Algebra Advisory/Comprehensive Examination September 2005

NAME____

Do exactly 10 problems. Do each problem on the sheet for that problem. Write your name on each page. Show all details and quote properly any theorems that you use. All problems are worth 10 points. We prefer complete solutions of a few problems to many partial solutions.

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Groups (Do at least two problems from G1-G4)

G1. Let G be a group and let $a \in G$ be an element of order n. Let $k \mid n$. What is the order of a^k? Justify your answer.

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G2. Prove that if G is a group in which every nonidentity element has order 2, then G has a nontrivial automorphism.

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G3. a) Prove that A_5 is a simple group.

b) Give an example of a nonabelian solvable group with 60 elements.

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G4. Let G be a group and let N be a subgroup of the center $Z(G)$ of G. Show that N is a normal subgroup of G and that if G/N is cyclic, then G is abelian.

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Rings (Do at least one problem from R5-R6)

R5. Prove that in a commutative ring R with identity, every maximal ideal is prime. (An ideal I is *prime* if $ab \in I$ implies that $a \in I$ or $b \in I$.)

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R6. Prove that any principal ideal domain R satisfies the **ascending chain condition**, that is, R cannot have a strictly increasing sequence of ideals

$$\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots$$

where each ideal is properly contained in the next one.

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Linear Algebra (Do at least two problems from L7-L10)

L7. Prove that any subspace of a vector space has a complement.

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L8. Let V be a finite-dimensional vector space over a field F. If τ and σ are linear operators on V, prove that $\sigma\tau = \iota$ (where ι is the identity) implies that τ and σ are invertible and that $\sigma = p(\tau)$ for some polynomial $p(x) \in F[x]$.

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L9. Let V be a finite-dimensional vector space and let T be a linear operator on V. Prove that T is injective if and only if it is surjective.

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L10. Let X be a 5×5 complex matrix. Find all solutions of the equation $X^2 - X = 0$.

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Fields (Do at least one problem from F11-F12)

F11. Show that the characteristic of a finite field is prime. (The *characteristic* of a field is the smallest positive integer n for which $n1 = 0$.)

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F12. Prove that a commutative ring R with identity is a field if and only if R has no nonzero proper ideals.