$$\vartheta = \left[\frac{(c_1-1)(\varsigma(c_1-1) - \varsigma(c_1-|b_1|))}{|b_1|-1} - \frac{(c_2-1)(\varsigma(c_2-1) - \varsigma(c_2-|b_2|))}{|b_2|-1}\right]$$

and $\varsigma(x) = \Gamma'(x)/\Gamma(x)$. *Then*

$$\left(L\begin{pmatrix} b_1, c_1 \\ b_2, c_2 \end{pmatrix}\right)(TR_{\hat{H}}(\beta)) \subset HUC(k, \alpha).$$

Proof. Note that when $a_1 = a_2 = 1$, P_2 in Theorem 2 reduces to

$$P_2 \leqslant (1-\beta)\left[\sum_{n=2}^{\infty}\left(k+1-\frac{(k+\alpha)}{n}\right)\frac{(|b_1|)_{n-1}}{(c_1)_{n-1}} + \sum_{n=1}^{\infty}\left(k+1+\frac{(k+\alpha)}{n}\right)\frac{(|b_2|)_{n-1}}{(c_2)_{n-1}}\right].$$

Using Lemma 7 and the given condition, the result follows. □

Theorem 5. *Let $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}, c_j > |a_j| + |b_j|$ for $j = 1, 2$. If for some k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$, the inequality*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leqslant 2$$

is satisfied, then

$$\Omega(THUC(k, \alpha)) \subset HUC(k, \alpha).$$

Proof. By adopting the proof of Theorem 2, definition of P_2, Lemma 2, Remark 1, and Lemma 3, we obtain

$$P_2 \leqslant (1 - \alpha)\left[\sum_{n=1}^{\infty} D_n + \sum_{n=0}^{\infty}\right] E_{n-1} \leqslant 1 - \alpha$$

by the given condition and this completes the proof. □

In the next result, we improve Theorem 5 by diluting the restrictions on the complex coefficients.

Theorem 6. *If $a_1, b_1 > -1, a_1 b_1 < 0, a_2, b_2 \in \mathbb{C} \setminus \{0\}, c_1 > a_1 + b_1, c_2 > |a_2| + |b_2|$, then a sufficient condition for*

$$\Omega(THUC(k, \alpha)) \subset HUC(k, \alpha)$$

is that

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geqslant 0$$

for any k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$.

Proof. Let $f = h + \bar{g} \in THUC(k, \alpha)$ with h and g in (4). Then

$$\Omega(f) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}. \quad (6)$$

This function can be rewritten as

$$\Omega(f) = z + \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}.$$

In view of Lemma 2, it suffices to prove that $P_3 \leqslant 1$ where

$$\begin{aligned} P_3 &:= \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{n(n(k+1) - (k+\alpha))}{1-\alpha} \left| \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}} |A_n| \right| \\ &\quad + \sum_{n=1}^{\infty} \frac{n(n(k+1) + (k+\alpha))}{1-\alpha} \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| \right|. \end{aligned}$$

Using the coefficient estimates in Remark 1, it follows that

$$\begin{aligned} P_3 &\leqslant \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-1}} + \sum_{n=1}^{\infty} E_{n-1} = \frac{|a_1 b_1|}{a_1 b_1} \sum_{n=1}^{\infty} \frac{(a_1)_n (b_1)_n}{(c_1)_n (1)_n} + \sum_{n=1}^{\infty} E_{n-1} \\ &= -F(a_1, b_1; c_1; 1) + 1 + F(|a_2|, |b_2|; c_2; 1) \leqslant 1 \end{aligned}$$

by the given coefficient condition. □

In our next result, we impose stronger conditions on the parameters $a_1, a_2, b_1, b_2, c_1, c_2$ and obtain a characterization for operator Ω which maps $THUC(k, \alpha)$ onto itself.

Theorem 7. *Let $a_j, b_j > 0$, $c_j > a_j + b_j$ $(j = 1, 2)$, $0 \leqslant k < \infty$, and $0 \leqslant \alpha < 1$. Then $\Omega(THUC(k, \alpha)) \subset THUC(k, \alpha)$ if and only if*

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leqslant 2. \quad (7)$$

Proof. Let $f = h + \bar{g} \in THUC(k,\alpha)$. In view of Remark 1, we only need to prove that $\Omega(f)$ given by (6) is in $THUC(k,\alpha)$ if and if $P_4 \leqslant 1$, where

$$P_4 := \sum_{n=2}^{\infty} \frac{n(n(k+1)-(k+\alpha))}{1-\alpha} \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| \right| + \sum_{n=1}^{\infty} \frac{n(n(k+1)+(k+\alpha))}{1-\alpha} \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| \right|.$$

Using the coefficient estimates stated in Remark 1, we obtain

$$P_4 \leqslant \sum_{n=1}^{\infty} D_n + \sum_{n=0}^{\infty} E_n \leqslant 1,$$

by the given condition. □

3. Connections with harmonic uniformly starlike mappings

In this section we shall look at the analogous results involving connections between various classes of planar harmonic mappings and $HUS^*(k,\alpha)$ by applying convolution operators Ω and L.

Lemma 8. *Let $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1). If for some k $(0 \leqslant k < \infty)$ and $0 \leqslant \alpha < 1$, the inequality*

$$\sum_{n=2}^{\infty} (n(k+1)-(k+\alpha))|A_n| + \sum_{n=1}^{\infty} (n(k+1)+(k+\alpha))|B_n| \leqslant 1-\alpha \tag{8}$$

is satisfied, then f is harmonic, sense-preserving, univalent in Δ, and $f \in HUS^*(k,\alpha)$.

Remark 2. The result in Lemma 8 is a special case of the corresponding result proved in [3]. However, for $k = 1$, Lemma 8 reduces to the result found in [18].

Remark 3. In [3], it is also shown that the $f = h + \bar{g}$ given by (4) is in the family $THUS^*(k,\alpha)$ if and only if the coefficient condition (8) holds. Moreover, if $f \in THUS^*(k,\alpha)$, then

$$|A_n| \leqslant \frac{1-\alpha}{n(k+1)-(k+\alpha)}, \quad n \geqslant 2,$$
$$|B_n| \leqslant \frac{1-\alpha}{n(k+1)+(k+\alpha)}, \quad n \geqslant 1.$$

Applying Lemmas 8, 1, 3 and using the technique of the proof of Theorem 1, we get

Theorem 8. *Let $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}, c_j > |a_j| + |b_j| + 2$ for $j = 1, 2$. If for some k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$ the inequality*

$$Q_4 F(|a_1|, |b_1|; c_1; 1) + R_4 F(|a_2|, |b_2|; c_2; 1) \leqslant 4(1-\alpha)$$

is satisfied, then

$$\Omega(K^0_{\hat{H}}) \subset HUS^*(k,\alpha),$$

where

$$Q_4 := (k+1)\frac{(|a_1|)_2(|b_1|)_2}{(c_1-|a_1|-|b_1|-2)_2} + (3k+4-\alpha)\frac{|a_1 b_1|}{c_1-|a_1|-|b_1|-1} + 2(1-\alpha),$$
$$R_4 := (k+1)\frac{(|a_2|)_2(|b_2|)_2}{(c_2-|a_2|-|b_2|-2)_2} + (3k+2+\alpha)\frac{|a_2 b_2|}{c_2-|a_2|-|b_2|-1}.$$

Analogous to Theorem 8, we next find connections of the classes $S^{0*}_{\hat{H}}$, $C^0_{\hat{H}}$, and $T^0_{\hat{H}}$ with $HUS^*(k,\alpha)$. However, we first need the following result which may be found in [1,8], or [23].

Lemma 9. *If* $f = h + \bar{g} \in C^0_{\hat{H}}$ $(S^{*0}_{\hat{H}}$ *or* $T^0_{\hat{H}})$ *with h and g as given by* (1) *with* $B_1 = 0$, *then*

$$|A_n| \leqslant \frac{(2n+1)(n+1)}{6},$$
$$|B_n| \leqslant \frac{(2n-1)(n-1)}{6}.$$

Theorem 9. *Let* $a_j, b_j \in \mathbb{C} \setminus \{0\}, c_j \in \mathbb{R}, c_j > |a_j| + |b_j| + 3$ *for* $j = 1, 2$. *If for some* k $(0 \leqslant k < \infty)$ *and* α $(0 \leqslant \alpha < 1)$ *the inequality*

$$Q_5 F(|a_1|, |b_1|; c_1; 1) + R_5 F(|a_2|, |b_2|; c_2; 1) \leqslant 12(1-\alpha)$$

is satisfied, then

$$\Omega(C^0_{\hat{H}}) \subset HUS^*(k,\alpha), \Omega(S^{*0}_{\hat{H}}) \subset HUS^*(k,\alpha), \quad \text{and} \quad \Omega(T^0_{\hat{H}}) \subset HUS^*(k,\alpha),$$

where

$$Q_5 := 2(k+1)\frac{(|a_1|)_3(|b_1|)_3}{(c_1 - |a_1| - |b_1| - 3)_3} + (13k + 15 - 2\alpha)\frac{(|a_1|)_2(|b_1|)_2}{(c_1 - |a_1| - |b_1| - 2)_2} + 3(5k + 8 - 3\alpha)\frac{|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} + 6(1-\alpha),$$

$$R_5 := 2(k+1)\frac{(|a_2|)_3(|b_2|)_3}{(c_2 - |a_2| - |b_2| - 3)_3} + (11k + 9 + 2\alpha)\frac{(|a_2|)_2(|b_2|)_2}{(c_2 - |a_2| - |b_2| - 2)_2} + 3(3k + 2 + \alpha)\frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1}.$$

Proof. Let $f = h + \bar{g} \in C^0_{\hat{H}}$ $(S^{*0}_{\hat{H}}$ or $T^0_{\hat{H}})$ where h and g are given by (1) with $B_1 = 0$. We need to prove that $\Omega(f) = H + \bar{g} \in HUS^*(k,\alpha)$ where H and G given by (2) with $B_1 = 0$ are analytic functions in Δ. In view of Lemma 8, we need to prove that $P_5 \leqslant 1 - \alpha$, where

$$P_5 := \sum_{n=2}^{\infty}(n(k+1) - (k+\alpha))\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n\right| + \sum_{n=2}^{\infty}(n(k+1) + (k+\alpha))\left|\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n\right|.$$

As an application of Lemma 8, we obtain

$$\begin{aligned} P_5 &\leqslant \frac{1}{6}\sum_{n=2}^{\infty}[(2n+1)(n+1)((k+1)n - (k+\alpha))D_{n-1} + (2n-1)(n-1)((k+1)n + (k+\alpha))E_{n-1}] \\ &= \frac{1}{6}\left[\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)^3 + (7k+9-2\alpha)(n-1)^2 + (6k+13-7\alpha)(n-1) + 6(1-\alpha)\right\}D_{n-1}\right] \\ &\quad + \frac{1}{6}\left[\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)^3 + (5k+3+2\alpha)(n-1)^2 + (2k+1+\alpha)(n-1)\right\}E_{n-1}\right] \\ &\leqslant 1 - \alpha, \end{aligned}$$

by using Lemma 3 and the given condition. □

Using Lemma 8, Corollary 3, Lemma 3, and steps similar to the proof of Theorem 3, we obtain

Theorem 10. *If all the restrictions and coefficient condition in Theorem 3 are satisfied, then*

$$\Omega(TN_{\hat{H}}(\beta)) \subset HUS^*(k,\alpha).$$

Remark 4. When $g \equiv 0$ in $f = h + \bar{g}$, then $HUS^*(k,\alpha)$ reduces to the class $US^*(k,\alpha)$ which consists of k-uniformly analytic starlike functions of order α $(0 \leqslant \alpha < 1)$. Furthermore, for $g \equiv 0$ and $k = 0$, the class $HUS^*(k,\alpha)$ reduces to $S^*(\alpha)$ - the class of analytic starlike functions of order α [10].

Analogous to Theorem 6, we state the following result without proof.

Theorem 11. *If all the restrictions and coefficient condition in Theorem 6 are satisfied, then*

$$\Omega(THUS^*(k,\alpha)) \subset THUS^*(k,\alpha).$$

Remark 5. Using the same restrictions on complex parameters and similar arguments as in Theorem 7, we find that (7) is also a characterization for $\Omega(THUS^*(k,\alpha)) \subset THUS^*(k,\alpha)$.

Finally, we investigate a connection of $THUS^*(k,\alpha)$ with $HUC(k,\alpha)$.

Theorem 12. *Let $a_j, b_j \in \mathbb{C}\setminus\{0\}, c_j \in \mathbb{R}$, and $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$. If for some k $(0 \leqslant k < \infty)$ and α $(0 \leqslant \alpha < 1)$, the inequality*

$$\sum_{j=1}^{2}\left(\frac{|a_jb_j|}{c_j - |a_j| - |b_j| - 1} + 1\right)F(|a_j|,|b_j|;c_j;1) \leqslant 2$$

is satisfied, then

$$\Omega(THUS^*(k,\alpha)) \subset HUC(k,\alpha).$$

Proof. Using the definition of P_4 in Theorem 7 and Remark 1, it suffices to show that $P_4 \leqslant 1$. On the other hand, as an application of coefficient estimates in Remark 3, we obtain

$$\begin{aligned} P_4 &\leqslant \sum_{n=2}^{\infty} nD_{n-1} + \sum_{n=1}^{\infty} nE_{n-1} \\ &= \frac{|a_1b_1|}{c_1}F(|a_1|+1,|b_1|+1;c_1+1;1) + F(|a_1|,|b_1|;c_1;1) - 1 \\ &\quad + \frac{|a_2b_2|}{c_2}F(|a_2|+1,|b_2|+1;c_2+1;1) + F(|a_2|,|b_2|;c_2;1). \end{aligned}$$

The result follows as an application of Lemma 3(i) and by using the given hypothesis. □

References

[1] O.P. Ahuja, Planar harmonic univalent and related mappings, J. Ineq. Pure Appl. Math. 6 (4) (2005) 1–18. Article 122.
[2] O.P. Ahuja, Planar harmonic convolution operators generated by hypergeometric functions, Integral Transform. Spec. Funct 18 (3) (2007) 165–177.
[3] O.P. Ahuja, R. Aghalary, S.B. Joshi, Harmonic univalent functions associated with k-uniformly starlike functions, Math. Sci. Res. J. 9 (1) (2005) 9–17.
[4] O.P. Ahuja, J.M. Jahangiri, Noshiro-type harmonic univalent functions, Sci. Math. Jpn. 6 (2) (2002) 253–259.
[5] O.P. Ahuja, J.M. Jahangiri, A subclass of harmonic univalent functions, J. Nat. Geom. 20 (1–2) (2001) 45–56.
[6] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985) 137–152.
[7] B.C. Carleson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15 (1984) 737–745.
[8] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A.I. Math. 9 (1984) 3–25.
[9] P.L. Duren, in: Harmonic Mappings in the Plane, Cambridge Tracts in Mathematics, vol. 156, Cambridge University Press, Cambridge, 2004, ISBN 0-521-64121-7.
[10] A.W. Goodman, Univalent Functions, vol. 1, Marina Publishing Co., Inc., Tempa, FL, 1983, XVII+246 ISBN: 0-936155-10-X.
[11] A.W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56 (1) (1991) 87–92.
[12] Y.E. Hohlov, Convolution operators preserving univalent functions, Ukrainian Mat. J. 37 (1985) 220–226, Russian.
[13] Y.C. Kim, J.M. Jahangiri, J.H. Choi, Certain convex harmonic functions, Int. J. Math. Sci. 29 (8) (2002) 459–465.
[14] S. Miller, P.T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, Proc. Amer. Math. Soc. 110 (2) (1990) 333–342.
[15] S. Owa, H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. 39 (1987) 1057–1077.
[16] S. Punnusamy, F. Rønning, Starlikeness properties for convolutions involving hypergeometric series, Ann. Univ. Mariae Curie-Sklodowska L. II 1 (16) (1998) 141–155.
[17] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118 (1993) 189–196.
[18] T. Rosy, B.A. Stephen, K.G. Subramanian, J.M. Jahangiri, Goodman-Rønning-type harmonic univalent functions, Kyungpook Math. J. 41 (1) (2001) 45–54.

[19] S. Ruscheweyh, V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl. 113 (1986) 1–11.
[20] H. Silverman, Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl. 220 (1998) 283–289.
[21] H. Silverman, E.M. Silvia, Subclasses of harmonic univalent functions, New Zealand J. Math 28 (2) (1999) 275–284.
[22] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Ellis Horwood Limited and John Wiley & Sons, New York, Chichester, Toronto, 1984.
[23] X.-T. Wang, X.-Q. Liang, Y.L. Zhang, Precise coefficient estimates for close-to-convex harmonic univalent mappings, J. Math. Anal. Appl. 263 (2) (2001) 501–509.