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Classes of Functions Whose Derivatives Have Positive Real Part

O. P. AHUJA

Department of Mathematics, University of Papua New Guinea, Box 320, University P.O., Papua, New Guinea

AND

H. SILVERMAN

Department of Mathematics, College of Charleston, Charleston, South Carolina 29424

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We investigate functions of the form $f(z)=z-\sum_{n=2}^{\infty} a_n z^n$, $a_n \geqslant 0$, that are analytic in $|z|<1$ and satisfy there the inequality

$$\operatorname{Re}\{(1-\lambda)f'+\lambda(1+zf''/f')\}>0 \qquad \text{for some real } \lambda$$

This family, denoted by $M(\lambda)$, is shown for all real λ to contain only starlike functions whose derivatives have positive real parts. Relationships between $M(\lambda)$ and other classes of univalent functions as well as coefficient and distortion bounds are found for $\lambda \geqslant 0$. A characterization for $M(\lambda)$ when $0 \leqslant \lambda \leqslant 1$ is also given

1. INTRODUCTION

For $f(z)$ normalized by $f(0)=f'(0)-1=0$ and analytic with $f'(z)\neq 0$ in $\Delta=\{|z|<1\}$, set

$$I(\lambda, f(z))=(1-\lambda)f'(z)+\lambda(1+zf''(z)/f'(z)), \tag{1}$$

where λ is a real number. Denote by $H(\lambda)$ the family of functions satisfying $\operatorname{Re}\{I(\lambda, f(z))\}>0$ for λ fixed and all $z\in\Delta$. Al-Amiri and Reade [2] showed that f in $H(\lambda)$, $\lambda\leqslant 0$, satisfies $\operatorname{Re} f'>0$ for $z\in\Delta$. Thus by a criterion of Noshiro [5] and Warschawski [9], f in $H(\lambda)$, $\lambda\leqslant 0$, must be univalent in Δ. Al-Amiri and Reade were unable to settle the question of univalence for $\lambda>0$ except, of course, for $\lambda=1$ when f is convex. Even in this case, Re f'

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need not be positive in Δ. In fact, the convex function $f(z)=z/(1-z)$ is not in $H(\lambda)$ for any real λ, $\lambda \neq 1$. To see this, observe for $z=e^{i\theta} \neq 1$ that

$$\operatorname{Re}\{I(\lambda, f(z))\} = \operatorname{Re}\left\{\frac{1-\lambda}{(1-z)^2} + \lambda\left(\frac{1+z}{1-z}\right)\right\}$$
$$= (1-\lambda)\operatorname{Re}\{(1-z)^{-2}\} = ((\lambda-1)\cos\theta)/2(1-\cos\theta).$$

Choosing $\theta_0 = \pi/3$ when $\lambda < 1$ and $\theta_0 = 2\pi/3$ when $\lambda > 1$, we see that $\operatorname{Re}\{I(\lambda, f(e^{i\theta_0})\} < 0$ and consequently $\operatorname{Re}\{I(\lambda, f(re^{i\theta_0})\} < 0$ for r sufficiently close to 1. Thus, $H(1) \not\subset H(\lambda)$, $\lambda \neq 1$.

These properties of $H(\lambda)$ are in contradistinction to those of the more "orderly" family $L(\lambda)$ of λ-convex functions, whose definition differs from that of $H(\lambda)$ in that $f'(z)$ is replaced by $zf'(z)/f(z)$ in the first expression on the right side of (1). Note that $L(0)$ and $L(1) = H(1)$ are, respectively, the families consisting of starlike and convex functions. It was shown [3] that if $f \in L(\lambda)$, λ real, then f is starlike and $L(\lambda) \subset L(\beta)$ for $0 \leqslant \beta \leqslant \lambda$. For additional properties and subclasses of $L(\lambda)$ see [1, 4, 8].

It is our purpose here to investigate a subfamily of $H(\lambda)$ that behaves in as orderly a fashion as $L(\lambda)$. In particular, we shall let $M(\lambda)$ denote the subclass of $H(\lambda)$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geqslant 0. \tag{2}$$

For $0 \leqslant \alpha < 1$, we let $T^*(\alpha)$ represent the family of functions of the form (2) that are starlike of order α; that is, $f \in T^*(\alpha)$ if and only if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, with all $a_n \geqslant 0$, and $f(z)$ satisfies $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha$ for $z \in \Delta$. It is known [7] that $f \in T^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} (n-\alpha)a_n/(1-\alpha) \leqslant 1. \tag{3}$$

In this note we determine various inclusion properties for $M(\lambda)$. We also establish a coefficient characterization for functions in $M(\lambda)$, $0 \leqslant \lambda \leqslant 1$, which enables us to find relationships between $M(\lambda)$ and $T^*(\alpha)$ as well as coefficient bounds and distortion properties.

2. Coefficient Inequalities

In the sequel, we shall assume—unless stated otherwise—that f is of the form (2). Then with I defined by (1), $\operatorname{Re}\{I(\lambda, f(z))\} > 0$ if and only if

$$\operatorname{Re}\left\{(1-\lambda)\sum_{n=2}^{\infty} na_n z^{n-1} + \lambda\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right\} < 1. \tag{4}$$

Thus, $f \in M(\lambda)$ if and only if (4) holds for all z in Δ. When $\lambda = 0$, this condition is equivalent to $\sum_{n=2}^{\infty} na_n \leqslant 1$ and hence $M(0) = T^*(0)$.

We now show that all functions in $M(\lambda)$ are starlike.

THEOREM 1. *$M(\lambda) \subset M(0) = T^*(0)$, λ real.*

Proof. For f to be in $M(\lambda)$, we must have $f'(z) = 1 - \sum_{n=2}^{\infty} na_n z^{n-1} \neq 0$ $(z \in \Delta)$. But $f'(r)$ is real with $f'(0) > 0$, so that $f'(r) > 0$ for all r, $0 < r < 1$. Letting $r \to 1^-$, we see that $\sum_{n=2}^{\infty} na_n \leqslant 1$ and $f \in M(0)$.

COROLLARY. *If $f \in M(\lambda)$, λ real, then* $\operatorname{Re} f' > 0$ *for* $z \in \Delta$.

Proof. For $f \in M(\lambda)$, $\operatorname{Re} f'(z) \geqslant f'(r) > 0$.

We may now prove the following inclusion property.

THEOREM 2. $M(\lambda) \subset M(\beta)$ $(\lambda \leqslant \beta \leqslant 0$ or $0 \leqslant \beta \leqslant \lambda)$.

Proof. For $\gamma \neq 0$, $f \in M(\gamma)$ if and only if for all z in Δ,

$$\begin{aligned} &\operatorname{Re}\{1 + zf''/f'\} > ((\gamma - 1)/\gamma) \operatorname{Re} f' \qquad (\gamma > 0), \\ &\operatorname{Re}\{1 + zf''/f'\} < ((\gamma - 1)/\gamma) \operatorname{Re} f' \qquad (\gamma < 0). \end{aligned} \tag{5}$$

The result now follows from Theorem 1 and its corollary upon noting that $g(\gamma) = (\gamma - 1)/\gamma$ is an increasing function of γ on both intervals $(-\infty, 0)$ and $(0, \infty)$.

Remark. The inclusion property of Theorem 2 does not extend through 0. For instance, $f(z) = (z - z^2/2) \in M(0)$, but for $z = e^{i\theta} \neq 1$ and $\lambda < 0$ the left hand side of (4) becomes $\operatorname{Re}\{(1 - \lambda)z + \lambda z/(1 - z)\} = (1 - \lambda) \cos\theta - \lambda/2$, which is greater than 1 for $\cos\theta > (2 + \lambda)/2(1 - \lambda)$. Thus, $f \notin M(\lambda)$ for any $\lambda < 0$.

That $(z - z^2/2)$ is not in $M(\lambda)$ for $\lambda > 0$ is a special case of the following theorem, which characterizes the family $M(\lambda)$ for $0 \leqslant \lambda \leqslant 1$.

THEOREM 3. (i) *The condition $\sum_{n=2}^{\infty} na_n \leqslant 1$ is necessary and sufficient for f to be in $M(0)$. When equality holds, $f \notin M(\lambda)$ for any $\lambda > 0$.*

(ii) *If $\sum_{n=2}^{\infty} na_n < 1$, then a necessary condition for f to be in $M(\lambda)$, $\lambda \neq 0$, is that*

$$(1 - \lambda) \sum_{n=2}^{\infty} na_n + \lambda \frac{\sum_{n=2}^{\infty} n(n-1) a_n}{1 - \sum_{n=2}^{\infty} na_n} \leqslant 1. \tag{6}$$

The condition (6) is also sufficient for $0 < \lambda \leqslant 1$.

Proof. The first part of (i) follows from (3) with $\alpha = 0$. When equality holds, let $z = r \to 1^-$ in (4) and observe that the LHS gets arbitrarily large if $\lambda > 0$. To prove (ii), set the bracketed part in (4) equal to $J(\lambda, z)$ and note that (6) is clearly satisfied for $\lambda < 0$ if $\sum_{n=2}^{\infty} n^2 a_n = \infty$ and that $\sum_{n=2}^{\infty} n^2 a_n < \infty$ for $\lambda > 0$ if $J(\lambda, r) < 1$ for all $z = r < 1$. The necessity of (6) for f to be in $M(\lambda)$ now follows from the continuity of $J(\lambda, r)$ at $r = 1$. The sufficiency of (6) for f to be in $M(\lambda)$ when $0 < \lambda \leqslant 1$ is a consequence of the inequality $\operatorname{Re} J(\lambda, z) \leqslant J(\lambda, r)$ for $0 < \lambda \leqslant 1$.

COROLLARY. *If $f \in M(\lambda)$, $\lambda \geqslant 0$, then*

$$a_n \leqslant A_n = \frac{2}{n[\lambda(n-2) + 2 + \sqrt{\lambda^2(n-2)^2 + 4\lambda(n-1)}]}. \tag{7}$$

The result is sharp for every n, with equality for $f_n(z) = z - A_n z^n$.

Proof. In view of (6), we must have

$$(1-\lambda) n a_n + \lambda n(n-1) a_n/(1 - n a_n) \leqslant 1. \tag{8}$$

The result follows upon noting that the LHS of (8) is an increasing function of a_n and is equal to 1 when $a_n = A_n$. The lemma preceding the next theorem shows that the extremal functions f_n are in $M(\lambda)$.

3. INCLUSION RELATIONS

Denote by $P'(\beta)$ the family consisting of functions of the form (2) for which $\operatorname{Re} f' > \beta$ for all $z \in \Delta$. It follows from Theorem 1 that $M(\lambda) \subset P'(0)$ for all real λ. The next theorem gives a sharp lower bound on $\operatorname{Re} f'$ when $\lambda \geqslant 0$. But first we need the following:

LEMMA. *The function $f(z) = z - az^n/n$, $0 \leqslant a < 1$, is in $M(\lambda)$, $\lambda \geqslant 0$, if and only if $(1-\lambda)a + \lambda a(n-1)/(1-a) \leqslant 1$.*

Proof. The necessity follows from Theorem 3. To prove sufficiency, according to (4) we need only show for $z^{n-1} = e^{i\sigma}$, σ real, that $\operatorname{Re}\{(1-\lambda) a e^{i\sigma} + \lambda a(n-1) e^{i\sigma}/(1 - a e^{i\sigma})\} \leqslant (1-\lambda)a + \lambda a(n-1)/(1-a)$ or, equivalently, that

$$\lambda \left(\frac{n-2+a}{1-a} + \cos\sigma - \frac{(n-1)(\cos\sigma - a)}{1 + a^2 - 2a\cos\sigma} \right) + 1 - \cos\sigma \geqslant 0$$

for all σ. This follows if

$$h_n(\sigma) = \frac{n-2+a}{1-a} + \cos\sigma - \frac{(n-1)(\cos\sigma - a)}{1 + a^2 - 2a\cos\sigma} \geqslant 0.$$

But for $n \geqslant 2$, $h_n(\sigma) \geqslant h_2(\sigma) = h(\sigma)$ and $h'(\sigma) = \sin\sigma((1-a^2)/(1+a^2-2a\cos\sigma)^2 - 1)$ can vanish only if $\sin\sigma = 0$ or $\cos\sigma_0 = ((1+a^2) - \sqrt{1-a^2})/2a$. Since $h(\pi) = 2a^2/(1-a^2) \geqslant 0$ and

$$h(\sigma_0) = a\left(\frac{3-a}{2(1-a)} + \frac{1}{1+\sqrt{1+a^2}}\right) \geqslant 0,$$

we see that $h(\sigma) \geqslant h(0) = 0$ for all real σ. This completes the proof.

Letting $n = 2$ and $a = 1/(1+\sqrt{\lambda})$ in the lemma, we see that the extremal function of the next theorem is in $M(\lambda)$.

THEOREM 4. *If $f \in M(\lambda)$, $\lambda \geqslant 0$, then $f \in P'(\beta_0)$ for $\beta_0 = \beta_0(\lambda) = \sqrt{\lambda}/(1+\sqrt{\lambda})$. The result is sharp, with extremal function $f(z) = z - z^2/2(1+\sqrt{\lambda})$.*

Proof. The case $\lambda = 0$ is trivial, so we may assume $\lambda > 0$. In [6] it was shown that a necessary and sufficient condition for f to be in $P'(\beta)$ is that

$$\sum_{n=2}^{\infty} na_n \leqslant 1 - \beta. \tag{9}$$

We must, therefore, show that (9) holds with $\beta = \beta_0$ whenever inequality (6) is valid. If $\sum_{n=2}^{\infty} na_n > 1 - \beta_0 = 1/(1+\sqrt{\lambda})$, then the LHS of (6) is bounded below by

$$(1-\lambda)\sum_{n=2}^{\infty} na_n + \lambda\frac{\sum_{n=2}^{\infty} na_n}{1-\sum_{n=2}^{\infty} na_n} = \lambda\sum_{n=2}^{\infty} na_n\left(\frac{\sum_{n=2}^{\infty} na_n}{1-\sum_{n=2}^{\infty} na_n}\right) + \sum_{n=2}^{\infty} na_n$$

$$> \lambda(1-\beta_0)\left(\frac{1-\beta_0}{\beta_0}\right) + (1-\beta_0)$$

$$= \frac{\lambda}{1+\sqrt{\lambda}}\left(\frac{\sqrt{\lambda}}{\lambda}\right) + \frac{1}{1+\sqrt{\lambda}} = 1,$$

which means that inequality (6) does not hold. This completes the proof.

As seen in Theorems 1 and 2, $f \in M(\lambda)$ is starlike for all real λ and is convex for $\lambda \geqslant 1$. In the next theorem we determine the order of starlikeness when $\lambda \geqslant 0$ and the order of convexity when $\lambda \geqslant 1$.

THEOREM 5. *If $f \in M(\lambda)$, $\lambda \geqslant 0$, then f is starlike of order*

$$\alpha_0 = \alpha_0(\lambda) = 2\sqrt{\lambda}/(1+2\sqrt{\lambda}) \tag{10}$$

and is convex of order $1-1/\sqrt{\lambda}$ *when* $\lambda \geqslant 1$. *The results are sharp, with extremal function*

$$f(z)=z-z^2/2(1+\sqrt{\lambda})=z-(1-\alpha_0)z^2/(2-\alpha_0).$$

Proof. The case $\lambda=0$ is trivial, so we may assume $\lambda>0$. We will show that (3) holds with $\alpha=\alpha_0$ whenever inequality (6) is valid. We may rewrite (6) as

$$\lambda\left(\frac{\sum_{n=2}^{\infty} n(n-2)a_n+(\sum_{n=2}^{\infty} na_n)^2}{1-\sum_{n=2}^{\infty} na_n}\right)+\sum_{n=2}^{\infty} na_n \leqslant 1.$$

Setting $A=\sum_{n=2}^{\infty} na_n$, we will show that if

$$\sum_{n=2}^{\infty} (n-\alpha_0)a_n/(1-\alpha_0)>1 \tag{11}$$

then

$$\lambda\left(\frac{\sum_{n=2}^{\infty} n(n-2)a_n+A^2}{1-A}\right)+A>1. \tag{12}$$

Now the LHS of (12) will attain a minimum when $\sum_{n=2}^{\infty} n(n-2)a_n=0$, that is, when $a_n=0$ for $n \geqslant 3$. In this case $A=2a_2$ and, in view of (11), $2a_2>2(1-\alpha_0/(2-\alpha_0)=1/(1+\lambda)$. Thus

$$\frac{\lambda A^2}{1-A}+A>\frac{\lambda}{(1+\sqrt{\lambda})^2}\frac{(1+\lambda)}{\sqrt{\lambda}}+\frac{1}{1+\sqrt{\lambda}}=1,$$

and inequality (12) follows. Hence $f \in T^*(\alpha_0)$ whenever $f \in M(\lambda)$, $\lambda \geqslant 0$.

If $\lambda \geqslant 1$, then we combine (5) with Theorem 4 to obtain

$$\operatorname{Re}\{1+zf''/f'\}>((\lambda-1)/\lambda)(\sqrt{\lambda}/(1+\sqrt{\lambda}))=1-1/\sqrt{\lambda}$$

for all z in Δ. This completes the proof.

Remark. In [7] it was shown that if f is convex of order α, then $f \in T^*(2/(3-\alpha))$. Since functions in $M(\lambda)$, $\lambda \geqslant 1$, are convex of order $1-1/\sqrt{\lambda}$, the order of starlikeness for functions in $M(\lambda)$, $\lambda \geqslant 1$, obtained in Theorem 5 may also be seen to be a consequence of the result in [7] with $\alpha=1-1/\sqrt{\lambda}$.

There is no converse to Theorems 4 or 5 in the following sense.

THEOREM 6. *For any* α, $\lambda(0 \leqslant \alpha<1,\ \lambda>0)$, $P'(\alpha) \not\subset M(\lambda)$ *and* $T^*(\alpha) \not\subset M(\lambda)$.

Proof. We will construct functions to show that inclusion does not hold for any fixed α. Set $f_n(z) = z - z^n/n^{3/2}$ and $g_n(z) = z - (1-\alpha)z^n/(n-\alpha)$. In view of (9) and (3), respectively, $f_n \in P'(\alpha)$ for $n > (1-\alpha)^{-2}$ and $g_n \in T^*(\alpha)$ for all n. However, in each case for $\lambda > 0$ the LHS of (6) gets arbitrarily large as $n \to \infty$.

4. Distortion Properties

We will make use of the following result.

LEMMA 7. *If $f \in T^*(\alpha)$ and $|z| \leqslant r < 1$, then*

$$r - (1-\alpha)r^2/(2-\alpha) \leqslant |f| \leqslant r + (1-\alpha)r^2/(2-\alpha)$$

and

$$1 - 2(1-\alpha)r/(2-\alpha) \leqslant |f'| \leqslant 1 + 2(1-\alpha)r/(2-\alpha).$$

Equality holds in all cases for $f(z) = z - (1-\alpha)z^2/(2-\alpha)$, $z = \pm r$.

THEOREM 7. *If $f \in M(\lambda)$, $\lambda \geqslant 0$, and $|z| \leqslant r < 1$, then*

$$r - r^2/2(1+\sqrt{\lambda}) \leqslant |f| \leqslant r + r^2/2(1+\sqrt{\lambda})$$

and

$$1 - r/(1+\sqrt{\lambda}) \leqslant |f'| \leqslant 1 + r/(1+\sqrt{\lambda}).$$

Equality holds in all cases for $f(z) = z - z^2/2(1+\sqrt{\lambda})$, $z = \pm r$.

Proof. By Theorem 5, $M(\lambda) \subset T^*(\alpha_0)$ where α_0 is defined by (10). Noting that the extremal function for $T^*(\alpha_0)$ is an element of $M(\lambda)$, the result follows from the lemma.

Letting $r \to 1^-$ in Theorem 7, we obtain the following covering result.

COROLLARY. *The disk $|z| < 1$ is mapped onto a domain that contains the disk $|w| < (1 + 2\sqrt{\lambda})/2(1+\sqrt{\lambda})$ by any $f \in M(\lambda)$, $\lambda \geqslant 0$.*

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