Cauchy Sequences, the Bolzano-Weierstrass Theorem, and Compactness

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The purpose of this note is to present some basic properties of compact sets. See [1] for the details.

1.1 Definition: A sequence (x_n) of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists an integer N such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N$.

Loosely speaking, a Cauchy sequence x_n is a sequence whose elements get arbitrarily close to one another for sufficiently large n. Naturally, a converging sequence is a Cauchy sequence.

1.2 Proposition: *A convergent sequence is a Cauchy sequence.*

Proof: Let (x_n) be a sequence converging to x, and let $\varepsilon > 0$ be given. Then there exists an integer N such that $|x_n - x| < \varepsilon/2$ for $n \geq N$. Then for $n, m \geq N$ we have that

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that (x_n) is a Cauchy sequence. ■

The next theorem is known as the Bolzano-Weierstrass theorem.

1.3 Theorem: (Bolzano-Weierstrass) *Every bounded sequence in $\mathbb{R}$ has a subsequence that converges to some point in $\mathbb{R}$.*

Proof: Suppose that (x_n) is a bounded sequence and that a and b are numbers satisfying $a \leq x_n \leq b$. Put $c = (a + b)/2$, i.e., the mid-point of the interval $[a, b]$. If $x_n \in [a, c]$ for infinitely many n then set $a_1 = a$ and $b_1 = c$, otherwise set $a_1 = c$ and $b_1 = b$. In either case $[a_1, b_1] \subset [a, b]$ and $b_1 - a_1 = (b - a)/2$. Let n_1 be an integer such that $a_1 \leq x_{n_1} \leq b_1$. Now let $c_1 = (a_1 + b_1)/2$. If $x_n \in [a_1, c_1]$ for infinitely many n then set $a_2 = a_1$ and $b_2 = c_1$, otherwise set $a_2 = c_1$ and $b_2 = b_1$. In either case $[a_2, b_2] \subset [a_1, b_1]$ and $b_2 - a_2 = (b - a)/2^2$. Let $n_2 > n_1$ be an integer such that $a_2 \leq x_{n_2} \leq b_2$. Such an integer exists since $x_n \in [a_2, b_2]$ holds for infinitely many n. We can repeat this same construction and obtain sequences (a_k), (b_k), and (x_{n_k}) such that $b_k - a_k = 2^{-k}(b - a)$ and $a_k \leq x_{n_k} \leq b_k$. By construction, the sequence (a_k) is non-decreasing and the sequence b_k is non-increasing. As (a_k) is also bounded it converges to $L = \sup_k a_k$ (this is where we use completeness of $\mathbb{R}$). Note that since (a_k) is bounded, $L \in \mathbb{R}$. Now for any pair of integers j and k, $a_j < b_k$. Indeed, if $j < k$ then $a_j \leq a_k < b_k$, and if $j \geq k$ then $a_j < b_j \leq b_k$. This means that b_k is an upper-bound for the sequence (a_j), and thus $L \leq b_k$. Therefore, $L \in [a_k, b_k]$ for each k, and hence

$$|L - x_{n_k}| \leq b_k - a_k = 2^{-k}(b - a),$$

which implies that $\lim_{k\to\infty} x_{n_k} = L$. Thus (x_{n_k}) is a subsequence of (x_n) converging to $L \in \mathbb{R}$. ■

The Bolzano-Weierstrass theorem on $\mathbb{R}$ can be easily extended to $\mathbb{R}^n$ by noticing that a bounded sequence in $\mathbb{R}^n$ has components that are bounded (see Theorem 1.18). Next we prove an important theorem in analysis, namely that every Cauchy sequence in $\mathbb{R}$ converges to an element of $\mathbb{R}$. Before given the proof, we need the following lemmas.

1.4 LEMMA: *Every Cauchy sequence is bounded.*

Proof: Let (x_n) be a Cauchy sequence and suppose $\varepsilon > 0$. There exists an integer N such that $|x_n - x_m| < \varepsilon$ for $n, m \geq N$. Therfore, for all $n \geq N$

$$x_N - \varepsilon < x_n < x_N + \varepsilon.$$

Let $m = \min\{x_1, x_2, \ldots, x_{N-1}, x_N - \varepsilon\}$ and $M = \max\{x_1, x_2, \ldots, x_{N-1}, x_N + \varepsilon\}$. Then we clearly have that $m \leq x_n \leq M$ for all n, proving that (x_n) is bounded. ■

1.5 REMARK: Proposition 1.2 together with Lemma 1.4 show that a convergent sequence is also bounded.

1.6 LEMMA: *If a subsequence of a Cauchy sequence converges to x, then the sequence itself converges to x.*

Proof: Let (x_n) be a Cauchy sequence and suppose that (x_{n_k}) is a subsequence of (x_n) converging to x. Since (x_n) is Cauchy, there exists an integer N such that if $n, m \geq N$ then $|x_n - x_m| < \varepsilon/2$. Since x_{n_k} converges to x, there exists an integer $n_0 > N$ such that $|x_{n_0} - x| < \varepsilon/2$. If $n > N$ then

$$|x_n - x| = |x_n - x_{n_0} + x_{n_0} - x| \leq |x_n - x_{n_0}| + |x_{n_0} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

proving that (x_n) converges to x also. ■

1.7 THEOREM: *Every Cauchy sequence in $\mathbb{R}$ converges to an element of $\mathbb{R}$.*

Proof: By Lemma 1.4, a Cauchy sequence is bounded and thus contains a convergent subsequence by the Bolzano-Weierstrass theorem. By Lemma 1.6 the Cauchy sequence converges to the same limit point of the subsequence. ■

By replacing the absolute value with a metric in the proofs of Proposition 1.2, Lemma 1.4, and Lemma 1.6, the following proposition can be easily proved.

1.8 PROPOSITION: *Let (X, d) be a metric space.*

(i) *Every convergent sequence is a Cauchy sequence.*
(ii) *A Cauchy sequence is bounded.*
(iii) *If a subsequence of a Cauchy sequence converges to a point $x \in X$, then the Cauchy sequence itself converges to x.*

Suppose that M is a bounded and closed subset of $\mathbb{R}$, and let (x_n) be a sequence in M. Then by the Bolzano-Weierstrass theorem, (x_n) has a subsequence converging to a point $x \in \mathbb{R}$. Since M is closed, $x \in M$. Thus, a closed and bounded subset of $\mathbb{R}$ has the following property: Every sequence in M has a subsequence converging to a point in M. This motivates the following definition.

1.9 DEFINITION: Suppose that (X, d) is a metric space. A subset $K \subset X$ is said to be **sequentially compact** if every sequence in K has a subsequence that converges to a point in K.

Thus, a closed and bounded subset of $\mathbb{R}$ is sequentially compact. We would like to know under what conditions a subset of a general metric space is sequentially compact. We begin with two definitions.

1.10 DEFINITION: A subset A of a metric space X is called **compact** if every open cover of A has a finite subcover.

1.11 DEFINITION: A subset A of a metric space X is called **totally bounded** if for each $\varepsilon > 0$ there is a finite set $\{x_1, \ldots, x_n\}$ in X such that $A \subset \cup_{i=1}^{n} D(x_i, \varepsilon)$.

A totally bounded set A is bounded. Indeed, suppose that $A \subset \cup_{i=1}^{n} D(x_i, \varepsilon)$. Since $D(x_i, \varepsilon) \subset D(x_1, \varepsilon + d(x_i, x_1))$ then if $R = \varepsilon + \max\{d(x_2, x_1), \ldots, d(x_n, x_1)\}$ then $A \subset D(x_1, R)$.

To prove our main result we need the following lemmas.

1.12 LEMMA: *A compact set $K \subset X$ is closed.*

Proof: We will show that $X \backslash K$ is open. Let $x \in X \backslash K$ and consider the sets $U_n = \{y \mid d(x, y) > 1/n\}$. It is clear that $U_n \subset U_{n+1}$. Now since for $y \neq x$ we have that $d(x, y) > 0$, $\{U_n\}$ is an open cover for K. Compactness of K implies that this open cover contains a finite subcover of A. Thus there exists some N such that $A \subset U_N$, and thus $D(x, 1/N) \subset X \backslash K$, proving that $X \backslash K$ is open. ■

1.13 LEMMA: *If (X, d) is a compact metric space and $K \subset X$ is closed, then K is compact.*

Proof: Let $\{U_i\}$ be an open cover of K. Then $\{U_i, X \backslash K\}$ is an open cover for X. Since X is compact there exists a finite subcover $\{U_{j_1}, U_{j_2}, \ldots, U_{j_k}, X \backslash K\}$ of X, and thus $\{U_{j_1}, U_{j_2}, \ldots, U_{j_k}\}$ is a finite subcover of K. ■

Here is the main result.

1.14 THEOREM: (Bolzano-Weierstrass) *A subset of a metric space is compact iff it is sequentially compact.*

Proof: Let K compact subset of the metric space X. Suppose that (x_n) is a sequence in K that has no convergent subsequences. Therefore, (x_n) has infinitely many distinct points, say, $y_1, y_2, y_3, \ldots$. Since (x_n) contains no convergent subsequences, there exists some neighbourhood U_k of y_k containing no other y_i $(i \neq k)$. Indeed, if every neighbourhood of y_k contained another y_i $(i \neq k)$ then we can choose the neighbourhoods $D(y_k, 1/m)$, $m = 1, 2, 3, \ldots$, and obtain a subsequence converging to y_k. The set $\{y_1, y_2, \ldots\}$ is closed because it does not contain any accumulation points by the assumption that there are no convergent subsequences of (x_n). By Lemma 1.13, $\{y_1, y_2, \ldots\}$ is a compact subset of X. But $\{U_k\}$ is an open covering of $\{y_1, y_2, \ldots\}$ that contains no finite subcover, a contradiction. Therefore, (x_n) must have a convergent subsequence. Since a compact set is closed (Lemma 1.12), the limit of the convergent subsequence must belong to K. This proves that K is sequentially compact. Now suppose that K is sequentially compact and that $\{U_i\}_{i \in I}$ is an open cover for K. To prove compactness of K we need the following lemmas.

1.15 Lemma: *There is an $r > 0$ such that for each $y \in K$, $D(y, r) \subset U_i$ for some U_i.*

Proof: Suppose that no such r exists. Then for each $\varepsilon_n = 1/n$, there exists some $y_n \in A$ such that $D(y_n, \varepsilon_n) \not\subseteq U_i$ for any U_i. The sequence (y_n) contains a convergent subsequence whose limit lies in A. Call this subsequence (y_{n_k}), and set $y = \lim_{k \to \infty} y_{n_k} \in A$. Then there is a set U_j such that $y \in U_j$, and thus $D(y, 2\varepsilon) \subset U_j$ for some $\varepsilon > 0$. Now choose n_k large enough so that $d(y_{n_k}, y) < \varepsilon$ and $\varepsilon_{n_k} < \varepsilon$. Then $D(y_{n_k}, \varepsilon_{n_k}) \subset D(z, 2\varepsilon) \subset U_j$, a contradiction. Therefore, such an $r > 0$ exists. ■

The number r in the preceding lemma is called the Lebesgue number of the covering $\{U_i\}$.

1.16 Lemma: *A sequentially compact set is totally bounded.*

Proof: Let A be a sequentially compact subset of a metric space X. If A is not totally bounded, then there exists a $\varepsilon > 0$ such that A cannot be covered by a finite number of ε disks. So, let $y_1 \in A$ and choose $y_2 \in A \backslash D(y_1, \varepsilon)$. Since $D(y_1, \varepsilon) \cup D(y_2, \varepsilon)$ does not contain all of A, choose $y_3 \in A \backslash (D(y_1, \varepsilon) \cup D(y_2, \varepsilon))$. We can continue this process to obtain a sequence (y_n) in A such that $d(y_k, y_j) \geq \varepsilon$ for $k \neq j$. Such a sequence does not have a convergent subsequence, a contradiction. ■

Now we finish the proof of the Theorem 1.14. By Lemma 1.15, there exists a $r > 0$ such that $y \in D(y, r) \subset U_i$ for some i. By Lemma 1.16, there exists a finite set $\{y_1, \ldots, y_n\}$ such that $A \subset \cup_{k=1}^{n} D(y_k, r)$. The theorem follows since $D(y_k, r) \subset U_{i_k}$ for some i_k. ■

The next theorem relates compactness with completeness.

1.17 Theorem: *A metric space is compact iff it is complete and totally bounded.*

Proof: Let X be a compact metric space. Then it is sequentially compact by the Bolzano-Weierstrass theorem and thus totally bounded by Lemma 1.16. If (x_n) is a Cauchy sequence, it has a subsequence converging to a point in X, and by Lemma 1.6 the original sequence must also converge to the same point. Conversely, suppose that X is a complete and totally bounded set. It is enough to show that X is sequentially compact by the Bolzano-Weierstrass theorem. So let (x_n) be a sequence in X. For $\varepsilon_1 = 1$, $X \subset \cup_{i=1}^m D(y_{1i}, \varepsilon_1)$ for some finite set $\{y_{11}, y_{12}, \ldots, y_{1m}\} \subset X$. One of the discs $D(y_{1i}, \varepsilon_1)$ must contain infinitely many element of the sequence. Assume, wlog, that it is $D(y_{11}, \varepsilon_1)$ and choose $x_{n_1} \in D(y_{11}, \varepsilon_1)$. Now let $\varepsilon_2 = 1/2$. Then the disc $D(y_{11}, \varepsilon_1)$ will be covered by a finite number of discs of radius ε_2, that is, $D(y_{11}, \varepsilon_1) \subset \cup_{i=1}^s D(y_{2i}, \varepsilon_2)$ for some finite set $\{y_{21}, \ldots, y_{2s}\}$. One of these subdiscs, say $D(y_{21}, \varepsilon_2)$, will contain infinitely many elements of the sequence (x_n). So let $x_{n_2} \in D(y_{21}, \varepsilon_2)$, with $n_2 > n_1$. By construction, $d(x_{n_1}, x_{n_2}) < 1$. We can continue this process to obtain a subsequence (x_{n_k}) such that $d(x_{n_{k+1}}, x_{n_k}) < \frac{1}{k}$. Therefore, for $j > l$

$$d(x_{n_j}, x_{n_l}) \leq d(x_{n_j}, x_{n_{j-1}}) + \cdots + d(x_{n_{l+1}}, x_{n_l}) < \frac{1}{j-1} + \cdots + \frac{1}{l},$$

which can be made arbitrarily small for sufficiently large l and j. This proves that (x_{n_k}) is a Cauchy sequence, and thus must converge by the completeness assumption. Therefore, (x_n) has a convergent subsequence, proving that X is sequentially compact, i.e., it is compact. ■

Here is a summary of the above results. In a metric space the following are true:

- Every convergent sequence is a Cauchy sequence.
- A Cauchy sequence is bounded.
- If a Cauchy sequence contains a convergent subsequence, then the whole sequence converges to the same point as the subsequence.
- A compact set is closed.
- A closed subset of a compact set is also compact.
- Sequentially compact spaces are totally bounded.
- A set is compact iff it is sequentially compact.
- A compact set is totally bounded, and thus also bounded.
- A set is compact iff it is complete and totally bounded.

Roughly speaking, compact sets are bounded and contain no "holes", and thus any sequence in a compact space will contain a convergent subsequence.

There is a special case of compact sets that arise frequently. These are the compact sets in $\mathbb{R}^n$.

1.18 Theorem: (Heine-Borel) *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof: We have already seen that a compact set is bounded (actually it is totally bounded) and closed. The converse is an easy consequence of the Bolzano-Weierstrass theorem and

Theorem 1.14. Indeed, suppose that A is a closed and bounded subset of $\mathbb{R}^n$, and let $(x_k) = (x_k^1, x_k^2, \ldots, x_k^n)$ be a sequence in A. The sequence (x_k^1) is bounded, and so contains a convergent subsequence $(x_{f_1(k)}^1)$. The subsequence $(x_{f_1(k)}^2)$ of (x_k^2) is bounded and so contains a convergent subsequence, say $(x_{f_2(k)}^2)$. So we have convergent sequence $(x_{f_2(k)}^1, x_{f_2(k)}^2)$ in $\mathbb{R}^2$. We generalize this process in $\mathbb{R}^n$ to obtain a convergent subsequence $(x_{f_n(k)}) = (x_{f_n(k)}^1, x_{f_n(k)}^2, \ldots, x_{f_n(k)}^n)$ of (x_k). This proves that A is sequentially compact, and by Theorem 1.14 it is compact. ■

Note that if we had taken an arbitrary convergent subsequence of (x_k^j) for each $j = 1, 2, \ldots, n$, say $(x_{f_j(k)}^j)$, then it is clear that in general $(x_{f_1(k)}^1, x_{f_2(k)}^2, \ldots, x_{f_n(k)}^n)$ will not be a subsequence of (x_k).

To end this note, we state one last theorem called the Nested Set Property.

1.19 Theorem: (Nested Set Property) *Let $\{A_k\}_{k=1}^\infty$ be a sequence of compact non-empty sets in a metric space X such that $A_{k+1} \subset A_k$ for all $k = 1, 2, \ldots$. Then $\cap_{k=1}^\infty A_k$ is non-empty.*

Proof: Let (x_k) be any sequence such that $x_k \in A_k$ for all $k = 1, 2, \ldots$. We can assume without loss of generality that the elements of (x_k) are all distinct. Since A_1 is compact, (x_k) has a convergent subsequence (x_{n_k}) whose limit x lies in A_1. For each $j = 1, 2, \ldots$, the sequence $\{x_{n_k}\}_{k=j}^\infty$ lies in A_{n_j} and converges to x by uniqueness. By compactness of A_{n_j}, $x \in A_{n_j}$. Since $n_j \geq j$, we have that $x \in A_{n_j} \subset A_j$ for all j. This proves the theorem. ■

References

[1] J.E. Marsden and M.J. Hoffman *Elementary Classical Analysis*, 2nd Edition, W.H. Freeman and Company, New York.