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Stability for the class of uniformly starlike functions with respect to symmetric points

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Abstract In this paper we investigate the problem of stability for the class of uniformly starlike functions with respect to symmetric points and we give the lower bounds of their radius of stability.

Keywords Stability of Hadamard product · Convolution · Uniformly starlike and convex functions · Symmetric points

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1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. As usual, we denote by S the subclass of $\mathcal{A}$ consisting of functions which are univalent in U. Also let K and S^* denote the family of all convex and starlike functions in U.

Sakaguchi [14] once introduced the concept of starlike functions with respect to the symmetric points, that is, the class of functions $f(z)$ analytic in U and satisfying

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$f(0) = 0$, $f'(0) = 1$, and

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0 \quad (|z| < 1).$$

Following him, many mathematicians discussed this class and its subclasses, see [3,7]. In this paper by following the recent work of Reddy and Reddy [8] we consider the following class:

For $\beta > 0$, let

$$SP(\beta) = \left\{f \in \mathcal{A} : \left|\frac{2zf'(z)}{f(z)-f(-z)} - \beta\right| \le Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) + \beta, z \in U\right\}.$$

This implies that $f \in SP(\beta)$ if and only if $\frac{2zf'(z)}{f(z)-f(-z)}$ lies in the region Ω bounded by a parabola with vertex at the origin and parameterized by $\frac{t^2+4i\beta t}{4\beta}$ for any real t. Rønning in [9] has shown that the function

$$Q_\beta(z) = \beta\left[1 + \frac{4}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right]$$

maps the unit disc U onto the parabolic region Ω (The branch of square root is chosen so that $Im\sqrt{z} \ge 0$). Then from the above definition $f \in \mathcal{A}$ is in the class $SP(\beta)$ if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec Q_\beta(z)$$

where $\prec$ denotes subordination.

The basic operations that we shall encounter frequently is the usual Hadamard product (or convolution) $f * g$ of two analytic functions $f, g \in \mathcal{A}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \Rightarrow (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Note that $f * g$ is in $\mathcal{A}$. Also the integral convolution is defined by

$$(f \otimes g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n.$$

We use the notation $\mathcal{A}_1 * \mathcal{A}_2$ to denote the set of all $f * g$ where $f \in \mathcal{A}_1$ and $g \in \mathcal{A}_2$. Here $\mathcal{A}_1$ and $\mathcal{A}_2$ are two subclasses of $\mathcal{A}$.

The concept of dual set has proved to be very useful in the study of properties of analytic functions, (see for example [1,2,4–6]).

Recall that the dual set of $V \subset \mathcal{A}$ (cf. [10,11,13]) is denoted by V^* and is defined by

$$V^* = \left\{g \in \mathcal{A} : \frac{(f * g)(z)}{z} \neq 0, \quad \forall f \in V, \quad \forall z \in U\right\}.$$

Denote by $G(\beta)$ the dual set of $SP(\beta)$. It is easy to see that

$$f \in SP(\beta) \Leftrightarrow \frac{(f * g)(z)}{z} \neq 0, \quad \forall g \in G(\beta), \quad \forall z \in U. \tag{1.1}$$

For $\delta \geq 0$ Ruscheweyh in [12] defined N_δ-neighborhood of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ by

$$N_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} n|b_n - a_n| \leq \delta \right\}.$$

We denote by $N_\delta(\mathcal{A}_1)$, $\mathcal{A}_1 \subset \mathcal{A}$, the union of all neighborhoods $N_\delta(f)$ ranging over the class $\mathcal{A}_1$.

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the subclasses of the class $\mathcal{A}$ and $\mathcal{A}_1 * \mathcal{A}_2 \subseteq \mathcal{A}_3$. The Hadamard product is called stable on the pair of classes $(\mathcal{A}_1, \mathcal{A}_2)$ if there exists $\delta > 0$ such that $N_\delta(\mathcal{A}_1) * N_\delta(\mathcal{A}_2) \subset \mathcal{A}_3$ and unstable otherwise (see [4]). Stability of the pair of classes $(\mathcal{A}_1, \mathcal{A}_2)$ for different choices of the subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ are studied by Kanas [4] and Bednarz and Sokół in [2].

Finally let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the subclasses of the class $\mathcal{A}$ and $\mathcal{A}_1 * \mathcal{A}_2 \subseteq \mathcal{A}_3$. Then a constant $\delta(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{A}_3) = \sup\{\delta : N_\delta(\mathcal{A}_1) * N_\delta(\mathcal{A}_2) \subset \mathcal{A}_3\}$ is called the radius of stability of the convolution pair $(\mathcal{A}_1, \mathcal{A}_2)$.

Motivation of the recent works of Kanas [4] and Bednarz and Sokół [2] in this paper we investigate the problem of stability for the class of uniformly starlike functions with respect to symmetric points and we give the lower bounds of their radius of stability.

2 Main results

In order to establish our main results, we shall require the following lemmas.

Lemma 2.1 (see [8]) *A function f in $\mathcal{A}$ is in $SP(\beta)$ if and only if for all z in U and for all h_β, $\frac{(f*h_\beta)(z)}{z} \neq 0$ where*

$$h_\beta(z) = \frac{4\beta}{4\beta - 4i\beta t - t^2}\left[\frac{z}{(1-z)^2} - \frac{t^2 + 4i\beta t}{4\beta}\frac{z}{(1-z^2)}\right].$$

Lemma 2.2 (see [8]) *Let $h_\beta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in G(\beta)$, then $|c_n| \leq n$ for $n = 2, 3, \ldots$.*

Lemma 2.3 (see [8]) *For $f \in \mathcal{A}$ and for every $\epsilon \in \mathbb{C}$ such that $|\epsilon| < \delta$ if $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in SP(\beta)$, then for every $h_\beta \in G(\beta)$ we have $\left|\frac{(f*h_\beta)(z)}{z}\right| \neq \delta$, $(z \in U)$.*

Lemma 2.4 (see [8]) *Let $f \in K$ and $g \in SP(\beta)$. Then $f * g \in SP(\beta)$.*

From these lemmas we have the following corollaries:

Corollary 2.1 *Let $f(z) = z + Az^n$ then*

$$|A| \leq \frac{1}{n} \Rightarrow f \in SP(\beta).$$

Corollary 2.2 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If $\sum_{n=2}^{\infty} n|a_n| < 1$, then $f \in SP(\beta)$*

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and $h_\beta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in G(\beta)$, then we have

$$\left|\frac{(f*h_\beta)(z)}{z}\right| = \left|1 + \sum_{n=2}^{\infty} a_n c_n z^{n-1}\right|$$
$$\geq 1 - \sum_{n=2}^{\infty} |a_n||c_n||z| > 1 - \sum_{n=2}^{\infty} |a_n||c_n| > 0.$$

Thus $\frac{(f*h_\beta)(z)}{z} \neq 0$ and from Lemma 2.1 we have $f \in SP(\beta)$. □

Definition 2.1 A function $f \in \mathcal{A}$ is said to be in the class $CSP(\beta)$ if for all $z \in U$, $zf'(z) \in SP(\beta)$.

Lemma 2.5 *Let $f(z) \in CSP(\beta)$, then for ϵ with $(|\epsilon| < \frac{1}{4})$, $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in SP(\beta)$.*

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CSP(\beta)$, then

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon} = \frac{(1+\epsilon)z + \sum_{n=2}^{\infty} a_n z^n}{1+\epsilon}.$$

Now by using the property $f(z) * \frac{z}{1-z} = f(z)$ we have

$$F_\epsilon(z) = \frac{f(z) * ((1+\epsilon)z + \sum_{n=2}^{\infty} z^n)}{1+\epsilon} = (f * \phi)(z), \tag{2.1}$$

where $\phi(z) = \frac{z - \frac{\epsilon}{1+\epsilon} z^2}{1-z}$. Also we have

$$f(z) = zf' * \log \frac{1}{1-z},$$

and so

$$(f * \phi)(z) = zf' * \log \frac{1}{1-z} * \phi(z) = zf' * \left(\log \frac{1}{1-z} * \phi(z)\right). \tag{2.2}$$

For $|\epsilon| < \frac{1}{4}$, we have $\phi(z) \in S^*$, because

$$\frac{z\phi'(z)}{\phi(z)} = 1 - \frac{\epsilon}{1+\epsilon} \frac{z^2}{z - \frac{\epsilon}{1+\epsilon} z^2} + \frac{z}{1-z} = 1 + \frac{1}{1-z} - \frac{1}{1 - \frac{\epsilon}{1+\epsilon} z}.$$

Hence for $|\epsilon| < \frac{1}{4}$,

$$Re \frac{z\phi'(z)}{\phi(z)} = Re \left\{1 + \frac{1}{1-z} - \frac{1}{1 - \frac{\epsilon}{1+\epsilon} z}\right\} > \frac{3}{2} - \frac{1}{1 - \frac{|\epsilon|}{1-|\epsilon|}} > 0.$$

That means the function ϕ is starlike or the function Φ is given by $\Phi(z) = \phi(z) * \log \frac{1}{1-z} = \int_0^z \frac{\phi(t)}{t} dt$, is convex. Now by making use of (2.1) and (2.2) we have $F_\epsilon(z) = zf'(z) * \Phi(z)$. Since $\Phi \in K$, so Lemma 2.4 implies that $F_\epsilon \in SP(\beta)$ and the proof is complete. □

Lemma 2.6 *Let $f \in CSP(\beta)$ and $h_\beta \in G(\beta)$, then $\left|\frac{(f*h_\beta)(z)}{z}\right| \geq \frac{1}{4}$.*

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CSP(\beta)$ and $h_\beta(z) \in G(\beta)$, then from Lemma 2.5 for $|\epsilon| < \frac{1}{4}$ we have $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in SP(\beta)$. Thus

$$\frac{1}{z}[h_\beta(z) * F_\epsilon(z)] \neq 0, \quad |\epsilon| < \frac{1}{4}.$$

Now from the properties of Hadamard product we obtain

$$\frac{1+\epsilon}{z}\left[h_\beta(z) * \frac{f(z)+\epsilon z}{(1+\epsilon)}\right] = \frac{1}{z}[h_\beta(z) * (f(z)+\epsilon z)] = \frac{1}{z}[h_\beta(z) * f(z)] + \epsilon \neq 0, \quad |\epsilon| < \frac{1}{4}.$$

Hence for $|\epsilon| < \frac{1}{4}$, $\frac{1}{z}[h_\beta(z) * f(z)] \neq -\epsilon$, and so $|\frac{(f*h_\beta)(z)}{z}| \geq \frac{1}{4}$. □

Theorem 2.1 *For* $0 < \delta \le \sqrt{(1+\frac{4\beta}{\pi^2})^2+1}-(1+\frac{4\beta}{\pi^2})$ *we have*

$$N_\delta\,(SP(\beta)) \otimes N_\delta(K) \subset SP(\beta).$$

Proof Let $f_0(z) = z + \sum_{n=2}^{\infty} a_{0n} z^n \in SP(\beta)$ and $g_0(z) = z + \sum_{n=2}^{\infty} b_{0n} z^n \in K$. Also let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N_\delta(f_0)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in N_\delta(g_0)$. We know that $f \otimes g \in SP(\beta)$ if and only if

$$\frac{((f \otimes g) * h_\beta)(z)}{z} \neq 0, \quad (h_\beta \in G(\beta)).$$

From the Hadamard product properties we have

$$(f \otimes g) * h_\beta = (f_0 \otimes g_0) * h_\beta + [f_0 \otimes (g - g_0)] * h_\beta \\ + [(f - f_0) \otimes g_0] * h_\beta + [(f - f_0) \otimes (g - g_0)] * h_\beta, \tag{2.3}$$

and so we have

$$\left|\frac{((f \otimes g) * h_\beta)(z)}{z}\right| \ge \left|\frac{[(f_0 \otimes g_0)] * h_\beta}{z}\right| - \left|\frac{[f_0 \otimes (g - g_0)] * h_\beta}{z}\right| \\ - \left|\frac{[(f - f_0) \otimes g_0] * h_\beta}{z}\right| - \left|\frac{[(f - f_0) \otimes (g - g_0)] * h_\beta}{z}\right|. \tag{2.4}$$

In view of $f_0 \in SP(\beta)$, $g_0 \in K$ and Lemma 2.4 we conclude that $f_0 * g_0 \in SP(\beta)$ or $z(f_0 \otimes g_0)' \in SP(\beta)$ and this means $f_0 \otimes g_0 \in CSP(\beta)$. Now from Lemma 2.5 we have

$$\left|\frac{(f_0 \otimes g_0) * h_\beta}{z}\right| \ge \frac{1}{4}. \tag{2.5}$$

On the other hand it is well known that the coefficient of convex functions are less than or equal 1. Since $g_0 \in K$ thus $|b_{0n}| \le 1$. Also by making use of Theorem 1 in [15] and Lemma 2.2 for $h_\beta(z) = z + \sum_{n=2}^{\infty} c_n z^n$ we obtain $|a_{0n}| \le \frac{4\beta}{\pi^2}$ and $|c_n| \le n$ (respectively).

Now the definitions of $N_\delta(f_0)$ and $N_\delta(g_0)$ imply that

$$\sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||c_n|}{n} \le \delta \frac{2\beta}{\pi^2}, \tag{2.6}$$

$$\sum_{n=2}^{\infty} \frac{|a_{0n} - a_n||b_{0n}||c_n|}{n} \le \frac{\delta}{2}, \tag{2.7}$$

and

$$\sum_{n=2}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||c_n|}{n} \le \frac{\delta^2}{4} \tag{2.8}$$

By subsisting the relations (2.5), (2.6), (2.7) and (2.8) on (2.4) we have

$$\left|\frac{((f \otimes g) * h_\beta)(z)}{z}\right| \ge \frac{1}{4} - \frac{2\delta\beta}{\pi^2} - \frac{\delta}{2} - \frac{\delta^2}{4}. \tag{2.9}$$

The right side of (2.9) is positive whenever $0 \le \delta < \sqrt{(1+\frac{4\beta}{\pi^2})^2+1}-(1+\frac{4\beta}{\pi^2})$ and the proof is complete. □

Corollary 2.3 *We have* $\delta\,(SP(\beta)) \otimes CV, SP(\beta)) \ge \sqrt{(1+\frac{4\beta}{\pi^2})^2+1}-(1+\frac{4\beta}{\pi^2})$.

Let I be an identity function.

Theorem 2.2 *We have*

i) $I * I \subset SP(\beta)$,
ii) *for* $\delta = \sqrt{2}$, $N_\delta(I) * N_\delta(I) \subset SP(\beta)$,
iii) $\delta(I * I, SP(\beta)) \geq \sqrt{2}$.

Proof i) The proof of i) is easy and we omit it.
ii) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N_\delta(I)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in N_\delta(I)$. By making use of definition $N_\delta(I)$ we have $\sum_{n=2}^{\infty} n|a_n| \leq \delta$ and $\sum_{n=2}^{\infty} n|b_n| \leq \delta$. Now let $h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} c_n z^n$ then

$$\sum_{n=2}^{\infty} n|c_n| = \sum_{n=2}^{\infty} n|a_n||b_n| \leq \frac{\delta}{2} \sum_{n=2}^{\infty} n|b_n| \leq \frac{\delta^2}{2} \leq 1$$

and from Corollary 2.2, $h(z) \in SP(\beta)$.
iii) From ii) it is clear.

□

By using the same techniques as in the proof of Theorem 2.2 we obtain the following theorem and we omit details.

Theorem 2.3 *We have*

i) $I \otimes I \subset SP(\beta)$,
ii) *for* $\delta = 2$, $N_\delta(I) \otimes N_\delta(I) \subset SP(\beta)$
iii) $\delta(I \otimes I, SP(\beta)) \geq 2$.

Finally we proof

Theorem 2.4 *We have*

i) $I * SP(\beta) \subset SP(\beta)$,
ii) *for* $0 < \delta \leq \sqrt{\frac{16\beta^2}{\pi^4} + 2} - \frac{4\beta}{\pi^2}$, $N_\delta(I) * N_\delta(SP(\beta)) \subset SP(\beta)$,
iii) $\delta(I * SP(\beta), SP(\beta)) \geq \sqrt{\frac{16\beta^2}{\pi^4} + 2} - \frac{4\beta}{\pi^2}$,
iv) *for* $0 < \delta \leq 2\sqrt{\frac{4\beta^2}{\pi^4} + 1} - \frac{4\beta}{\pi^2}$, $N_\delta(I) \otimes N_\delta(SP(\beta)) \subset SP(\beta)$,
v) $\delta(I \otimes SP(\beta), SP(\beta)) \geq 2\sqrt{\frac{4\beta^2}{\pi^4} + 1} - \frac{4\beta}{\pi^2}$.

Proof i) Let $g \in SP(\beta)$, then $I * g = z \in SP(\beta)$.
ii) Let $f_0(z) = I(z) = z$ and $g_0(z) = z + \sum_{n=2}^{\infty} b_{0n} z^n \in SP(\beta)$. Also suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in N_\delta(f_0)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in N_\delta(g_0)$, then we have $\sum_{n=2}^{\infty} n|a_n| \leq \delta$ and $\sum_{n=2}^{\infty} n|b_n - b_{0n}| \leq \delta$.
Let $h_\beta(z) = z + \sum_{n=2}^{\infty} c_n z^n \in G(\beta)$ from (1.1) it is enough to show that

$$\frac{\big((f * g) * h_\beta\big)(z)}{z} \neq 0.$$

We have

$$(f * g) * h_\beta = (f_0 * g_0) * h_\beta + [f_0 * (g - g_0)] * h_\beta \\ + [(f - f_0) * g_0] * h_\beta + [(f - f_0) * (g - g_0)] * h_\beta,$$

therefore

$$\left|\frac{((f*g)*h_\beta)(z)}{z}\right| \geq \left|\frac{(f_0*g_0)*h_\beta}{z}\right| - \left|\frac{f_0*(g-g_0)*h_\beta}{z}\right| - \left|\frac{(f-f_0)*g_0*h_\beta}{z}\right| - \left|\frac{(f-f_0)*(g-g_0)*h_\beta}{z}\right|.$$

Since

$$f_0(z) = z \Rightarrow (f_0 * g_0) * h_\beta = z,$$

and

$$g(z) - g_0(z) = \sum_{n=2}^{\infty}(b_n - b_{0n})z^n,$$

hence

$$[f_0 * (g - g_0)] * h_\beta = 0.$$

On the other hand we have

$$\left|\frac{[(f-f_0)*g_0]*h_\beta}{z}\right| \leq \sum_{n=2}^{\infty}|a_n||b_{0n}||c_n| \leq \sum_{n=2}^{\infty} n|a_n||b_{0n}| \leq \frac{2\beta\delta}{\pi^2},$$

and

$$\left|\frac{[(f-f_0)*(g-g_0)]*h_\beta}{z}\right| \leq \sum_{n=2}^{\infty}|a_n||b_n-b_{0n}||c_n| \leq \sum_{n=2}^{\infty} n|a_n||b_n-b_{0n}| \leq \frac{\delta^2}{4}.$$

Now following the same techniques as in the proof of Theorem 2.2 we conclude the result and we omit details.

iii) from ii) it is clear.

The proofs of iv) and v) are similar to parts ii) and iii) and we omit details. □

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