Int. Journal of Math. Analysis, Vol. 1, 2007, no. 22, 1051 - 1057

Goodman-Salagean-Type Harmonic Univalent Functions with Varying Arguments

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Abstract. The aim of the present paper is to study a certain subclass of harmonic univalent functions with varying arguments defined by Salegean operator. For this class we determine a sufficient coefficient condition, representation theorem, distortion theorem, extreme points.

Mathematics Subject Classification: Primary 30C45; Secondary 30C80

Keywords: Harmonic function, uniformaly convex function, distortion, extreme point

1. Introduction

The harmonic functions are famous for their natural role in parameterizing minimal surface and have been studied by differential geometers such as Choqest [1], Kneser [7].

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. In this case, the Jacobian of $f = h + \bar{g}$, is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.

The mapping $z \mapsto f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f(z) > 0$ in D. The necessity of this condition is a result of Lewy [8].

Let denote by H the family of functions $f = h + \bar{g}$ which are harmonic, orientation preserving, and univalent in the open unit disc $U = \{z : |z| < 1\}$ where h and g are given

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

The differential operator D^m was introduced by Salagean. For $f = h + \bar{g}$ given by (1,1), Jahangiri et al. [3] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}, \tag{1.2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k, \quad and \quad D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

Goodman [2] defined uniformaly convex (UCV) functions and Kanas and Srivastava [6] generalized that class to the class K-ST consisting of functions so that its analytic characterization is $\psi \in K - ST$ if and only if

$$\Re\left(\frac{z\psi'(z)}{\psi(z)}\right) > k\left|\frac{z\psi'(z)}{\psi(z)}\right|, \quad k \geq 0.$$

Generalizing the class K-ST to include harmonic functions, we let $S_H(m;n;\alpha;\rho)$ denote the subclass of H consisting of functions $f = h+\bar{g}$ that satisfy the condition

$$\Re\left\{(1+\rho e^{i\gamma})\frac{D^m f(z)}{D^n f(z)} - \rho e^{i\gamma}\right\} > \alpha, \tag{1.3}$$

where $0 \leq \alpha < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, z \in U, 0 \leq \rho \leq 1$ and γ is real.

Also we let the subclass $V_{\bar{H}}(m;n;\alpha;\rho)$ consist of harmonic functions $f_m = h + \bar{g}_m$ in $S_H(m;n;\alpha;\rho)$ so that h and g_m are the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \tag{1.4}$$

and there exist ϕ such that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi, k \geq 2 \quad and \quad \arg(b_k) + (k+1)\phi \equiv (m-1)\pi, k \geq 1. \tag{1.5}$$

The classes $S_H(m;n;\alpha;\rho)$ and $V_{\bar{H}}(m;n;\alpha;\rho)$ includes a variety of well-known subclass of H which is studied by several authors. For example see, [3], [4], [5], [9], [10],[11], [12].

In [4] it is proved that $f_1 = h + \bar{g}_1 \in V_{\bar{H}}(1;0;\alpha;0)$ if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 1 - \frac{1+\alpha}{1-\alpha}|b_1|.$$

In this paper we provide necessary and sufficient condition, distortion theorems, and extreme points for the functions in $V_{\bar{H}}(m;n;\alpha;\rho)$.

2. Main Results

First we give a theorem that provides necessary coefficient bound for harmonic functions to be in $S_H(m;n;\alpha;\rho)$

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1,1). Furthermore, let*

$$\sum_{k=1}^{\infty}\left(\frac{(1+\rho)k^m - k^n(\alpha+\rho)}{1-\alpha}|a_k| + \frac{(1+\rho)k^m - (-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}|b_k|\right) \le 2, \tag{2.1}$$

where $a_1 = 1, 0 \le \alpha < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, z \in U, 0 \le \rho \le 1$. Then f is sense-preserving harmonic univalent function in U and $f \in S_H(m;n;\alpha;\rho)$.

Proof. If the inequality (2.1) holds for the coefficients of $f = h + \bar{g}$, then by (1.6), f is sense-preserving harmonic univalent function in U. So it remains to show that

$$\Re\left\{(1+\rho e^{i\gamma})\frac{D^m f(z)}{D^n f(z)} - \rho e^{i\gamma}\right\} > \alpha.$$

Using the fact $\Re\omega > \alpha$ if and only if $|1-\alpha+\omega| > |1+\alpha-\omega|$, it suffices to show that $A > B$ where $A = |(1-\alpha-\rho e^{i\gamma})D^n f(z) + (1+\rho e^{i\gamma})D^m f(z)|$ and $B = |(1+\alpha+\rho e^{i\gamma})D^n f(z) - (1+\rho e^{i\gamma})D^m f(z)|$. Substituting for D^n and D^m in A and B yields,

$$A - B > (2-\alpha)|z| - \sum_{k=2}^{\infty}[k^m + k^n(1-\alpha) + \rho(k^m - k^n)]|a_k||z| -$$

$$\sum_{k=1}^{\infty}[k^m + (-1)^{m-n}k^n(1-\alpha) + \rho(k^m - (-1)^{m-n}k^n)]|b_k||z| - \alpha|z|$$

$$-\sum_{k=2}^{\infty}[k^m(1+\rho) - k^n(1+\alpha+\rho)]|a_k||z|$$

$$+\sum_{k=1}^{\infty}[k^m(1+\rho) - (-1)^{m-n}k^n(1+\alpha+\rho)]|b_k||z|$$

$$= 2(1-\alpha)|z|\{1 - \sum_{k=2}^{\infty}\frac{(1+\rho)k^m - k^n(\alpha+\rho)}{1-\alpha}|a_k|$$

$$-\sum_{k=1}^{\infty}\frac{(1+\rho)k^m - (-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}|b_k|\}$$

The last expression is non-negative by (2.1) and so $f \in S_H(m;n;\alpha;\rho)$. □

Theorem 2.2. *Let $f_m = h + \bar{g}_m$ be given by (1.4). Then $f_m \in V_{\bar{H}}(m;n;\alpha;\rho)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{(1+\rho)k^m - k^n(\alpha+\rho)}{1-\alpha}|a_k| + \frac{(1+\rho)k^m - (-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}|b_k|\right) \leq 2, \tag{2.2}$$

where $a_1 = 1, 0 \leq \alpha < 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, z \in U, 0 \leq \rho \leq 1$.

Proof. Since $V_{\bar{H}}(m;n;\alpha;\rho) \in S_H(m;n;\alpha;\rho)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f_m \in V_{\bar{H}}(m;n;\alpha;\rho)$, we notice that the condition

$$\Re\left\{(1+\rho e^{i\gamma})\frac{D^m f_m(z)}{D^n f_m(z)} - \rho e^{i\gamma}\right\} > \alpha,$$

is equivalent to

$$\Re\frac{(1+\rho e^{i\gamma})[z + \sum_{k=2}^{\infty} k^m a_k z^k + (-1)^m \sum_{k=1}^{\infty} k^m \bar{b}_k \bar{z}^k]}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k}$$

$$-\Re\frac{(\alpha+\rho e^{i\gamma})[z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k]}{z + \sum_{k=2}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k} > 0$$

The above condition hold for all values of z in U and for all real γ. Upon choosing ϕ according (1.5) and substituting $z = re^{i\phi}, (0 < r < 1)$, and $\gamma = 0$, we must have

$$\frac{E}{1 - [\sum_{k=2}^{\infty} k^n|a_k| + (-1)^n \sum_{k=1}^{\infty} k^n(-1)^{m-1}|b_k|]r^{k-1}} > 0, \tag{2.3}$$

where

$$E = (1-\alpha) - \left(\sum_{k=2}^{\infty}[k^m(1+\rho) - (\rho+\alpha)k^n]|a_k|\right) r^{k-1}$$

$$-\left(\sum_{k=1}^{\infty}[(-1)^m k^m(1+\rho) - (-1)^n(\rho+\alpha)k^n](-1)^m|b_k|\right) r^{k-1}.$$

If the condition (2.2) does not hold then E is negative for r sufficiently close to 1. Hence there exist a $z_0 = r_0$ in (0,1) for which E is negative. But this is contradiction and so the proof is complete. □

The following theorem gives the distortion bounds for functions in $V_{\bar{H}}(m;n;\alpha;\rho)$, which yields covering result for this class.

Theorem 2.3. *If $f_m \in V_{\bar{H}}(m;n;\alpha;\rho)$, then*

$$|f_m(z)| \leq (1+|b_1|)r + \frac{1-\alpha-(1+\rho-(-1)^{m-n}(\alpha+\rho))|b_1|}{(1+\rho)2^m - (\alpha+\rho)2^n}r^2,$$

and

$$|f_m(z)| \geq (1+|b_1|)r - \frac{1-\alpha-(1+\rho-(-1)^{m-n}(\alpha+\rho))|b_1|}{(1+\rho)2^m-(\alpha+\rho)2^n} r^2.$$

Proof. We will only proof the right hand inequality. The proof for the left hand inequality is similar. Let $f_m \in V_{\bar{H}}(m;n;\alpha;\rho)$. Taking the absolute value of f_m, we obtain

$$\begin{aligned}
|f_m(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(1+\rho)2^m-(\alpha+\rho)2^n}[\sum_{k=2}^{\infty}\frac{(1+\rho)k^m-(\alpha+\rho)k^n}{1-\alpha}|a_k| \\
&+\sum_{k=2}^{\infty}\frac{(1+\rho)k^m-(-1)^{m-n}(\alpha+\rho)k^n}{1-\alpha}|b_k|]r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{(1+\rho)2^m-(\alpha+\rho)2^n}(1-\frac{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}{1-\alpha}|b_1|)r^2 \\
&= (1+|b_1|)r + \frac{1-\alpha-(1+\rho-(-1)^{m-n}(\alpha+\rho))|b_1|}{(1+\rho)2^m-(\alpha+\rho)2^n},
\end{aligned}$$

and the proof is complete. □

Theorem 2.4. *The closed convex hull of $V_{\bar{H}}(m;n;\alpha;\rho)$ denoted by $clcoV_{\bar{H}}(m;n;\alpha;\rho)$ is*

$$\{f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \in S_H(m;n;\alpha;\rho)$$

$$\sum_{k=1}^{\infty}(\frac{(1+\rho)k^m-k^n(\alpha+\rho)}{1-\alpha}|a_k| + \frac{(1+\rho)k^m-(-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}|b_k|) \leq 2\},$$

where $a_1 = 1$. Set $\lambda_m = \frac{1-\alpha}{(1+\rho)k^m-k^n(\alpha+\rho)}$ and $\mu_m = \frac{1-\alpha}{(1+\rho)k^m-(-1)^{m-n}k^n(\alpha+\rho)}$. For b_1 fixed, $0 < |b_1| < \frac{1-\alpha}{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}$, the extreme points for $V_{\bar{H}}(m;n;\alpha;\rho)$ are

$$\left\{z + \lambda_k x z^k + \bar{b}_1 z\right\} \bigcup \left\{\overline{z + \mu_k x z^k + b_1 z}\right\}, \tag{2.4}$$

where $k \geq 2$ and $|x| = 1 - \frac{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}{1-\alpha}$.

Proof. Let function f in $S_H(m;n;\alpha;\rho)$ is written as

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|e^{i\delta_k}z^k + \bar{b}_1 z + \overline{\sum_{k=1}^{\infty}|b_k|e^{i\beta_k}z^k},$$

where the coefficients satisfy the inequality (2.2).

Set $h_1(z)=z, g_1(z)=b_1z, h_m(z)=z+\lambda_m e^{i\delta_m}z^m, g_m(z)=b_1z+\mu_m e^{i\beta_m}z^m$, for $m=2,3,...$. Writing $X_k=\frac{|a_k|}{\lambda_k}, Y_k=\frac{|b_k|}{\mu_k}, k=2,3,...$ and $X_1=1-\sum_{k=2}^{\infty}X_k, Y_1=1-\sum_{k=2}^{\infty}Y_k$ we have,

$$f(z)=\sum_{k=1}^{\infty}(X_kh_k(z)+\overline{Y_kg_k(z)}).$$

In particular, setting

$$f_1(z)=z+\bar{b}_1z,$$

and

$$f_k(z)=z+\lambda_kxz^k+\overline{b_1(z)}+\overline{\mu_kyz^k},$$

$$(k\geq 2, |x|+|y|=1-\frac{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}{1-\alpha}|b_1|),$$

we see that extreme points of $V_{\bar{H}}(m;n;\alpha;\rho)$ are contained $\{f_m(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z)=\frac{1}{2}\{f_1(z)+\lambda(1-\frac{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}{1-\alpha}|b_1|)z^2\}$$

$$+\frac{1}{2}\{f_1(z)-\lambda(1-\frac{(1+\rho)-(-1)^{m-n}(\alpha+\rho)}{1-\alpha}|b_1|)z^2\},$$

a convex linear combination of functions $V_{\bar{H}}(m;n;\alpha;\rho)$. Next we will show if both $|x|\neq 0$ and $|y|\neq 0$, then f_k is not an extreme point. Without loss of generality, assume $|x|\geq|y|$. Choose $\epsilon>0$ small enough so that $\epsilon<\frac{|x|}{|y|}$. Set $A=1+\epsilon$ and $B=1-|\frac{\epsilon x}{y}|$, we then see that both

$$t_1(z)=z+\lambda_kAxz^k+\overline{b_1z+\mu_kyBz^k},$$

and

$$t_2(z)=z+\lambda_k(2-A)xz^k+\overline{b_1z+\mu_ky(2-B)z^k},$$

are in $V_{\bar{H}}(m;n;\alpha;\rho)$ and note that

$$f_k(z)=\frac{1}{2}(t_1(z)+t_2(z)).$$

The extremal coefficient bounds shows that functions of the form (2.4) are the extreme point for $V_{\bar{H}}(m;n;\alpha;\rho)$, and so the proof is complete. □

Now we will examine the closure properties of the class $V_{\bar{H}}(m;n;\alpha;\rho)$ under the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ which is defined by $L_c(f(z))=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad c>-1$.

Theorem 2.5. *Let $f_m\in V_{\bar{H}}(m;n;\alpha;\rho)$. Then $L_c(f_m(z))$ belongs to the class $V_{\bar{H}}(m;n;\alpha;\rho)$.*

Proof. From the representation of $L_c(f_m(z))$, it follows that

$$L_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1}\{h(t)+\bar{g}(t)\}dt$$
$$= \frac{c+1}{z^c}\int_0^z t^{c-1}\{t+\sum_{k=2}^{\infty} a_k t^k + \overline{\sum_{k=1}^{\infty} b_k t^k}\}dt$$
$$= z + \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}.$$

where $A_k = \frac{c+1}{c+k}a_k$, $B_k = \frac{c+1}{c+k}b_k$. Therefore,

$$\sum_{k=1}^{\infty}\left(\frac{(1+\rho)k^m - k^n(\alpha+\rho)}{1-\alpha}\frac{c+1}{c+k}|a_k| + \frac{(1+\rho)k^m-(-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}\frac{c+1}{c+k}|b_k|\right)$$

$$\leq \sum_{k=1}^{\infty}\left(\frac{(1+\rho)k^m - k^n(\alpha+\rho)}{1-\alpha}|a_k| + \frac{(1+\rho)k^m-(-1)^{m-n}k^n(\alpha+\rho)}{1-\alpha}|b_k|\right) \leq 2,$$

and the proof is complete. □

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Received: April 4, 2007