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ON THE SPIRAL-LIKENESS OF RATIONAL FUNCTIONS

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The object of this paper is to extend some results concerning the univalence, starlikeness, and convexity of rational functions recently obtained by Reade, Silverman, and Todorov. The domain of variability of $\log\{f(z)/z\}$ for a fixed z and for such functions f ranging over the class of λ-spirallike functions of order α are also determined.

1. Introduction.

A function f is called λ-spirallike of order α if f is analytic in the unit disk $\Delta = \{z : |z| < 1\}$, $f(0) = 0$, $f'(0) = 1$ and the inequality

$$\operatorname{Re}\{e^{i\lambda} z f'(z)/f(z)\} > \alpha \cos\lambda \tag{1.1}$$

holds for some real $\alpha, \lambda (0 \leq \alpha < 1, |\lambda| \leq \pi/2)$ and for all z in Δ. This family of functions, which we denote by $S^\lambda(\alpha)$, was first studied by Libera in [3]. We note that $\alpha < 1$ and we will only be interested in the case $\alpha \geq 0$. If $\alpha \geq 0$, then $S^\lambda(\alpha) \subset S^\lambda(0) = S^\lambda$, the class of λ-spirallike functions. It was shown by Spacek [7] that such functions are univalent in Δ. It is clear that $|\lambda| = \pi/2$ only for the identity function $f(z) = z$; and hence we exclude that case from further consideration. Associated with $S^\lambda(\alpha)$ is the family $C^\lambda(\alpha)$ of analytic functions f for which $zf'(z)$ belong to to $S^\lambda(\alpha)$. More precisely, f belongs to $C^\lambda(\alpha)$ if and only if f is analytic in Δ, $f(0) = 0$, $f'(0) = 1$ and

$$\operatorname{Re}\{e^{i\lambda}(1 + zf''(z)/f'(z))\} > \alpha \cos\lambda \tag{1.2}$$

for all z in Δ and with some α, λ as above. This class was studied by Chichra [2]. The family $C^\lambda(\alpha)$ was introduced by Robertson in [6]. A subclass of these

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functions has been studied by the first author in [1]. A function f in $C^\lambda(\alpha)$ is called a λ-Robertson function of order α.

We observe that $S^0(\alpha) = S^*(\alpha)$, the class of starlike functions of order α; and $C^0(\alpha) = K(\alpha)$, the class of convex functions of order α. These subclasses of the class of all analytic and univalent functions in Δ are well known in the literature of theory of univalent functions.

In [4], Mitrinovič established sufficient conditions for functions of the form

$$z/(1 + b_1 z + \ldots + b_n z^n), b_n \neq 0$$

to be univalent in Δ and obtained the radius of univalence of the rational function of the form

$$z/(1 + z^n)^k; n, k = 1, 2 \ldots, nk - 1 > 0. \tag{1.4}$$

Reade, Silverman and Todorov [5] have observed that Mitrinovič criterion for univalence of functions of the form (1.3) does not ensure starlikeness. These authors have extended Mitrinovič results by providing sufficient conditions for starlikeness and convexity for functions of the form

$$f(z) = z/(1 + \sum_{k=1}^{\infty} b_k z^k). \tag{1.5}$$

In this paper, we determine the conditions on the coefficients $\{b_k\}$ for functions of the form (1.5) that will guarantee (i) λ-spirallikeness of order α and (ii) λ-Robertson of order α in the unit disk Δ. We next describe the domain of variability of $\log\{f(z)/z\}$ for a fixed z and for function f of the form (1.5) which range over the class of λ-spirallike functions of order α. Finally, the radii of spirallikeness of order α and λ-Robertson of order α for the functions of the form (1.4) are determined.

2. The spiral-likeness of functions.

We first find a condition on the coefficients $\{b_k\}$ that will guarantee the spiral-likeness of a class of functions of the form (1.5).

THEOREM 1. *Let $f(z) = z/(1 + \sum_{k=1}^{\infty} b_k z^k)$, and let α, λ be constants, $0 \le \alpha < 1$, $-\pi/2 < \lambda < \pi/2$. If the coefficientis $\{b_k\}$ satisfy*

$$\sum_{k=1}^{\infty} [k + \{k^2 - 4(1 - \alpha)(k + \alpha - 1)\cos^2\lambda\}^{\frac{1}{2}}]|b_k| \le 2(1 - \alpha)\cos\lambda, \tag{2.1}$$

then f is λ-spirallike of order α in the unit disk Δ.

We must show that (1.1) is satisfied in Δ or, equivalently, that for $p(z) = zf'(z)/f(z)$ we have

$$\left|\frac{p(z)-1}{p(z)-1+2(1-\alpha)e^{-i\lambda}\cos\lambda}\right| < 1 \tag{2.2}$$

in Δ. Since

$$p(z)-1 = -\left(\sum_{k=1}^{\infty} kb_k z^k\right) \Big/ \left(1+\sum_{k=1}^{\infty} b_k z^k\right),$$

the inequality (2.2) is equivalent to

$$\left|\frac{\sum_{k=1}^{\infty} kb_k z^k}{2(1-\alpha)^{-i\lambda}\cos\lambda - \sum_{k=1}^{\infty}(k-2(1-\alpha)e^{-i\lambda}\cos\lambda)b_k z^k}\right| < 1. \tag{2.3}$$

Noting that the left side of inequality (2.3) is bounded above by the expression

$$\frac{\sum_{k=1}^{\infty} k|b_k|}{2(1-\alpha)\cos\lambda - \sum_{k=1}^{\infty}|k-2(1-\alpha)e^{-i\lambda}\cos\lambda||b_k|}$$

in Δ, it follows that

$$\sum_{k=1}^{\infty} k|b_k| \leq 2(1-\alpha)\cos\lambda - \sum_{k=1}^{\infty}|k-2(1-\alpha)e^{-i\lambda}\cos\lambda||b_k|$$

which is equivalent to the coefficient inequality (2.1). This completes the proof of the theorem.

The following example illustrates that the converse of Theorem 1 may not hold.

EXAMPLE. Consider the functions

$$f(z) = z/(1+e^{i\beta}z), \beta \quad \text{real.}$$

These function satisfy (2.1) if and only if $\lambda = 0$. If $\lambda \neq 0$, then these functions are λ-spirallike in the disk $|z| < \cos\lambda$ because

$$\text{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}_{z=re^{i\theta}} = \frac{\cos\lambda + r\cos(\beta+\theta-\lambda)}{1+r^2+2r\cos(\beta+\theta)} \geq \frac{\cos\lambda - r}{(1+r)^2} > 0$$

if $\cos\lambda > r$.

Setting $\lambda = 0$ in Theorem 1, we have the following.

COROLLARY *The function* $f(z) = z/\left(1+\sum_{k=1}^{\infty} b_k z^k\right)$ *is starlike of order* α *in the disk* Δ *if*

$$\sum_{k=2}^{\infty}(k-1+\alpha)|b_k| \le \begin{cases}(1-\alpha)-(1-\alpha)|b_1|, 0 \le \alpha \le \frac{1}{2}, \\ (1-\alpha)-\alpha|b_1|, \frac{1}{2} \le \alpha < 1.\end{cases} \tag{2.5}$$

The result in the above Corollary is established in [5].

Let $G^\lambda(\alpha)$ be the set of functions of the form $f(z) = z/(1+\sum_{k=1}^{\infty} b_k z^k)$ that satisfy the condition (1.1) for some $\alpha, \lambda (0 \le \alpha < 1, -\pi/2 < \lambda < \pi/2)$ and for all z in Δ. Then $G^\lambda(\alpha) \subset S^\lambda(\alpha)$. Letting $f(z) = z/g(z)$ where $g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$, we have

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = e^{i\lambda} - e^{i\lambda}\frac{zg'(z)}{g(z)}.$$

The introduction of the appropriate normalizing factors enable us to write

$$\frac{(1-\alpha)\cos\lambda - e^{i\lambda}zg'(z)/g(z)}{(1-\alpha)\cos\lambda} = 1 + \sum_{n=1}^{\infty} p_n z^n \equiv p(z)$$

where $\mathrm{Re}\,p(z) > 0$ in Δ. Hence it follows that f belongs to $G^\lambda(\alpha)$ if and only if

$$\log(f(z)/z) = (1-\alpha)e^{-i\lambda}\cos\lambda \int_0^z \{(p(\xi)-1)/\xi\}d\xi \tag{2.6}$$

in Δ. This formula enables us to obtain the convex hull of the image of $\log(f(z)/z)$, for a fixed z, $|z| \le r < 1$ when $f \in G^\lambda(\alpha)$.

THEOREM 2. *The region of variability of* $\log(f(z)/z)$ *for a fixed* z, $|z| \le r < 1$, *and* f *ranging over the class* $G^\lambda(\alpha)$ *lies in the image of* $|z| \le r$ *under the mapping*

$$\omega = (1-\alpha)e^{-i\lambda}\cos\lambda \cdot \log(1-\theta z)^{-2}, \quad |\theta| = 1. \tag{2.7}$$

Proof. It is well known that every function $p(z)$ with the above properties has a Harglotz's representation.

(2.8) $$p(\xi) = \int_{-\pi}^{\pi} \frac{1+\xi e^{it}}{1-\xi e^{it}} d\mu(t),$$

where $\mu(t)$ is real-valued nondecreasing function defined for

$$-\pi \leq t \leq \pi \quad \text{with} \quad \int_{-\pi}^{\pi} d\mu(t) = 1.$$

Combining (2.6) and (2.8), the structural formula for $G^{\lambda}(\alpha)$ becomes

(2.9) $$\log(f(z)/z) = -2(1-\alpha)e^{-i\lambda}\cos\lambda \int_{-\pi}^{\pi} \log(1-ze^{it})d\mu(t).$$

Let $H(z) = \int_0^z \{(p(\xi)-1)/\xi\}d\xi$ and $K(z,t) = \log(1-ze^{it})^{-2}$.
Then (2.8) yields

(2.10) $$H(z) = \int_{-\pi}^{\pi} K(z,t)d\mu(t).$$

Therefore, (2.9) can be written as

(2.11) $$\log(f(z)/z) = (1-\alpha)e^{-i\lambda}\cos\lambda\, H(z).$$

Note that the kernal $K(z,t)$ is analytic for z in Δ and continuous in t in $[-\pi,\pi]$.

From (2.10) we know that for a fixed z, $|z| \leq r < 1$, the region of variability for the set of points $H(z)$ is the convex hull of the curve $C : \omega_1 = K(z,t)$, $-\pi \leq t \leq \pi$. Further, the function $\omega_1 = K(z,t)$ maps Δ univalently onto a convex domain which is independent of t because

$$\text{Re}\left\{1 + \frac{zK''(z,t)}{K'(z,t)}\right\} = \text{Re}\left\{\frac{1}{1-ze^{it}}\right\} > \frac{1}{2}.$$

In view of (2.11), it now follows that the set of all points $\log(f(z)/z)$, for a fixed $z(|z| \leq r < 1)$, lie in the image $|z| \leq r$ under the mapping (2.7). This completes the proof of theorem.

Remark.

(i) Note that the function $\log(1-z)^{-2}$ maps $|z| \leq r < 1$ onto a convex domain which is symmetric with respect to the real axis. Therefore, by Theorem 2, the region of variability of $\log(f(z)/z)$ is a convex domain which is symmetric with respect to the line $\arg\omega = -\lambda$.

(ii) The function

$$g(z) = z/(1-z)^{2(1-\alpha)e^{-i\lambda}\cos\lambda}$$

belongs to $G^{\lambda}(\alpha)$ and thus it shows that the result in Theorem 2 is sharp in the sense that the boundary points are attained by the function g.

(iii) Fixing $\lambda = 0$, Theorem 2 will describe the region of variability of $\log(f(z)/z)$ for a fixed $z(|z| \leq r < 1)$, when f of the form (1.5) is a starlike function of order λ

3. A sufficient condition for λ-Robertson functions.

Our next result provides a sufficient condition for a function f of the form. (1.5) to be λ-Robertson function of order α.

THEOREM 3. *Let* $f(z) = z/(1+\sum_{k=1}^{\infty} b_k z^k)$, *let* α, λ *be costants*, $0 \leq \alpha < 1$, $-\pi/2 < \lambda < \pi/2$. *If the coefficients* $\{b_k\}$ *satisfy*

$$\frac{3+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}|b_1| + \sum_{k=2}^{\infty}\frac{(k-1)(3k+(1-\alpha)\cos\lambda)}{(1-\alpha)\cos\lambda}|b_k| \leq 1, \tag{3.1}$$

then f *is a* λ*-Robertson function of order* α *in* Δ.

Proof. An elementary computation yields

$$\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right) - \alpha\cos\lambda\right\} =$$

$$(1-\alpha)\cos\lambda - \operatorname{Re}\left\{\frac{e^{i\lambda}\sum_{k=1}^{\infty}2kb_k z^k}{1+\sum_{k=1}^{\infty}b_k z^k}\right\} - \operatorname{Re}\left\{\frac{e^{i\lambda}\sum_{k=2}^{\infty}k(k-1)b_k z^k}{1-\sum_{k=2}^{\infty}(k-1)b_k z^k}\right\}.$$

Now $\operatorname{Re}\{e^{i\lambda}(1 + zf''(z)/f'(z)) - \alpha\cos\lambda\} \geq 0$ in λ when both the inequalities

$$\left|\frac{\sum_{k=1}^{\infty} e^{i\lambda}2kb_k z^k}{1+\sum_{k=1}^{\infty} b_k z^k}\right| \leq \frac{\sum_{k=1}^{\infty} 2k|b_k|}{1-\sum_{k=1}^{\infty}|b_k|} \leq a$$

and

$$\left|\frac{e^{i\lambda}\sum_{k=2}^{\infty} k(k-1)b_k z^k}{1-\sum_{k=2}^{\infty}(k-1)b_k z^k}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)|b_k|}{1-\sum_{k=2}^{\infty}(k-1)|b_k|} \leq b$$

hold for some a, b, where $a + b = (1-\alpha)\cos\lambda$ and $0 < a < (1-\alpha)\cos\lambda$. Therefore, f is a λ-Robertson function of order α if the inequalities

$$\sum_{k=1}^{\infty}\left(\frac{2k+1}{a}\right)|b_k| \leq 1 \tag{3.2}$$

and

$$\sum_{k=2}^{\infty}\frac{(k-1)\{k+(1-\alpha)\cos\lambda - a\}}{(1-\alpha)\cos\lambda - a}|b_k| \leq 1 \tag{3.3}$$

both hold for some $a, 0 < a < (1-\alpha)\cos\lambda$. The smallest value of a for which the condition

$$\frac{2k+a}{a} \leq \frac{(k-1)\{k+(1-\alpha)\cos\lambda - a\}}{(1-\alpha)\cos\lambda - a}$$

holds for all $k \geq 2$ is $a = (2/3)(1-\alpha)\cos\lambda$. Setting this value of a, the inequalities in (3.2) and (3.3), respectively, reduce to

$$\frac{3+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}|b_1| + \sum_{k=2}^{\infty}\frac{3k+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}|b_k| \leq 1 \tag{3.4}$$

and

$$\sum_{k=2}^{\infty}\frac{(k-1)\{3k+(1-\alpha)\cos\lambda\}}{(1-\alpha)\cos\lambda}|b_k| \leq 1. \tag{3.5}$$

The last two inequalities hold whenever (3.1) is satisfied. This completes the proof of the theorem.

Setting $\lambda = 0$ in Theorem 3, we have

COROLLARY *The function* $f(z) = z/(1+\sum_{k=1}^{\infty} b_k z^k)$ *is convex of order* α *in* Δ *when*

$$\frac{4-\alpha}{1-\alpha}|b_1| + \sum_{k=2}^{\infty} \frac{(k-1)(3k+1-\alpha)}{1-\alpha}|b_k| \leq 1. \tag{3.6}$$

Remark. The case $\lambda = 0 = \alpha$ was established in [5].

4. The radius problems of $K(z) = z/(1+z^n)^k$.

In this section, we shall determine the radii of λ-spirallike of order α and λ-Robertson of order α of the function $K(z) = z/(1+z^n)^k$.

THEOREM 4. *Let* $K(z) = z/(1+z^n)^k$, *where* n, k *are a pair of fixed positive integers and* $nk - 1 > 0$. *Then* $K(z)$ *is* λ*-spirallike of order* α *for* $|z| < r_\lambda(\alpha)$, *where*

$$r_\lambda(\alpha) = \left[\frac{2(1-\alpha)\cos\lambda}{(nk-2+2\alpha)\cos\lambda + nk\sin\lambda + \{n^2k^2 + nk(nk-2+2\alpha)\sin 2\lambda\}^{\frac{1}{2}}}\right]^{1/n}.$$

This result is sharp.

Proof. At $z = re^{i\theta}$, we have

$$\operatorname{Re}\left\{e^{i\lambda}(1 + \frac{zK'(z)}{K(z)}\right\} = (1 - nk/2)\cos\lambda + \frac{nk}{2}\left\{\frac{(1-r^{2n})\cos\lambda + 2r^n \sin n\theta \sin\lambda}{1 + r^{2n} + 2r^n \cos n\theta}\right\} \geq$$

$$(1 - nk/2)\cos\lambda + \frac{nk}{2}\left\{\frac{(1-r^{2n})\cos\lambda - 2r^n \sin\lambda}{(1+r^n)^2}\right\}.$$

Hence f is λ-spirallike function of order α in the disk $|z| < r_\lambda(\alpha)$ if $r_\lambda(\alpha)$ is the smallest positive root of the equation

$$(nk - 1 + \alpha)\cos\lambda \cdot r^{2n} + \{(nk-2+2\alpha)\cos\lambda + nk\sin\lambda\}r^n - (1-\alpha)\cos\lambda = 0.$$

The result in the theorem is sharp for $z = r_\lambda(\alpha)e^{i2\pi p/n}$, $p = 0, 1, 2, \ldots, n-1$ when $\lambda = 0$ and for $z = r_\lambda(\alpha)e^{i\theta}$ when $\lambda \neq 0$, where θ is a solution of the equation

$$\frac{(1-r^{2n})cos\lambda + 2r^n \sin n\theta \sin\lambda}{1 + r^{2n} + 2r^n \cos n\theta} = \frac{(1-r^{2n})\cos\lambda - 2r^n \sin\lambda}{(1+r^n)^2}.$$

Remark. Theorem 4 immediately yields

$$r_0(\alpha) = \left[\frac{1-\alpha}{nk-1+\alpha}\right]^{1/n},$$

the radius of starlikeness of order α found in [5]; and $r_0(0) = 1/(nk-1)^{1/n}$, the radius of univalence found by Mitrinovič [4].

THEOREM 5. *Let $K(z) = z/(1+z^n)^k$, where n, k are a pair of fixed positive integers and $nk-1>0$. Then $K(z)$ is λ-Robertson function of order α for $|z|<R$, where R is the unique positive root of the fourth degree equation*

$$Ar^{4n} + Br^{3n} + Cr^{2n} + Dr^n - 2(1-\alpha)\cos\lambda = 0, \tag{4.1}$$

where

$A = 2(nk-1)^2(nk-1+\alpha)\cos\lambda,$

$B = 2(nk-1)[\cos\lambda\{nk(nk-n+2\alpha-5)+4(1-\alpha)\}+n\sin\lambda\{(k+1)(nk-1)+1\}],$

$C = 2\cos\lambda[(nk-1)^2(-n+\alpha-1)+(nk-1)(-2nk+5-4\alpha)+(n+\alpha)]-4nk(nk-1)\sin\lambda,$

$D = 2\cos\lambda[(nk-1)(n+2-2\alpha)+n(k+1)-2(1-\alpha)]+2n(n+1)k\sin\lambda.$

Proof. At $z = re^{i\theta}$, we have

$$\mathrm{Re}\left\{e^{i\lambda}(1+\frac{zK''(z)}{K'(z)}\right\} =$$

$$= \left(\frac{2-nk}{2}\right)\cos\lambda + \frac{n(k+1)}{2}\left\{\frac{(1-r^{2n})\cos\lambda + 2r^n\sin\lambda\sin n\theta}{1+r^{2n}+2r^n\cos n\theta}\right\} -$$

$$-\frac{n}{2}\left\{\frac{\cos\lambda(1-(nk-1)^2r^{2n}) - 2(nk-1)r^n\sin\lambda\sin n\theta}{1+(nk-1)^2r^{2n}-2(nk-1)r^n\cos n\theta}\right\} \geq$$

$$\geq \left(\frac{2-nk}{2}\right)\cos\lambda + \frac{n(k+1)}{2}\left\{\frac{(1-r^{2n})\cos\lambda - 2r^n\sin\lambda}{(1+r^n)^2}\right\} -$$

$$-\frac{n}{2}\left\{\frac{\cos\lambda(1-(nk-1)^2r^{2n}) + 2(nk-1)\sin\lambda\cdot r^n}{1+(nk-1)^2r^{2n}-2(nk-1)r^n}\right\}.$$

Thus, after some lengthy computations, it follows that

$$\mathrm{Re}\left\{e^{i\lambda}(1+\frac{zK''(z)}{K'(z)}\right\} > \alpha\cos\lambda$$

for $|z|<R$, where R is the smallest positive root in $(0,1)$ of the equation (4.1).

Remarks.

(i) When $\lambda = 0$, the result in Theorem 5 is sharp for $z = R\exp(i2p\pi/n)$, $p = 0,1,2,\ldots\ldots,n-1$. When $\lambda \neq 0$, we are not able to determine the extremal function.

(ii) For $\lambda = 0$, we obtain radius of convexity of order α found in [5].

Concluding Comments.

It would be of interest to obtain sufficient as well as necessary conditions in terms of the coefficients $\{b_k\}$ for $f(z)$ of the form $z/(1+\sum_{k=1}^{\infty} b_k z^k)$ to be (i) λ-spirallike of order α and (ii) λ-Robertson of order α of the unit disk Δ.

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