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# INEQUALITIES FOR ANALYTIC FUNCTIONS DEFINED BY CERTAIN LINEAR OPERATORS

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**Abstract**

In the present investigation, we obtain inequalities for analytic functions in the open unit disk which are associated with the Dziok-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n,\lambda)$.

## 1. Introduction

Let $A_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in C : |z| < 1\}) \tag{1}$$

and let $A_1 := A$. For two functions $f(z)$ given by (1) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{2}$$

For $\alpha_j \in C \quad (j = 1,2,\ldots,l)$ and $\beta_j \in C - \{0,-1,-2,\ldots\} (j = 1,2,\ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$ is defined by the infinite series

$${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

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$$(l \le m+1; l, m \in N_0 := \{0,1,2,\ldots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in N := \{1,2,3\ldots\}). \end{cases}$$

Corresponding to the function $h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) := z_l^p F_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$, the Dziok-Srivastava operator [5] (see also [16]) $H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)$ is defined by the Hadamard product

$$H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) := h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) * f(z)$$

$$= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p}\cdots(\alpha_l)_{n-p}}{(\beta_1)_{n-p}\cdots(\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!}. \tag{3}$$

To make the notation simple, we write

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z).$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [6], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [14], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [1], [7], [10]) and the Srivastava-Owa fractional derivative operators (*cf.* [12], [13]).

Motivated by the multiplier transformation on $A$, we define the operator $I_p(n, \lambda)$ on $A_p$ by the following infinite series

$$I_p(n,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k \quad (\lambda \ge 0). \tag{4}$$

The operator $I_p(n,\lambda)$ is closely related to the Salagean derivative operators [15]. The operator $I_\lambda^n := I_1(n,\lambda)$ was studied recently by Cho and Srivastava [3] and Cho and Kim [4]. The operator $I_n := I_1(n,1)$ was studied by Uralegaddi and Somanatha [17].

In our present investigation, we extend Theorem 2.3h of Miller and Mocanu [11] for functions associated with the Dziok-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n, \lambda)$. Similar result for meromorphic functions defined through a linear operator is considered by Liu and Owa [8].

To prove our results, we need the following lemma due to Miller and Mocanu.

**Lemma 1.** [11] *Let* $w(z) = a + w_m z^m + \cdots$ *be analytic in* $\Delta$ *with* $w(z) \not\equiv a$ *and* $m \geq 1$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, *then* $z_0 w'(z_0) = k w(z_0)$ *and* $\Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k$, *where* $k$ *is real and* $k \geq m$.

## 2. Inequalities Associated with Dziok-Srivastava linear operator

We begin with the following definition for a class of functions which we require in our first result.

**Definition 1.** Let $G_1$ be the set of complex-valued functions $g(r,s,t) : C^3 \to C$ such that

1. $g(r,s,t)$ is continuous in a domain $D \subset C^3$,
2. $(0,0,0) \in D$ and $|g(0,0,0)| < 1$,
3. $\left|g\left(e^{i\theta}, \frac{k+\alpha_1-p}{\alpha_1} e^{i\theta}, \frac{L+(1+\alpha_1-p)(2k+\alpha_1-p)e^{i\theta}}{\alpha_1(\alpha_1+1)}\right)\right| \geq 1,$

whenever $\left(e^{i\theta}, \frac{k+\alpha_1-p}{\alpha_1} e^{i\theta}, \frac{L+(1+\alpha_1-p)(2k+\alpha_1-p)e^{i\theta}}{\alpha_1(\alpha_1+1)}\right) \in D$ with $\Re(e^{-i\theta}L) \geq k(k-1)$ for real $\theta, \alpha_1 \in C$ and real $k \geq p$.

Making use of the Lemma 1, we first prove

**Theorem 1.** *Let* $g(r,s,t) \in G_1$. *If* $f(z) \in A_p$ *satisfies*

$$\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z)\right) \in D \subset C^3$$

*and*

$$\left|g\left(H_p^{l,m}[\alpha_1]f(z), H_p^{l,m}[\alpha_1+1]f(z), H_p^{l,m}[\alpha_1+2]f(z)\right)\right| < 1, \quad (z \in \Delta),$$

*then we have*

$$\left|H_p^{l,m}[\alpha_1]f(z)\right| < 1, \qquad (z \in \Delta).$$

**Proof.** Define $w(z)$ by

$$w(z) := H_p^{l,m}[\alpha_1]f(z).$$

Then we have $w(z) \in A_p$ and $w(z) \neq 0$ at least for one $z \in \Delta$. By making use of

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z), \tag{5}$$

we get

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = zw'(z) + (\alpha_1 - p)w(z)$$

and

$$\alpha_1(\alpha_1+1)H_p^{l,m}[\alpha_1+2]f(z) = z^2 w''(z) + 2(1+\alpha_1-p)zw'(z) + (\alpha_1-p)(\alpha_1+1-p)w(z).$$

If $|H_p^{l,m} f(z)| < 1$ is false, then there exists $z_0$ with $|z_0| = r_0 < 1$ such that

$$|w(z_0)| = \max_{|z| \le |z_0|} |w(z)| = 1.$$

Letting $w(z_0) = e^{i\theta}$ and using Lemma 1, we see that

$$H_p^{l,m}[\alpha_1]f(z_0) = e^{i\theta},$$

$$H_p^{l,m}[\alpha_1+1]f(z_0) = \frac{k+\alpha_1-p}{\alpha_1}e^{i\theta}, \text{ and}$$

$$H_p^{l,m}[\alpha_1+2]f(z_0) = \frac{L+(1+\alpha_1-p)(2k+\alpha_1-p)e^{i\theta}}{\alpha_1(\alpha_1+1)},$$

where $L = z_0^2 w''(z_0)$ and $k \ge p$. Further, by an application of Lemma 1, we have

$$\Re\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \Re\left\{\frac{z_0^2 w''(z_0)}{ke^{i\theta}}\right\} \ge k-1,$$

or $\Re\{e^{-i\theta}L\} \ge k(k-1)$.

Since $g(r,s,t) \in G_1$, we have

$$\left|g\left(H_p^{l,m}[\alpha_1]f(z_0), H_p^{l,m}[\alpha_1+1]f(z_0), H_p^{l,m}[\alpha_1+2]f(z_0)\right)\right|$$

$$= \left|g\left(e^{i\theta}, \frac{k+\alpha_1-p}{\alpha_1}e^{i\theta}, \frac{(1+\alpha_1-p)(2k+\alpha_1-p)e^{i\theta}+L}{\alpha_1(\alpha_1+1)}\right)\right| \ge 1,$$

which contradicts the hypothesis of Theorem 1. Therefore we conclude that

$$|w(z)|=\left|H_p^{l,m}[\alpha_1]f(z)\right|<1 \quad (z\in\Delta).$$

This completes the proof of Theorem 1.

**Corollary 3.** *If $f(z)\in A_p$ satisfies*

$$|H_p^{l,m}[\alpha_1+1]f(z)|<1 \quad (\Re\alpha_1\geq(p-k)/2; k\geq p),$$

then

$$|H_p^{l,m}[\alpha_1]f(z)|<1.$$

### 3. Inequalities Associated with Multiplier Transform

In this section, we prove a result similar to Theorem 1 for functions defined by multiplier transform. We need the following:

**Definition 2.** Let $G_2$ be the set of complex-valued functions $g(r,s,t):C^3\to C$ such that

1. $g(r,s,t)$ is continuous in a domain $D\subset C^3$,
2. $(0,0,0)\in D$ and $|g(0,0,0)|<1$,
3. $\left|g\left(e^{i\theta},\frac{k+\lambda}{p+\lambda}e^{i\theta},\frac{L+[(1+2\lambda)k+\lambda^2]e^{i\theta}}{(p+\lambda)^2}\right)\right|\geq 1$,

whenever $\left(e^{i\theta},\frac{k+\lambda}{p+\lambda}e^{i\theta},\frac{L+[(1+2\lambda)k+\lambda^2]e^{i\theta}}{(p+\lambda)^2}\right)\in D$, with $\Re(e^{-i\theta}L)\geq k(k-1)$ for real $\theta,\lambda\geq 0$ and real $k\geq p$.

**Theorem 2.** *Let $g(r,s,t)\in G_2$. If $f(z)\in A_p$ satisfies*

$$(I_p(n,\lambda)f(z),I_p(n+1,\lambda)f(z),I_p(n+2,\lambda)f(z))\in D\subset C^3$$

and

$$\left|g(I_p(n,\lambda)f(z),I_p(n+1,\lambda)f(z),I_p(n+2,\lambda)f(z))\right|<1, \quad (z\in\Delta),$$

then we have

$$|I_p(n,\lambda)f(z)|<1, \qquad (z\in\Delta).$$

**Proof.** Define $w(z)$ by $w(z):=I_p(n,\lambda)f(z)$. Then we have $w(z)\in A_p$ and $w(z)\neq 0$ at least for one $z\in\Delta$. By making use of

$$(p+\lambda)I_p(n+1,\lambda)f(z)=z[I_p(n,\lambda)f(z)]'+\lambda I_p(n,\lambda)f(z) \qquad (1)$$

we obtain

$$(p+\lambda)I_p(n+1,\lambda)f(z)=zw'(z)+\lambda w(z)$$
$$\text{and } (\lambda+p)^2 I_p(n+2,\lambda)f(z)=z^2w''(z)+(1+2\lambda)zw'(z)+\lambda^2 w(z).$$

If $|I_p(n,\lambda)f(z)|<1$ is false, then there exists $z_0$ with $|z_0|=r_0<1$ such that

$$|w(z_0)|=\max_{|z|\le|z_0|}|w(z)|=1.$$

Letting $w(z_0)=e^{i\theta}$ and using Lemma 1, we see that

$$I_p(n,\lambda)f(z_0)=e^{i\theta},$$
$$I_p(n+1,\lambda)f(z_0)=\frac{k+\lambda}{p+\lambda}e^{i\theta},$$
$$\text{and } I_p(n+2,\lambda)f(z_0)=\frac{[(2\lambda+1)k+\lambda^2]e^{i\theta}+L}{(p+\lambda)^2},$$

where $L=z_0^2w''(z_0)$ and $k\ge p$. Further, an application of Lemma 1, we obtain that

$$\Re\left\{\frac{z_0w''(z_0)}{w'(z_0)}\right\}=\Re\left\{\frac{z_0^2w''(z_0)}{ke^{i\theta}}\right\}\ge k-1,$$

or $\Re\{e^{-i\theta}L\}\ge k(k-1)$. Since $g(r,s,t)\in G_2$, we have

$$\left|g(I_p(n,\lambda)f(z_0),I_p(n+1,\lambda)f(z_0),I_p(n+2,\lambda)f(z_0))\right|$$
$$=\left|g\left(e^{i\theta},\frac{k+\lambda}{\lambda+p}e^{i\theta},\frac{[(2\lambda+1)k+\lambda^2]e^{i\theta}+L}{(p+\lambda)^2}\right)\right|\ge 1,$$

which contradicts the hypothesis of Theorem 2. Therefore we conclude that $|w(z)|=|I_p(n,\lambda)f(z)|<1$, for all $z\in\Delta$. This completes the proof of assertion of Theorem 2.

**Corollary 1.** *If* $f(z) \in A_p$ *satisfies*

$$|I_p(n,\lambda+1)f(z)|<1,$$

then

$$|I_p(n,\lambda)f(z)|<1.$$

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