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Multiplicative Relations in Powers of Euler's Product

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In a recent paper, Cooper, Hirschhorn, and Lewis conjecture many relations among the coefficients of certain products of powers of Euler's product. Here we use the theory of modular forms with complex multiplication to prove these conjectures. © 2001 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

Define the infinite product $(q)_\infty := \prod_{n=1}^{\infty} (1-q^n)$. In a recent paper [C–H–L], Cooper, Hirschhorn, and Lewis study identities between the coefficients of products of powers of this product. At the end of their paper, they formulate many conjectures (see Conjectures 1–3 below). The goal in this paper is to prove

THEOREM 1. *Conjectures* 1–3 *are true.*

In Theorems 1–4 of [C–H–L] the authors prove many relations similar to those described in the conjectures. In their work, they make heavy use of the Macdonald identities [M]. Our approach is quite different; we notice that each conjectured identity is equivalent to the fact that a certain modular form is annihilated by a family of Hecke operators. To prove the latter fact, we show that the relevant modular forms are in fact linear combinations of Hecke forms, and so are necessarily annihilated by every Hecke operator $T(p)$ for which p is inert in all of the relevant quadratic fields. It should be noted that all of the original theorems in [C–H–L], as well as many other similar identities, can be proved in this fashion. We now state the conjectures.

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Conjecture 1. Suppose that r and s are integers and that p is prime. Let $(q)^r_\infty (q^2)^s_\infty = \sum_{n=0}^{\infty} a(n)\, q^n$. Then the coefficients $a(n)$ satisfy

$$a\left(pn + \frac{(r+2s)(p^2-1)}{24}\right) = \varepsilon p^{(r+s)/2-1}\, a\left(\frac{n}{p}\right)$$

for the following values of r, s, p, and ε.

	(r, s)	p	ε
1.	(9, 9)	$p \equiv 7 \pmod 8$	1
2.	(7, 3), (3, 7)	$p \equiv 7, 11 \pmod{12}$	$\begin{cases} 1 \text{ if } p \equiv 7 \pmod 8 \\ -1 \text{ if } p \equiv 3 \pmod 8 \end{cases}$
3.	(7, 1), (1, 7)	$p = 31$	-1
4.	(3, -1), (-1, 3)	$p \equiv 7, 11 \pmod{24}$	$\begin{cases} 1 \text{ if } p \equiv 7 \pmod{24} \\ -1 \text{ if } p \equiv 11 \pmod{24} \end{cases}$
5.	(7, -1), (-1, 7)	$p \equiv 7, 11 \pmod{12}$	$\begin{cases} 1 \text{ if } p \equiv 7 \pmod 8 \\ -1 \text{ if } p \equiv 3 \pmod 8 \end{cases}$
6.	(13, -5), (-5, 13)	$p = 31$	-1
7.	(15, -5), (-5, 15)	$p \equiv 19, 23 \pmod{24}$	$\begin{cases} 1 \text{ if } p \equiv 23 \pmod{24} \\ -1 \text{ if } p \equiv 19 \pmod{24} \end{cases}$
8.	(16, -6), (-6, 16)	$p \equiv 11 \pmod{12}$	1
9.	(17, -7), (-7, 17)	$p \equiv 7 \pmod 8$	1
10.	(18, -8), (-8, 18)	$p \equiv 11 \pmod{12}$	1
11.	(19, -9), (-9, 19)	$p \equiv 19, 23 \pmod{24}$	$\begin{cases} 1 \text{ if } p \equiv 23 \pmod{24} \\ -1 \text{ if } p \equiv 19 \pmod{24} \end{cases}$

Conjecture 2. Suppose that r and s are integers and that p is prime. Let $(q)^r_\infty (q^3)^s_\infty = \sum_{n=0}^{\infty} a(n)\, q^n$. Then the coefficients $a(n)$ satisfy

$$a\left(pn + \frac{(r+3s)(p^2-1)}{24}\right) = \varepsilon p^{(r+s)/2-1}\, a\left(\frac{n}{p}\right)$$

for the following values of r, s, p, and ε:

	(r, s)	p	ε
1.	(2, 2)	$p \equiv 5 \pmod 6$	-1
2.	(5, 3), (3, 5)	$p \equiv 5 \pmod 6$	$\begin{cases} 1 \text{ if } p \equiv 5 \pmod{12} \\ -1 \text{ if } p \equiv 11 \pmod{12} \end{cases}$
3.	(3, -1), (-1, 3)	$p = 5 \pmod 6$	1
4.	(5, -1), (-1, 5)	$p \equiv 5 \pmod 6$	$\begin{cases} 1 \text{ if } p \equiv 5 \pmod{12} \\ -1 \text{ if } p \equiv 11 \pmod{12} \end{cases}$
5.	(9, -1), (-1, 9)	$p \equiv 11 \pmod{12}$	-1
6.	(10, -2), (-2, 10)	$p = 5 \pmod 6$	-1
7.	(11, -3), (-3, 11)	$p \equiv 11 \pmod{12}$	-1

Conjecture 3. Suppose that r and s are integers and that p is prime. Let $(q)^r_\infty\,(q^4)^s_\infty=\sum_{n=0}^\infty a(n)\,q^n$. Then the coefficients $a(n)$ satisfy

$$a\left(pn+\frac{(r+4s)(p^2-1)}{24}\right)=\varepsilon p^{(r+s)/2-1}\,a\left(\frac{n}{p}\right)$$

for the following values of r, s, p, and ε:

	(r, s)	p	ε
1.	$(5, 5)$	$p\equiv 19, 23 \pmod{24}$	1
2.	$(7, -1), (-1, 7)$	$p=7 \pmod 8$	1

In the next section, we recall some basic facts from the theory of modular forms. In the following section, we recast the conjectures in this new language, and in the last section we prove them.

2. FACTS ON MODULAR FORMS

If k and N are positive integers and χ is a Dirichlet character modulo N, then we denote by $M_k(\Gamma_0(N),\chi)$ (resp. $S_k(\Gamma_0(N),\chi)$) the usual space of modular forms (resp. cusp forms) of weight k and Nebentypus character χ on the congruence subgroup $\Gamma_0(N)$(see, for example, [K] for full definitions). If $f(z)\in M_k(\Gamma_0(N),\chi)$, then f has a Fourier expansion $f(z)=\sum_{n=0}^\infty a(n)\,q^n$, where, as always, $q:=e^{2\pi iz}$. If p is prime, then define the usual Hecke operator $T(p)$: $M_k(\Gamma_0(N),\chi)\to M_k(\Gamma_0(N),\chi)$; if $f(z)=\sum_{n=0}^\infty a(n)\,q^n$, then we have

$$f(z)\mid T(p)=\sum_{n=0}^\infty\left(a(pn)+\chi(p)\,p^{k-1}a\left(\frac{n}{p}\right)\right)q^n. \tag{2.1}$$

Now recall the definition of Dedekind's eta-function,

$$\eta(z):=q^{1/24}\prod_{n=1}^\infty(1-q^n).$$

The conjectured identities can all be written in terms of eta-quotients; that is, expressions of the form

$$f(z)=\prod_{\delta\mid N}\eta^{r_\delta}(\delta z), \tag{2.2}$$

where $r_\delta\in\mathbb{Z}$. We recall the following theorem regarding modular properties of such quotients (see, for example, [G–H]).

THEOREM 2. *Suppose that* $f(z)$ *is an eta-quotient as in* (2.2). *Suppose that* $k := \frac{1}{2}\sum_{\delta \mid N} r_\delta$ *is a positive integer, and that*

$$\sum_{\delta \mid N} \delta r_\delta \equiv \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Suppose further that for all positive divisors μ *of* N *we have*

$$\sum_{\delta \mid N} \frac{(\mu, \delta)^2 r_\delta}{\delta} \geqslant 0. \tag{2.3}$$

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, *where* χ *is defined by*

$$\chi(d) := \left(\frac{(-1)^k s}{d}\right), \qquad s := \prod_{\delta \mid N} \delta^{r_\delta}.$$

Further, if the sums in (2.3) *are all positive, then* f *is a cusp form.*

Finally, we recall some facts regarding Hecke character forms (see [S, Sect. 1.2; R, Sect. 3]). Let K be an imaginary quadratic field of discriminant D, and let $\mathcal{O}_K$ be its ring of integers. Let $k \geqslant 1$ be an integer, let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_K$, and let c be a Hecke character (i.e. Grössencharacter) of exponent $k-1$ which is defined modulo $\mathfrak{m}$. In other words, c is a homomorphism $I(\mathfrak{m}) \to \mathbb{C}^\times$ (where $I(\mathfrak{m})$ is the group of fractional ideals of K prime to $\mathfrak{m}$) which satisfies

$$c(\alpha\mathcal{O}_K) = \alpha^{k-1} \quad \text{if } \alpha \in K^\times \qquad \text{and} \qquad \alpha \equiv 1 \bmod^\times \mathfrak{m}.$$

A Hecke character c defines a Dirichlet character ε_c by

$$\varepsilon_c(d) := \frac{c(d\mathcal{O}_K)}{d^{k-1}} \qquad \text{for} \quad d \in \mathbb{Z} \text{ prime to } \mathfrak{m}.$$

We then define the Hecke form

$$\varphi_{K,c}(z) := \sum_{\substack{(\mathfrak{a}, \mathfrak{m}) = 1 \\ \mathfrak{a} \text{ integral}}} c(\mathfrak{a})\, q^{N(\mathfrak{a})},$$

where $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$. If $k \geqslant 2$ then $\varphi_{K,c}(z)$ is a cusp form of weight k and character

$$\chi(d) := \left(\frac{D}{d}\right) \varepsilon_c(d)$$

on $\Gamma_0(|D| \cdot N(\mathfrak{m}))$. Moreover, $\varphi_{K,c}(z)$ is a normalized eigenform of each Hecke operator $T(p)$ with $p \nmid |D| \cdot N(\mathfrak{m})$. It follows that if such a prime p is inert in K, then $\varphi_{K,c}$ is annihilated by the Hecke operator $T(p)$; this is a consequence of the fact that the eigenvalue of $T(p)$ is the pth Fourier coefficient of the form $\varphi_{K,c}(z)$, together with the fact that there are no ideals of norm p in K. Therefore if f is a linear combination of such Hecke forms (i.e. if f is a CM form), then f is annihilated by $T(p)$ for every prime p which is inert in all of the relevant quadratic fields. This fact is responsible for the truth of most of the conjectures.

3. REVISED CONJECTURES

In this section we recast the conjectures in the language of the last section. If d is a squarefree integer, then we denote by χ_d the Kronecker character for $\mathbb{Q}(\sqrt{d})$. We claim that the following imply the corresponding conjectures in the first section.

Conjecture 1′. Each eta-quotient lies in the indicated space and is annihilated by the Hecke operators $T(p)$ for the indicated primes.

	Eta-quotient	Space	p
1.	$\eta^9(8z)\,\eta^9(16z)$	$S_9(\Gamma_0(2^7), \chi_{-2})$	$p \equiv 7 \pmod 8$
2.	$\eta^7(24z)\,\eta^3(48z), \eta^7(48z)\,\eta^3(24z)$	$S_5(\Gamma_0(2^7 \cdot 3^2), \chi_{-2})$	$p \equiv 7, 11 \pmod{12}$
3.	$\eta^7(8z)\,\eta(16z), \eta^7(16z)\,\eta(8z)$	$S_4(\Gamma_0(2^7), \chi_2)$	$p \equiv 31$
4.	$\dfrac{\eta^3(24z)}{\eta(48z)}, \dfrac{\eta^3(48z)}{\eta(24z)}$	$S_1(\Gamma_0(2^7 \cdot 3^2), \chi_{-2})$	$p \equiv 7, 11 \pmod{24}$
5.	$\dfrac{\eta^7(24z)}{\eta(48z)}, \dfrac{\eta^7(48z)}{\eta(24z)}$	$S_3(\Gamma_0(2^7 \cdot 3^2), \chi_{-2})$	$p \equiv 7, 11 \pmod{12}$
6.	$\dfrac{\eta^{13}(8z)}{\eta^5(16z)}, \dfrac{\eta^{13}(16z)}{\eta^5(8z)}$	$S_4(\Gamma_0(2^7), \chi_2)$	$p \equiv 31$
7.	$\dfrac{\eta^{15}(24z)}{\eta^5(48z)}, \dfrac{\eta^{15}(48z)}{\eta^5(24z)}$	$S_5(\Gamma_0(2^7 \cdot 3^2), \chi_{-2})$	$p \equiv 19, 23 \pmod{24}$
8.	$\dfrac{\eta^{16}(6z)}{\eta^6(12z)}, \dfrac{\eta^{16}(24z)}{\eta^6(12z)}$	$S_5(\Gamma_0(2^4 \cdot 3^2), \chi_{-1})$	$p \equiv 11 \pmod{12}$
9.	$\dfrac{\eta^{17}(8z)}{\eta^7(16z)}, \dfrac{\eta^{17}(16z)}{\eta^7(8z)}$	$S_5(\Gamma_0(2^7), \chi_{-2})$	$p \equiv 7 \pmod 8$
10.	$\dfrac{\eta^{18}(12z)}{\eta^8(24z)}, \dfrac{\eta^{18}(12z)}{\eta^8(6z)}$	$S_5(\Gamma_0(2^4 \cdot 3^2), \chi_{-1})$	$p \equiv 11 \pmod{12}$
11.	$\dfrac{\eta^{19}(24z)}{\eta^9(48z)}, \dfrac{\eta^{19}(48z)}{\eta^9(24z)}$	$S_5(\Gamma_0(2^7 \cdot 3^2), \chi_{-2})$	$p \equiv 19, 23 \pmod{24}$

Conjecture 2′. Each eta-quotient lies in the indicated space and is annihilated by the Hecke operators $T(p)$ for the indicated primes.

	Eta-quotient	Space	p
1.	$\eta^2(3z)\,\eta^2(9z)$	$S_2(\Gamma_0(3^3))$	$p \equiv 5 \pmod 6$
2.	$\eta^5(12z)\,\eta^3(36z), \eta^5(12z)\,\eta^3(4z)$	$S_4(\Gamma_0(2^4 \cdot 3^2), \chi_3)$	$p \equiv 5 \pmod 6$
3.	$\frac{\eta^3(z)}{\eta(3z)}, \frac{\eta^3(9z)}{\eta(3z)}$	$M_1(\Gamma_0(9), \chi_{-3})$	$p \equiv 5 \pmod 6$
4.	$\frac{\eta^5(12z)}{\eta(36z)}, \frac{\eta^5(36z)}{\eta(12z)}$	$S_2(\Gamma_0(2^4 \cdot 3^3), \chi_3)$	$p \equiv 5 \pmod 6$
5.	$\frac{\eta^9(4z)}{\eta(12z)}, \frac{\eta^9(36z)}{\eta(12z)}$	$S_4(\Gamma_0(2^4 \cdot 3^2), \chi_3)$	$p \equiv 11 \pmod{12}$
6.	$\frac{\eta^{10}(6z)}{\eta^2(18z)}, \frac{\eta^{10}(18z)}{\eta^2(6z)}$	$S_4(\Gamma_0(2^2 \cdot 3^3))$	$p \equiv 5 \pmod 6$
7.	$\frac{\eta^{11}(12z)}{\eta^3(36z)}, \frac{\eta^{11}(12z)}{\eta^3(4z)}$	$S_4(\Gamma_0(2^4 \cdot 3^2), \chi_3)$	$p \equiv 11 \pmod{12}$

Conjecture 3′. Each eta-quotient lies in the indicated space and is annihilated by the Hecke operators $T(p)$ for the indicated primes.

	Eta-quotient	Space	p
1.	$\eta^5(24z)\,\eta^5(96z)$	$S_5(\Gamma_0(2^8 \cdot 3^2), \chi_{-1})$	$p \equiv 19, 23 \pmod{24}$
2.	$\frac{\eta^7(8z)}{\eta(32z)}, \frac{\eta^7(32z)}{\eta(8z)}$	$S_3(\Gamma_0(2^8), \chi_{-1})$	$p \equiv 7 \pmod 8$

It is not difficult to show that these imply the conjectures in the first section. To verify that each eta-quotient lies in the indicated space, we use Theorem 2; we then use (2.1) together with some straightforward q-series manipulations to complete the verification. We will give one example here to illustrate the latter computations. Namely, consider Conjecture 1, part (8). Define

$$\sum_{n=1}^{\infty} a(n)\, q^n := \frac{(q)_\infty^{16}}{(q^2)_\infty^6}, \qquad \sum_{n=1}^{\infty} b(n)\, q^n := \frac{n^{16}(6z)}{n^6(12z)}.$$

Then we have $a(n) = b(6n+1)$ for all $n \geqslant 0$. If Conjecture 1′, part (8) is true, then we have

$$b(np) = -\chi_{-1}(p)\, p^4 b\left(\frac{n}{p}\right) = p^4\, b\left(\frac{n}{p}\right) \qquad \text{for all} \quad p \equiv 11 \pmod{12}.$$

It follows for such p that

$$a\left(np+\frac{p^2-1}{6}\right)=b(6np+p^2)=p^4 b\left(6\frac{n}{p}+1\right)=p^4 a\left(\frac{n}{p}\right).$$

This is the statement in Conjecture 1.

After similar analysis in each case, we conclude that it suffices to prove the statements in Conjectures 1′–3′.

4. PROOFS

If χ is a quadratic character, then the Fricke involution

$$f(z)\mapsto N^{-k/2}\, z^{-k} f\left(\frac{-1}{Nz}\right)$$

preserves the space $M_k(\Gamma_0(N),\chi)$. Moreover, this involution commutes with the Hecke operators $T(n)$ with $(n, N)=1$. Using the functional equation $\eta(\frac{-1}{z})=\sqrt{z/i}\,\eta(z)$, it can be checked that the pairs listed in Conjectures 1′–3′ form cycles (up to scalar multiplication) under the action of the Fricke involution. Therefore it suffices to check only that one member of each pair is annihilated by the relevant Hecke operators.

We describe briefly a method to construct Hecke characters. Given an imaginary quadratic field K and a non-zero ideal $\mathfrak{m}$ of $\mathcal{O}_K$, let χ be a character of $(\mathcal{O}_K/\mathfrak{m})^\times$; χ may be viewed as a character of the multiplicative group of those $\alpha\in K^\times$ which are prime to $\mathfrak{m}$. If $\chi(\varepsilon)\,\varepsilon^{k-1}=1$ for all units ε in $\mathcal{O}_K$, then we can define a Hecke character c of exponent $k-1$ by

$$c(\mathfrak{a}) := \chi(\alpha)\,\alpha^{k-1} \qquad \text{for principal ideals} \quad \mathfrak{a}=(\alpha). \tag{4.1}$$

The extension of c to non-principal ideals can then be accomplished using the structure of the ideal class group. We will be working in the fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-6})$. The first three have class number one; since $\mathbb{Q}(\sqrt{-6})$ has class number two, each character as given in (4.1) will have two extensions to non-principal ideals. During the construction, it will be useful to recall that if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then we have

$$|(\mathcal{O}_K/\mathfrak{p}^n)^\times| = N(\mathfrak{p})^{n-1}\,(N(\mathfrak{p})-1).$$

It is known that two modular forms in $M_k(\Gamma_0(N), \chi)$ whose Fourier expansions agree up to

$$(4.2) \qquad 1+\frac{Nk}{12}\prod_{p\mid N}\left(1+\frac{1}{p}\right)$$

terms must in fact be equal. This fact will be used repeatedly to verify identities between modular forms.

We now turn to the proofs. We must separate several categories.

Conjecture 1′, *parts* (3) *and* (6). Using (2.1) and (4.2), it is easy to prove these by checking enough Fourier coefficients. ▮

We now turn to the two cases in which the eta-quotient has weight one.

Conjecture 1′, *part* (4). In [G–R] it is proved that $\sum_{n=1}^{\infty} b(n)\, q^n := \frac{\eta^3(24z)}{\eta(48z)}$ is a linear combination of Hecke forms from the field $\mathbb{Q}(i)$. It follows that if n is not the norm of a non-zero ideal in $\mathbb{Z}[i]$, then $b(n)=0$. This is certainly true of any n which is divisible exactly once by some prime $p\equiv 3 \pmod 4$. So if p is such a prime, then we have

$$b(np)=b\left(\frac{n}{p}\right)=0 \qquad 0\leqslant n\leqslant p-1.$$

The bound (4.2) in this case is 193. Therefore, using (2.1), we see that $T(p)$ annihilates $\frac{\eta^3(24z)}{\eta(48z)}$ if $p\equiv 3 \pmod 4$ and $p\geqslant 193$. This can be checked case by case for each prime $3<p<193$ such that $p\equiv 3 \pmod 4$. The conjecture follows. ▮

Conjecture 2′, *part* (3). In [G–H] it is proved that $\sum_{n=1}^{\infty} b(n)\, q^n := \frac{\eta^3(9z)}{\eta(3z)}$ is a Hecke form arising from the field $\mathbb{Q}(\sqrt{-3})$. Therefore if n is divisible exactly once by some prime $p\equiv 5 \pmod 6$, then $b(n)=0$. If p is such a prime, it follows as in the last item that $\frac{\eta^3(9z)}{\eta(3z)}$ is annihilated by $T(p)$. The conjecture follows. ▮

We now turn to the cases where the weight is at least two; our task is to exhibit the relevant eta-quotients as explicit linear combinations of Hecke forms (recall that in this case, the Hecke forms are in fact eigenforms). As it turns out, Gordon and Hughes [G–H] and Gordon and Robins [G–R] have accomplished this task for the remainder of Conjecture 1′ as well as for the first few items of Conjecture 2′. We shall give more details on this later; for now we concern ourselves with items (5)–(7) in Conjecture 2′ and all of Conjecture 3′.

Conjecture 2′, *part* (5). Consider the form $f(z) := \frac{\eta^9(4z)}{\eta(12z)}$. We claim that f is a linear combination of 6 Hecke forms arising from the fields $\mathbb{Q}(i)$ and

$\mathbb{Q}(\sqrt{-3})$. To begin, let $K=\mathbb{Q}(i)$ and $\mathfrak{m}=(6)$. Then $|(\mathcal{O}_K/\mathfrak{m})^\times|=16$; in fact, if $\alpha\in\mathcal{O}_K$ is prime to 6, then there exist unique a and b such that

$$\alpha\equiv i^a \pmod{2\mathcal{O}_k}, \qquad a\in\mathbb{Z}/2\mathbb{Z},$$

$$\alpha\equiv (1-i)^b \pmod{3\mathcal{O}_K}, \qquad b\in\mathbb{Z}/8\mathbb{Z}.$$

Let $\zeta := 1/\sqrt{2}+i/\sqrt{2}$ be a primitive eighth root of unity, and define Hecke characters $c^{\pm}$ by

$$c^{\pm}(\mathfrak{a}) := (-1)^a\,(\pm\zeta)^{3b}\,\alpha^3 \qquad \text{if} \quad \mathfrak{a}=(\alpha)$$

(it is straightforward to check that this definition, as well as those which follow, is independent of the generator chosen).

Next, let $L=\mathbb{Q}(\sqrt{-3})$ and let $\mathfrak{m}=(4)$; then $|(\mathcal{O}_L/\mathfrak{m})^\times|=12$. In fact, if $\zeta := 1/2+\sqrt{-3}/2$ and $\alpha\in\mathcal{O}_L$ is prime to 4, then we have

$$\alpha\equiv\zeta^a(1-2\zeta)^b \pmod 4, \qquad a\in\mathbb{Z}/6\mathbb{Z},\ b\in\mathbb{Z}/2\mathbb{Z}.$$

Define Hecke characters c_1 and c_2 by

$$c_1(\mathfrak{a}) := (-1)^a\,\alpha^3 \qquad \text{if} \quad \mathfrak{a}=(\alpha)$$

$$c_2(\mathfrak{a}) := (-1)^{a+b}\,\alpha^3 \qquad \text{if} \quad \mathfrak{a}=(\alpha).$$

Finally, let c_1' and c_2' be the imprimitive characters obtained by defining c_1 and c_2 modulo the larger ideal $(4\sqrt{-3})$.

Let $\varphi_{K,c^{\pm}}(z)$, $\varphi_{L,c_1}(z)$, $\varphi_{L,c_2}(z)$, $\varphi_{L,c_1'}(z)$, and $\varphi_{L,c_2'}(z)$ be the Hecke forms attached to our characters as described in Section 2; each of these forms lies in the space $S_4(\Gamma_0(2^4\cdot 3^2),\chi_3)$. Moreover, we have the following identity, which can be checked by comparing a sufficient number of Fourier coefficients:

$$\frac{\eta^9(4z)}{\eta(12z)}=\frac{1}{2\sqrt{2}i}(\varphi_{K,c^+}(z)-\varphi_{K,c^-}(z))+\frac{1}{4}(\varphi_{K,c^+}(z)+\varphi_{K,c^-}(z))$$
$$+\frac{3}{4}(\varphi_{L,c_1}(z)+\varphi_{L,c_2}(z))-\frac{1}{2}(\varphi_{L,c_1'}(z)+\varphi_{L,c_2'}(z))$$

The conjecture follows since every prime $p\equiv 11 \pmod{12}$ is inert in both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. ∎

Conjecture 2′, *part* (6). This example is simpler than the last. Let $K=\mathbb{Q}(\sqrt{-3})$ and let $\mathfrak{m}=(6)$. Then $|(\mathcal{O}_K/\mathfrak{m})^\times|=18$; in particular if $\alpha\in\mathcal{O}_k$ is prime to 6 and $\zeta := 1/2+\sqrt{-3}/2$ then we have

$$\alpha\equiv\zeta^a \pmod{3\mathcal{O}_k},\qquad a\in\mathbb{Z}/6\mathbb{Z},$$
$$\alpha\equiv(\zeta^2)^b \pmod{2\mathcal{O}_k},\qquad b\in\mathbb{Z}/3\mathbb{Z}.$$

Define Hecke characters

$$c_1(\mathfrak{a}) := \zeta^{a+4b}\,\alpha^3,\quad c_2(\mathfrak{a}) := \zeta^{5a+2b}\alpha^3 \qquad \text{if}\quad \mathfrak{a}=(\alpha),$$

and let $\varphi_{K,c_1}, \varphi_{K,c_2}$ be the corresponding Hecke forms; these lie in the space $S_4(\Gamma_0(2^2\cdot 3^3))$. Then the following identity can be checked:

$$\frac{\eta^{10}(6z)}{\eta^2(18z)}=\frac{1}{2}(\varphi_{K,c_1}(z)+\varphi_{K,c_2}(z)).$$

The conjecture follows since primes $p\equiv 5 \pmod 6$ are inert in $\mathbb{Q}(\sqrt{-3})$. ▮

Conjecture 2′, *part* (7). Define Hecke forms as in part (6). Then the following identity can be checked:

$$\frac{\eta^{11}(12z)}{\eta^3(36z)}=\frac{1}{4}(\varphi_{K,c^+}(z)+\varphi_{K,c^-}(z)+\varphi_{K,c_1}(z)+\varphi_{K,c_2}(z)).$$

The conjecture follows. ▮

Conjecture 3′, *part* (1). This is the most involved of our constructions. Let $K=\mathbb{Q}(i)$ and let $\mathfrak{m}=(24)$. Then $|(\mathcal{O}_K/\mathfrak{m})^\times|=256$. In fact, if $\alpha\in\mathcal{O}_K$ is prime to 24 then there exist unique a, b, c, and d such that

$$\alpha\equiv i^a\,5^b(1+2i)^c \pmod{8\mathcal{O}_K},\qquad a,c\in\mathbb{Z}/4\mathbb{Z},\ b\in\mathbb{Z}/2\mathbb{Z},$$
$$\alpha\equiv(1-i)^d \pmod{3\mathcal{O}_K},\qquad d\in\mathbb{Z}/8\mathbb{Z}.$$

Define four Hecke characters $c^{\pm\pm}$ by

$$c^{\pm\pm}(\mathfrak{a}) := (-1)^a\,(\pm i)^c\,(\pm i)^d\,\alpha^4 \qquad \text{if}\quad \mathfrak{a}=(\alpha), \tag{4.3}$$

and define

$$f := \tfrac{1}{4}(\varphi_{K,c^{++}}+\varphi_{K,c^{+-}}+\varphi_{K,c^{-+}}+\varphi_{K,c^{--}})\in S_5(\Gamma_0(2^8\cdot 3^2),\chi_{-1}). \tag{4.4}$$

(Notice that if α has either c or d odd in (4.3), then there is no contribution from the ideal (α) in (4.4); using this fact it can be checked that the Fourier expansion of $f(z)$ is supported on exponents of the form $24n+1$.)

Next, let $L=\mathbb{Q}(\sqrt{-6})$ and let $\mathfrak{m}=(24)$, so that $|(\mathcal{O}_K/\mathfrak{m})^\times|=192$. If $\alpha\in\mathcal{O}_K$ is prime to 24 then we have

$$\alpha\equiv 5^a\cdot 7^b\cdot 23^c\cdot(1+\sqrt{-6})^d \pmod{24\mathcal{O}_K},\qquad a, b, c\in\mathbb{Z}/2\mathbb{Z},\ d\in\mathbb{Z}/24\mathbb{Z}.$$

Define Hecke characters c_1 and c_2 for principal ideals by

$$c_1(\mathfrak{a}):=(-1)^b\,\alpha^4,\quad c_2(\mathfrak{a}):=(-1)^{b+d}\,\alpha^4\qquad \text{if}\quad \mathfrak{a}=(\alpha).$$

Each of these has two extensions $c_i^\pm$ to non-principal ideals. However, if $\mathfrak{a}$ is non-principal then $c_i^+(\mathfrak{a})+c_i^-(\mathfrak{a})=0$. Let $\varphi_{L,c_i^\pm}$ be the corresponding Hecke forms, and define

$$g:=\tfrac{1}{4}(\varphi_{L,c_1^+}+\varphi_{L,c_1^-}+\varphi_{L,c_2^+}+\varphi_{L,c_2^-})\in S_5(\Gamma_0(2^8\cdot 3^2),\chi_{-1}).$$

(Note that, by our comment above, we do not need to consider non-principal ideals when we are computing the coefficients of g.) By computing a sufficient number of terms, we can check that

$$\eta^5(24z)\,\eta^5(96z)=\tfrac{1}{1920}(g(z)-f(z)).$$

The conjecture follows since every prime $p\equiv 19, 23 \pmod{24}$ is inert in both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-6})$. ∎

Conjecture 3′, *part* (2). Let $K=\mathbb{Q}(i)$ and let $\mathfrak{m}=(8)$; then $|(\mathcal{O}_K/\mathfrak{m})^\times|=32$. If $\alpha\in\mathcal{O}_k$ is prime to 2 then we have

$$\alpha\equiv i^a\,5^b(1+2i)^c \pmod{8\mathcal{O}_K},\qquad a, c\in\mathbb{Z}/4\mathbb{Z},\ b\in\mathbb{Z}/2\mathbb{Z},\ c\in\mathbb{Z}/4\mathbb{Z}.$$

Define

$$c^\pm(\mathfrak{a}):=(-1)^a\,(\pm i)^c\,\alpha^2\qquad \text{if}\quad \mathfrak{a}=(\alpha),$$

and let $\varphi_{K,c^\pm}$ be the corresponding Hecke forms.

Next let $L=\mathbb{Q}(\sqrt{-2})$ and let $\mathfrak{m}=(8)$. If $\alpha\in\mathcal{O}_K$ is prime to 2 then we have

$$\alpha\equiv(-1)^a\,5^b(1+\sqrt{-2})^c \pmod{8\mathcal{O}_K},\qquad a, b\in\mathbb{Z}/2\mathbb{Z},\quad c\in\mathbb{Z}/8\mathbb{Z}.$$

Define

$$c_1(\mathfrak{a}):=(-1)^b\,\alpha^2,\quad c_2(\mathfrak{a}):=(-1)^{b+c}\,\alpha^2,\qquad \text{if}\quad \mathfrak{a}=(\alpha),$$

and let $\varphi_{L,c_1}, \varphi_{L,c_2}$ be the corresponding Hecke forms. We have the following identity:

$$\frac{\eta^7(8z)}{\eta(32z)} = \frac{1}{4}\left(\varphi_{K,c^+}(z) + \varphi_{K,c^-}(z) + \varphi_{L,c_1}(z) + \varphi_{L,c_2}(z)\right).$$

The conjecture follows since primes $p \equiv 7 \pmod 8$ are inert in both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$. ▮

Conjecture 1′, *parts* (1)–(3) *and* (5)–(11). Gordon and Robins [G–R] have compiled the complete list of all lacunary eta-quotients of the form $\eta^r(z)\,\eta^s(2z)$ with $r, s \in \mathbb{Z}$. In the course of their work, they exhibit each of these eta-quotients as a linear combination of Hecke forms. Conjecture 1′ follows as above from this work; we omit the details here. ▮

Conjecture 2′, *parts* (1), (2), *and* (4). In these cases we appeal to work of Gordon and Hughes [G–H]; they exhibit the relevant eta-quotients as linear combinations of Hecke forms, and the conjectures follow. ▮

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