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Boundedness of generalized Cesáro averaging operators on certain function spaces

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Abstract

We define a two-parameter family of Cesáro averaging operators $\mathscr{P}^{b,c}$,

$$\mathscr{P}^{b,c} f(z) = \frac{\Gamma(b+1)}{\Gamma(c)\Gamma(b+1-c)} \int_0^1 t^{c-1}(1-t)^{b-c}(1-tz)F(1,b+1;c;tz)f(tz)\,\mathrm{d}t,$$

where $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on the unit disc Δ, and $F(a,b;c;z)$ is the classical hypergeometric function. In the present article the boundedness of $\mathscr{P}^{b,c}$, $\mathrm{Re}\,(b+1) > \mathrm{Re}\,c > 0$, on various function spaces such as Hardy, BMOA and a-Bloch spaces is proved. In the special case $b = 1+\alpha$ and $c = 1$, $\mathscr{P}^{b,c}$ becomes the α-Cesáro operator $\mathscr{C}^{\alpha}$, $\mathrm{Re}\,\alpha > -1$. Thus, our results connect the special functions in a natural way and extend and improve several well-known results of Hardy-Littlewood, Miao, Stempak and Xiao.

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Keywords: Gaussian hypergeometric functions; Cesáro operators

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1. Introduction

Suppose that $A=(t_{ij})_{i,j\geqslant 0}$ is an infinite matrix with complex entries. Then we can consider A as a transformation which carries a complex sequence $a=\{a_i\}_{i\geqslant 0}$ into a complex sequence $b=\{b_i\}_{i\geqslant 0}$ through the system $Aa=b$, where

$$\sum_{j\geqslant 0} t_{ij}a_j = b_i, \quad i=0,1,2,\ldots, \tag{1.1}$$

and where we assume that the series converge. Since each sequence $\{a_i\}_{i\geqslant 0}$ can be uniquely associated with a power series $f(z)=\sum_{i=0}^{\infty} a_i z^i$, the matrix A maps a power series into another power series defined by $g(z)=\sum_{n=0}^{\infty} b_i z^i$, where b_i is given by (1.1). Assuming that the matrix A transforms each power series convergent in the unit disc $\Delta=\{z : |z|<1\}$ into a power series convergent in Δ, the following problem arises:

Problem 1.2. What are the function spaces $\mathscr{F}$, consisting of analytic functions in the unit disc Δ, on which the operators defined through the given matrices are bounded?

Our main result is motivated by this problem. Special functions provide a valuable testing ground for analytical methods in complex variable theory. The surprising use of the hypergeometric functions in the proof of Bieberbach conjecture by de Branges, has prompted renewed interest in the hypergeometric functions—the core of special functions. Moreover, many special functions encountered in physics, engineering and probability theory are special cases of the Gaussian hypergeometric function [1,2,16] defined by the power series expansion

$${}_2F_1(a,b;c;z) := F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!} \quad (|z|<1), \tag{1.2}$$

where a,b,c are complex numbers such that $c\neq -m, m=0,1,2,3,\ldots$, and (a,n) is the shifted factorial defined by Appel's symbol

$$(a,n) := a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n\in\mathbb{N}=\{1,2,\ldots\}$$

and $(a,0)=1$ for $a\neq 0$. We assume $c\neq -m, m=0,1,2,3,\ldots,$ to prevent the denominators vanishing. Clearly, $F(a,b;c;z)$ belongs to $\mathscr{H}$, the space of all analytic functions in Δ. The asymptotic behaviour of $F(a,b;c;z)$ near $z=1$ can be obtained from standard texts (see [1,4,17]). Many other properties of the hypergeometric series including the relations for contiguous functions (differing by 1 in the parameter values) and its various generalizations are gathered together in standard texts such as [2,4,13,16,17]. Asymptotic expansions and inequalities for hypergeometric functions are also discussed in [12]. The following proposition is simple and is the basis for our investigation.

Proposition 1.4. *We have*

$$(a+b-c)zF(a,b+1;c+1;z)+cF(a-1,b;c;z)=c(1-z)F(a,b+1;c;z). \tag{1.3}$$

Proof. This identity follows easily if we compare the coefficients of z^n on both sides of (1.3). □

Eq. (1.3) is a contiguous relation. If $a=1$ and $c=a+b$, then (1.3) reduces to the trivial equation $c=c$. Therefore, our interest lies in the following three cases:

(i) $a=1$ and $c \neq a+b=1+b$,
(ii) $a \neq 1$ and $c=a+b$ (see Section 4),
(iii) $a \neq 1$ and $c \neq a+b$.

We produce a number of results concerning case (i) only and leave the other two cases open, although we shall at least outline the problem for case (ii) in Section 4 ahead.

2. Generalization of Cesáro means

If $a=1$ and $c \neq a+b=1+b$, then (1.3) is equivalent to

$$\frac{((1+b-c)/c)zF(1,b+1;c+1;z)+1}{1-z}=F(1,b+1;c;z). \tag{2.1}$$

A comparison of the coefficients of z^n on both sides shows that for $n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} = 1,$$

where

$$A_k^{a,b;c} = \frac{(a,k)(b,k)}{(c,k)(1,k)}, \qquad A_k^{b;c} := A_k^{1,b;c} = \frac{(b,k)}{(c,k)}$$

and b_k is the coefficient of z^k in $[(1+b-c)/c]zF(1,b+1;c+1;z)+1$ given by $b_0=1$, and for $k \geqslant 1$

$$b_k = \frac{(1+b-c)}{c} A_{k-1}^{b+1;c+1} = \frac{(1+b-c)}{b} A_k^{b;c},$$

the second identity being well defined only when $b \neq 0$ (otherwise we have to treat the second identity as a limiting case if $b=0$). This basic property suggests that for a given sequence of complex numbers $\{a_k\}_{k \geqslant 0}$, we can consider the Cesáro mean of type $(1,b;c)$ which we define by

$$\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} a_k, \qquad n \in \mathbb{N} \cup \{0\}.$$

We call this a generalized Cesáro mean because in the special case $b=1+\alpha$ ($\operatorname{Re} \alpha > -1$) and $c=1$, the above mean becomes the classical Cesáro mean of order α (or simply α-Cesáro mean). If $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}$, then we define

$$\mathscr{P}^{b,c} f(z) := \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} b_{n-k} a_k \right) z^n \tag{2.2}$$

and we call this Cesáro operator of type $(1,b;c)$, or simply a *generalized* Cesáro operator. It is not hard to see that the right-hand side of (2.2) defines an analytic function on Δ. The fact that $\mathscr{P}^{b,c} f$ is analytic becomes clear from the integral representation which we derive below. In the notation of Stempak [15], we find that

$$\mathscr{P}^{1+\alpha,1} f = \mathscr{C}^{\alpha} f \quad (\operatorname{Re}(\alpha+1)>0)$$

and, in particular, for $\alpha=0$, that

$$\mathscr{P}^{1,1} f = \mathscr{C}^{0} f := \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n,$$

where $\mathscr{C}^{\alpha} f$ is called the Cesáro operator of order α, or simply the α-Cesáro operator acting on f (see [15]). If $\alpha=0$, the α-Cesáro operator $\mathscr{C}^{\alpha} f$ is simply the classical Cesáro operator $\mathscr{C}$. To find an integral representation for $\mathscr{P}^{b,c} f(z)$, we rewrite (2.2) as

$$\begin{aligned}
\mathscr{P}^{b,c} f(z) &:= \left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k} a_k\right) z^n\right] * \sum_{n=0}^{\infty} \frac{1}{A_n^{b+1;c}} z^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k} a_k\right) z^n * \sum_{n=0}^{\infty} A_n^{c;b+1} z^n \\
&= \frac{(1+b-c)}{c}[zF(1,b+1;c+1;z)+1] f(z) * F(1,c;b+1;z)
\end{aligned}$$

and, by (2.1), we obtain that

$$\mathscr{P}^{b,c} f(z) = [f(z)(1-z)F(1,b+1;c;z)] * F(1,c;b+1;z),$$

where $*$ denotes the Hadamard product (or convolution) of power series. That is, if $f(z)=\sum_{i=0}^{\infty} a_i z^i$ and $g(z)=\sum_{i=0}^{\infty} b_i z^i$ are two analytic functions in $|z|<R$ then $f*g$ is defined by $(f*g)(z)=\sum_{i=0}^{\infty} a_i b_i z^i$ and this series converges for $|z|<R^2$. Moreover,

$$(f*g)(z) = \frac{1}{2\pi \mathrm{i}} \int_{|w|=\rho} f(w) g(z/w) \frac{\mathrm{d}w}{w}, \quad |z|<\rho R<R^2.$$

In particular, if f, g are in $\mathscr{H}$, we have

$$(f*g)(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{\mathrm{i}t}) g(z e^{-\mathrm{i}t})\,\mathrm{d}t, \quad 0<\rho<1.$$

We recall the Euler's representation for $F(a,b;c;z)$, namely,

$$F(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t, \tag{2.3}$$

which is valid when $\operatorname{Re} c > \operatorname{Re} b > 0$. It follows easily for each $g(z)=\sum_{k=0}^{\infty} b_k z^k \in \mathscr{H}$, that

$$F(1,b;c;z) * g(z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1} g(tz)\,\mathrm{d}t$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$. Using this formula with b replaced by c and c replaced by $b+1$, and with $g(z) = f(z)(1-z)F(1, b+1; c; z)$, we have the integral representation

$$\mathscr{P}^{b,c} f(z) = \frac{1}{B(c, b+1-c)} \int_0^1 t^{c-1}(1-t)^{b-c}(1-tz)F(1, b+1; c; tz) f(tz)\, \mathrm{d}t$$

which is valid for $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$. In view of the well-known Gauss identity [2,17]

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z), \tag{2.4}$$

we can rewrite the previous equation to obtain the following result.

Proposition 2.5. For $b, c \in \mathbb{C}$ with $\operatorname{Re}(b+1) > \operatorname{Re} c > 0$, we have

$$\begin{aligned}
\mathscr{P}^{b,c} f(z) &= \frac{1}{B(c, b+1-c)} \int_0^1 t^{c-1}(1-t)^{b-c} \frac{f(tz)}{(1-tz)^{b+1-c}} F(c-1, c-b-1; c; tz)\, \mathrm{d}t \\
&= \frac{z^{-b}}{B(c, b+1-c)} \int_0^z \zeta^{c-1}(z-\zeta)^{b-c} \frac{f(\zeta)}{(1-\zeta)^{b+1-c}} F(c-1, c-b-1; c; \zeta)\, \mathrm{d}\zeta.
\end{aligned}$$

In particular, if $c = 1$ and $b = 1 + \alpha$, we find that the representation for the classical Cesáro operator of order α is given by

$$\mathscr{C}^{\alpha} f(z) := \mathscr{P}^{1+\alpha,1} f(z) = (1+\alpha) \int_0^1 f(tz) \frac{(1-t)^{\alpha}}{(1-tz)^{1+\alpha}}\, \mathrm{d}t,$$

as in [15]. Thus, $\mathscr{P}^{b,c} f(z)$ is clearly a natural generalization of $\mathscr{C}$.

3. Boundedness of the generalized Cesáro operator

For $f \in \mathscr{H}$, $0 \leqslant r < 1$, the integral means $M_p(r, f)$ are defined by

$$M_p(r, f) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|^p\, \mathrm{d}t \right)^{1/p}, \quad 0 < p < \infty$$

and are known to be increasing with r. The standard Hardy space H^p $(0 < p < \infty)$ is the space of all $f \in \mathscr{H}$ for which

$$\|f\|_p := \sup_{r \in [0,1)} M_p(r, f) = \lim_{r \to 1^-} M_p(r, f) < \infty.$$

For $p = \infty$, $H^p = H^\infty$ denotes the space of all bounded analytic functions on Δ, i.e., $f \in \mathscr{H}$ satisfying $\|f\|_\infty = \sup_{z \in \Delta} |f(z)| < \infty$.

The boundedness of $\mathscr{C}$ on H^p was investigated by a number of authors, Hardy–Littlewood [9] for $1 < p < \infty$, Siskakis [14] for $p = 1$, Miao [10] for $0 < p < 1$, and Danikas and Siskakis [6] for $p = \infty$. For $\alpha \in (0, \infty)$, the boundedness of $\mathscr{C}^\alpha$ on H^p, $0 < p \leqslant 2$, was obtained by Stempak [15] and the case $2 < p \leqslant \infty$ remained open. The case $2 < p < \infty$ was recently settled by Xiao [18] affirmatively. The main aim of this paper is to discuss the boundedness of the general operator $\mathscr{P}^{b,c}$ on H^p for $0 < p < 1$. The boundedness of $\mathscr{P}^{b,c}$ on H^p for $1<p<\infty$ is yet to be proved as we have faced some difficulties in getting integral representation for the adjoint of $P^{b,c}$. For the moment, we leave this problem open. In order to prove our result, we will make use of the following lemma as in [15].

Lemma 3.1. (i) *For* $0<p<\infty$,

$$\int_0^{2\pi} \sup_{0\leqslant r<1} |f(re^{i\theta})|^p \, d\theta \leqslant B\|f\|_p^p$$

with $B=B_p$ *independent of* $f\in H^p$.

(ii) *If* $0<p<\infty$, *and* $n>1$, *then*

$$\int_0^1 (M_{np}(r,f))^p(1-r)^{-1/n}\, dr \leqslant C\|f\|_p^p$$

with $C=C_p$ *independent of* $f\in H^p$.

(iii) *If* $s>1$, *then*

$$\int_0^{2\pi} |1-re^{i\theta}|^{-s}\, d\theta \leqslant D(1-r)^{-s+1}$$

with $D=D_s$ *independent of* r, $0<r<1$.

Lemma 3.1 is due to Hardy and Littlewood and Lemma 3.1(i) is well-known as the name Hardy–Littlewood maximal theorem. We refer to pp. 12, 65 of [7] for parts (i) and (iii) of Lemma 3.1 whereas for Lemma 3.1(ii), see [9, p. 412]). The first main result we shall prove here is the following

Theorem 3.2. *Let* $b,c\in\mathbb{C}$ *be such that* $\mathrm{Re}\,(b+1)>\mathrm{Re}\,c>0$. *Then* $\mathscr{P}^{b,c}$ *is a bounded operator on* H^p, $0<p\leqslant 1$.

Proof. Our main aim is to show that

$$[M_p(r,\mathscr{P}^{b,c}f)]^p \leqslant K\|f\|_p^p$$

for some constant $K>0$, depending only on b, c, and p. We provide the proof only for the case of reals b, c with $b+1>c>0$. For the proof of the complex case, we simply require to note the following for $t\in(0,1)$:

$$|t^{c-1}|=t^{\mathrm{Re}\,(c)-1} \quad \text{and} \quad |(1-t)^{b-c}|=(1-t)^{\mathrm{Re}\,(b-c)}.$$

Here we choose the principal argument for $\arg(1-tz)$ such that $\arg(1-tz)=0$ at $z=0$, and we note that $|\arg(1-tz)|<\pi/2$ for $z\in\varDelta$. Moreover, the integral $\int_0^1 t^{c-1}(1-t)^{b-c}\,dt$ converges by the hypotheses $\mathrm{Re}\,(b+1-c)>0$ and $\mathrm{Re}\,c>0$, therefore, we observe that it suffices to assume b and c are real, and that $b+1>c>0$ in the remaining part of the proof.

Let $t_k=1-2^{-k}$ for each $k\geqslant 1$. We will show that

$$0\leqslant \int_{t_{k-1}}^{t_k} t^{c-1}(1-t)^{b-c}\,dt \leqslant K_1 2^{-k(b+1-c)}. \tag{3.1}$$

For this, we need to consider two cases $k=1$ and $k>1$ separately. For $k=1$,

$$0 \leqslant \int_{t_0}^{t_1} t^{c-1}(1-t)^{b-c}\,\mathrm{d}t < 2\int_0^{1/2} t^{c-1}\,\mathrm{d}t = \frac{1}{c2^{c-1}},$$

because $(1-t)^{b-c} = (1-t)^{-1}(1-t)^{b+1-c} \leqslant 2(1-t)^{b+1-c} \leqslant 2$. For $k>1$,

$$\begin{aligned}
0 &\leqslant \int_{t_{k-1}}^{t_k} t^{c-1}(1-t)^{b-c}\,\mathrm{d}t \leqslant 2\int_{t_{k-1}}^{t_k} (1-t)^{b-c}\,\mathrm{d}t \\
&= \frac{2}{b+1-c}\left[2^{-(k-1)(b+1-c)} - 2^{-k(b+1-c)}\right] \\
&= \frac{2}{b+1-c}\left[2^{b+1-c} - 1\right]2^{-k(b+1-c)} \\
&< 2\frac{2^{b+1-c}}{b+1-c}2^{-k(b+1-c)},
\end{aligned}$$

since $t^{c-1} = t^{-1}t^c \leqslant 2t^c \leqslant 2$. Putting together the two cases gives the required inequality (3.1) for some constant K_1. Now, as in [10,15], we suppose $f \in H^p$ and

$$G_k(r,\theta) = \sup_{t\in(t_{k-1},t_k)} \left|\frac{f(tr\,\mathrm{e}^{\mathrm{i}\theta})}{(1-tr\,\mathrm{e}^{\mathrm{i}\theta})^{b+1-c}}\right|.$$

Since $b+1>c$, the boundedness of $F(c-1, c-b-1; c; z)$ on $|z| \leqslant 1$ and the above calculations show that

$$\begin{aligned}
&|B(c, b+1-c)\mathscr{P}^{b,c}f(r\mathrm{e}^{\mathrm{i}\theta})| \\
&\quad = \left|\int_0^1 t^{c-1}(1-t)^{b-c}\frac{f(tr\,\mathrm{e}^{\mathrm{i}\theta})}{(1-tr\,\mathrm{e}^{\mathrm{i}\theta})^{b+1-c}}F(c-1, c-b-1; c; tr\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}t\right| \\
&\quad \leqslant K_2\sum_{k=1}^{\infty}\int_{t_{k-1}}^{t_k} t^{c-1}(1-t)^{b-c}\left|\frac{f(tr\,\mathrm{e}^{\mathrm{i}\theta})}{(1-tr\,\mathrm{e}^{\mathrm{i}\theta})^{b+1-c}}\right|\mathrm{d}t \\
&\quad \leqslant K_2\sum_{k=1}^{\infty} G_k(r,\theta)\int_{t_{k-1}}^{t_k} t^{c-1}(1-t)^{b-c}\,\mathrm{d}t \\
&\quad \leqslant K_1K_2\sum_{k=1}^{\infty} G_k(r,\theta)2^{-k(b+1-c)}
\end{aligned}$$

and therefore, the last inequality with $K_3 = K_1^p K_2^p$ gives

$$\begin{aligned}
&2\pi(B(c, b+1-c))^p[M_p(r, \mathscr{P}^{b,c} f)]^p \\
&\quad = \int_0^{2\pi} |B(c, b+1-c)\mathscr{P}^{b,c} f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta \\
&\quad \leqslant K_3 \sum_{k=1}^{\infty} 2^{-k(b+1-c)p} \int_0^{2\pi} G_k^p(r,\theta)\,\mathrm{d}\theta \quad (\text{since } 0<p\leqslant 1) \\
&\quad \leqslant K_3 \sum_{k=1}^{\infty} 2^{-k(b+1-c)p} \int_0^{2\pi} \sup_{t\in(0,t_k)} \left|\frac{f(\mathrm{tr}\,\mathrm{e}^{\mathrm{i}\theta})}{(1-\mathrm{tr}\,\mathrm{e}^{\mathrm{i}\theta})^{b+1-c}}\right|^p \mathrm{d}\theta \\
&\quad \leqslant K_3 B \sum_{k=1}^{\infty} 2^{-k(b+1-c)p} M_p\left(t_k r, \frac{f}{(1-z)^{b+1-c}}\right)^p, \quad \text{Lemma 3.1(i)}, \\
&\quad \leqslant K_3 B K_4 \sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} (1-t)^{p(b+1-c)-1} M_p\left(\mathrm{tr}, \frac{f}{(1-z)^{b+1-c}}\right)^p \mathrm{d}t \\
&\quad \leqslant K_5 \int_0^1 (1-t)^{p(b+1-c)-1} M_p\left(t, \frac{f}{(1-z)^{b+1-c}}\right)^p \mathrm{d}t,
\end{aligned} \tag{3.2}$$

where $K_4 = p(b+1-c)/(1-2^{-p(b+1-c)})$ and $K_5 = K_3 B K_4$. Now, we choose $n>1$ such that $1-(b+1-c)p < 1/n < 1$ and $1/n + 1/m = 1$. Then by Hölders's inequality

$$\begin{aligned}
\left[M_p\left(t, \frac{f}{(1-z)^{b+1-c}}\right)\right]^p &= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f(t\mathrm{e}^{\mathrm{i}\theta})}{(1-t\mathrm{e}^{\mathrm{i}\theta})^{b+1-c}}\right|^p \mathrm{d}\theta \\
&\leqslant \frac{1}{2\pi}\left(\int_0^{2\pi} |f(t\mathrm{e}^{\mathrm{i}\theta})|^{np}\,\mathrm{d}\theta\right)^{1/n} \left(\int_0^{2\pi} |1-t\mathrm{e}^{\mathrm{i}\theta}|^{-(b+1-c)pm}\,\mathrm{d}\theta\right)^{1/m} \\
&\leqslant (2\pi)^{1/n-1} D^{1/m} M_{np}(t,f)^p (1-t)^{(-(b+1-c)pm+1)/m} \\
&\quad \text{(by Lemma 3.1(iii))}, \\
&= K_6 M_{np}(t,f)^p (1-t)^{-(b+1-c)p+1-1/n},
\end{aligned} \tag{3.3}$$

where $K_6 = (2\pi)^{1/n-1} D^{1/m}$. Finally, combining inequalities (3.2) and (3.3) we can quickly obtain

$$\begin{aligned}
2\pi(B(c,b+1-c))^p[M_p(r,\mathscr{P}^{b,c} f)]^p &\leqslant K_5 K_6 \int_0^1 (1-t)^{-1/n} M_{np}(t,f)^p\,\mathrm{d}t \\
&\leqslant C K_5 K_6 \|f\|_p^p,
\end{aligned}$$

where the second inequality is a consequence of Lemma 3.1(ii). This completes the proof. □

For $b = 1+\alpha$ and $c = 1$, we have

Corollary 3.6. *For any* α, $\operatorname{Re}\alpha > -1$, *the operator* $\mathscr{C}^{\alpha}$ *is bounded on* H^p, $0 < p \leqslant 1$.

In addition to the Hardy spaces, we are interested in two other spaces, namely, the Bloch space and the space BMOA.

The a-Bloch spaces B^a are defined for $a>0$ as

$$B^a=\left\{f\in\mathscr{H}:\|f\|_{B^a}=\sup_{z\in\Delta}(1-|z|)^a|f'(z)|<\infty\right\}.$$

In particular, the spaces B^a become the classical Lipschitz and Bloch spaces whenever $a\in(0,1)$ and $a=1$, respectively.

The space BMOA is defined to be the class of all analytic functions in H^2 such that

$$\sup_{z\in\Delta}\left\|f\left(\frac{z+a}{1+\bar{a}z}\right)-f(a)\right\|_2<\infty.$$

One of the basic properties of BMOA is that it is contained in the Bloch space. Moreover, the space BMOA equipped with the norm

$$\|f\|_{\mathrm{BMOA}}:=|f(0)|+\sup_{z\in\Delta}\left\|f\left(\frac{z+a}{1+\bar{a}z}-f(a)\right)\right\|_2,$$

is a Banach space.

We recall the property of Bloch function with nonnegative coefficients from [3,5].

Proposition 3.7. *If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ with $a_n\geqslant 0$, then f is Bloch if and only if $\sum_{n=N}^{2N}a_n=\mathrm{O}(1)$.*

For instance, $f(z)=-\log(1-z)=zF(1,1;2;z)$ is a Bloch function. Also, we note that f is convex univalent in Δ. Univalence of f is trivial because $\mathrm{Re}\, f'(z)>\frac{1}{2}>0$ in Δ. It is well known that an analytic function f is univalent and Bloch if and only if $f\in$ BMOA. Thus, $f\in$ BMOA.

A simple consequence of a result of Zhu [19, Proposition 7] is the following.

Proposition 3.8. *Suppose that $a>1$. Then f is in B^a if and only if $(1-|z|)^{a-1}f(z)$ is bounded in Δ.*

In view of Proposition 3.8, for $a>1$, we have an equivalent definition for B^a in the following form:

$$B^a=\left\{f\in\mathscr{A}:\|f\|'_{B^a}=\sup_{z\in D}(1-|z|)^{a-1}|f(z)|<\infty\right\}.$$

We require this equivalent form in the proof of part (ii) of Theorem 3.9.

Theorem 3.9. *Let $b,c\in\mathbb{C}$. Then we have the following*:

(i) *For* $\mathrm{Re}\,(b+1)>\mathrm{Re}\,c>0$, *there exists* $f\in$ BMOA *such that* $\mathscr{P}^{b,c}f$ *does not belong to* BMOA.

(ii) *For* $\mathrm{Re}\,(b+1)>\mathrm{Re}\,c\geqslant 1$, $\mathscr{P}^{b,c}$ *is a bounded operator from* B^a *to* B^a *for all a in* $(1,\infty)$.

Proof. (i) Consider the function

$$f_1(z)=-z^{-1}\log(1-z)=F(1,1;2;z).$$

Then, we note that f_1 is univalent and Bloch and therefore it is BMOA. Indeed univalence of f_1 follows from [11, Corollary 1.9(5)] and it belongs to BMOA because [8]

$$\frac{f_1^{(n)}(0)}{n!}=\frac{1}{n}.$$

Our aim is to prove that $\mathscr{P}^{b,c} f_1(z)$ is not in BMOA. As each function in BMOA is Bloch, to show that $\mathscr{P}^{b,c} f_1(z)$ is not in BMOA, it suffices to show that $\mathscr{P}^{b,c} f_1(z)$ is not in the Bloch space. For convenience, we assume b, c real and $b+1>c>0$. Now put $a_k = 1/(k+1)$ in the definition $\mathscr{P}^{b,c} f_1(z)$ so that

$$\mathscr{P}^{b,c} f_1(z) = \frac{1+b-c}{b} \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{b+1;c}} \sum_{k=0}^{n} \frac{A_{n-k}^{b;c}}{k+1} \right) z^n.$$

It is known that [4]

$$A_n^{b;c} = \frac{\Gamma(c)}{\Gamma(b)}\, n^{b-c} \left\{ 1 + \mathrm{O}\left(\frac{1}{n}\right) \right\} \quad \text{as} \quad n \to \infty.$$

To complete the proof, by Proposition 3.7, it is enough to show that

$$S_N = \sum_{n=N}^{2N} B_n \neq \mathrm{O}(1), \quad \text{where} \quad B_n = \frac{1}{n^{b+1-c}} \sum_{k=0}^{n} \frac{(n-k)^{b-c}}{k+1}.$$

Now proceeding exactly as in [18] we complete the proof.

(ii) By using Proposition 3.8, to prove the theorem it is sufficient to show that

$$|\mathscr{P}^{b,c} f(z)| \leqslant \frac{K \|f\|'_{B^a}}{(1-|z|)^{a-1}}$$

for some positive constant K. As usual, we deal with the case when b, c are real and $b+1>c\geqslant 1$ since the proof for the complex case follows easily, for example, as in Theorem 3.2. Define

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \lambda(t) = \frac{1}{B(c, b+1-c)} t^{c-1}(1-t)^{b-c}.$$

Then the integral representation of $P^{b,c} f(z)$ takes an equivalent form given by

$$\mathscr{P}^{b,c} f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{(1-tz)^{b+1-c}} F(c-1, c-b-1; c; tz)\, \mathrm{d}t.$$

Now

$$\begin{aligned}
|\mathscr{P}^{b,c} f(z)| &\leqslant \int_0^1 \lambda(t) \frac{|f(tz)|}{|(1-tz)|^{b+1-c}} |F(c-1, c-b-1; c; tz)|\, \mathrm{d}t \\
&\leqslant K \int_0^1 t^{c-1}(1-t)^{b-c} \frac{|f(tz)|}{|1-tz|^{b+1-c}}\, \mathrm{d}t \\
&\leqslant K \|f\|'_{B^a} \int_0^1 \frac{t^{c-1}(1-t)^{b-c}}{|1-tz|^{b+1-c}(1-t|z|)^{a-1}}\, \mathrm{d}t \\
&= K \|f\|'_{B^a} \int_0^{|z|} \frac{t^{c-1}(1-t)^{b-c}}{|1-tz|^{b+1-c}(1-t|z|)^{a-1}}\, \mathrm{d}t \\
&\quad + \int_{|z|}^1 \frac{t^{c-1}(1-t)^{b-c}}{|(1-tz)|^{b+1-c}(1-t|z|)^{a-1}}\, \mathrm{d}t.
\end{aligned}$$

For $t \in (0,1)$ and $|z|<1$, we have $|1-tz|^{-1}<(1-t)^{-1}$ and $(1-|z|)^{-1} \geqslant |1-tz|^{-1}$. Therefore, we see that

$$|\mathscr{P}^{b,c} f(z)| \leqslant K \|f\|'_{B^a} \left[\int_0^{|z|} t^{c-1}(1-t)^{-a}\,\mathrm{d}t + \int_{|z|}^1 \frac{t^{c-1}(1-t)^{b-c}}{(1-|z|)^{a+b-c}}\,\mathrm{d}t \right].$$

For $0<t<|z|$, the first integral on the right of the last inequality gives the estimate

$$\int_0^{|z|} t^{c-1}(1-t)^{-a}\,\mathrm{d}t \leqslant \frac{1}{(a-1)(1-|z|)^{a-1}}$$

and, for $|z|<t\leqslant 1$, the second integral gives the estimate

$$\int_{|z|}^1 \frac{t^{c-1}(1-t)^{b-c}}{(1-|z|)^{a+b-c}}\,\mathrm{d}t \leqslant \frac{1}{(1-|z|)^{a-1}}.$$

Using these two inequalities, we can easily obtain that

$$|\mathscr{P}^{b,c} f(z)| \leqslant \frac{K'\|f\|'_{B^a}}{(1-|z|)^{a-1}}.$$

The desired conclusion follows if we use the definition of the norm on B^a. $\square$

Remark. We note that the operator $\mathscr{P}^{b,c}$ does not map H^∞ functions to H^∞. This may be seen by applying $\mathscr{P}^{b,c}$ to the function $f \equiv 1$, arguing as in [18].

4. Remarks and an open question

If $a \neq 1$ and $c=a+b$, then (1.3) is equivalent to

$$\frac{F(a-1,b;a+b;z)}{1-z} = F(a,b+1;a+b;z), \quad a \neq 1. \tag{4.1}$$

A comparison of the coefficient of z^n on both sides of (4.1) shows that

$$\frac{1}{A_n^{a,b+1;a+b}} \sum_{k=0}^n A_{n-k}^{a-1,b;a+b} = 1, \quad n \in \mathbb{N}\cup\{0\}.$$

As in the case $a=1$, $c \neq a+b$, we can consider the mean

$$\frac{1}{A_n^{a,b+1;a+b}} \sum_{k=0}^n A_{n-k}^{a-1,b;a+b} a_k, \quad n \in \mathbb{N}\cup\{0\}$$

and form the set of new averaging operators defined by

$$\mathscr{Q}^{b,c} f(z) := \sum_{n=0}^\infty \left(\frac{1}{A_n^{a,b+1;a+b}} \sum_{k=0}^n A_{n-k}^{a-1,b;a+b} a_k \right) z^n, \tag{4.2}$$

where $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is analytic on the unit disc Δ. Again, it is not difficult to show that the right-hand side of (4.2) represents an analytic function on Δ. In fact, (4.2) can be rewritten as

$$\mathcal{Q}^{b,c}f(z)=[f(z)(1-z)F(a-1,b;a+b;z)]*{}_3F_2(1,1,a+b;a,b+1;z),$$

where $a\neq 1$. Here ${}_pF_q$ represents the generalized hypergeometric function defined by

$${}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;z)=\sum_{n=0}^{\infty}\frac{(a_1,n)\cdots(a_p,n)}{(c_1,n)\cdots(a_q,n)}\frac{z^n}{n!}.$$

We remark that for $p\leqslant q$, the series converges for $z\in\mathbb{C}$. When $p>q+1$, the series diverges for $z\in\mathbb{C}\setminus\{0\}$ unless the series breaks off into a polynomial. In the interesting case where $p=q+1$, the series converges for $|z|<1$. If Re $(\sum_{j=1}^{q}c_j-\sum_{j=1}^{q+1}a_j)>0$, then ${}_{q+1}F_q$ converges also at the point $z=1$. It would be interesting to know whether $\mathcal{Q}^{b,c}f$ is bounded on Hardy spaces and other function spaces.

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