PROOF Use the theorem to write

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$$
$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz.$$

□

This is the deformation principle; if you can continuously deform $C_1$ to $C_2$, without crossing points where $f$ is not analytic, then the value of $\int_C f(z)dz$ is preserved.

EXAMPLE Suppose $C$ is a simple, closed contour which contains the origin. Then

$$\int_C \frac{dz}{z} = 2\pi i.$$

Indeed, we know this is true for the unit circle, thus for any circle, and in fact for any simple closed contour.

## 33 Cauchy Integral Formula

Lecture 2

We start with a slight extension of Cauchy's theorem.

LEMMA Let $C$ be a simple closed contour. Suppose that $f$ is analytic on an inside $C$, *except* at a point $z_0$ which satisfies

$$\lim_{z \to z_0} |(z - z_0) f(z)| = 0.$$

Then $\int_C f(z)dz = 0$.

PROOF Let $C_r$ be the contour which wraps around the circle of radius $r$ centered at $z_0$ exactly once in the clockwise direction. By the deformation principle, if $r$ is sufficiently small (i.e., if $C_r$ is in the interior of $C$), then $\int_C f(z)dz = \int_{C_r} f(z)dz$.

Given any $\epsilon > 0$, there exists a $\delta > 0$ so that if $|z - z_0| < \delta$, then $|(z - z_0)f(z)| < \epsilon$, and $|f(z)| < \frac{\epsilon}{|z-z_0|}$. If necessary, shrink $\delta$ so that $N_\delta(z_0)$ is in the interior of $C$. If $0 < r < \delta$, then

$$\begin{aligned}
\left|\int_C f(z)dz\right| &= \left|\int_{C_r} f(z)dz\right| \\
&\le \int_{C_r} |f(z)||dz| \\
&< \int_{C_r} \frac{\epsilon}{z - z_0}|dz| \\
&= \frac{\epsilon}{r}\int_{C_r} |dz| \\
&= \frac{\epsilon}{r} 2\pi r \\
&= 2\pi\epsilon.
\end{aligned}$$

We have shown that $|\int_C f(z)dz| < 2\pi\epsilon$ for all $\epsilon$, so that $\int_C f(z)dz = 0$. □

We can use this to prove the Cauchy integral formula.

THEOREM Suppose $f$ is analytic everywhere inside and on a simple closed positive contour $C$. If $z_0$ is any point interior to $C$, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz.$$

PROOF Since $f$ is analytic at $z_0$, we have

$$\lim_{z\to z_0}\left|(z - z_0)\frac{f(z) - f(z_0)}{z - z_0}\right| = 0.$$

By the slightly souped-up Cauchy's theorem, we have

$$\int_C \frac{f(z) - f(z_0)}{z - z_0}dz = 0.$$

But we can rewrite this integral as

$$\begin{aligned}\int_C \frac{f(z) - f(z_0)}{z - z_0}dz &= \int_C \frac{f(z)}{z - z_0}dz - \int_C \frac{f(z_0)}{z - z_0}dz\\ \int_C \frac{f(z)}{z - z_0}dz &= f(z_0)\int_C \frac{1}{z - z_0}dz\\ &= f(z_0)2\pi i.\end{aligned}$$

□

APPLICATIONS We can often use this to evaluate contour integrals. Here are couple of problems from the book: Let $C$ be the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Let $C'$ correspond to the square $x = \pm 4$, $y = \pm 4$. Compute the following four integrals: *The point with all these is to identify each of them as $\int_C \frac{g(z)}{z-z_0}$, and thus realize them as $g(z_0)$:*

a. $\int_C \frac{\exp(-z)}{z-(\pi/2)}dz$. Use $f(z) = \exp(-z)$; analytic everywhere, and in particular on and inside $C$. Then the integral in question is

$$\begin{aligned}I_a &= \int_C \frac{f(z)}{z - \pi/2}dz\\ &= 2\pi i f(\pi/2)\\ &= 2\pi i \exp(-\pi/2)\end{aligned}$$

b. $\int_C \frac{z}{2z+1} dz$.

$$\begin{aligned}\int_C \frac{z}{2z+1} dz &= \frac{1}{2}\int_C \frac{z}{z-(-\frac{1}{2})} \\ &= \frac{1}{2} 2\pi i f(-1/2) \text{ where } f(z) = z \\ &= -\pi i/2\end{aligned}$$

c. $\int_C \frac{\cos(z)}{z^3+9z} dz$. Inside the contour $C$, there's just a single root of the denominator, namely, $z = 0$. So

$$\int_C \frac{\cos(z)}{z^3+9z} dz = \int_C \frac{f(z)}{z} dz$$

where

$$\begin{aligned}f(z) &= \frac{\cos(z)}{z^2+9} \\ \int_C \frac{f(z)}{z} dz &= 2\pi i f(0) \\ &= 2\pi i/9\end{aligned}$$

d. $\int_{C'} \frac{\cos(z)}{z^3+9z} dz$. For this one, there are three poles to worry about. Let $C_1$ be a small loop around $3i$; $C_2$ is a small loop around $0$; and $C_3$ a small loop around $-3i$. Then by the deformation principle,

$$\begin{aligned}\int_{C'} \frac{\cos(z)}{z^3+9z} dz &= \sum_j \int_{C_j} \frac{\cos(z)}{z^3+9z} dz \\ \int_{C_1} \frac{\cos(z)}{z^3+9z} dz &= \int_{C_1} \frac{\cos(z)/(z(z+3i)}{z-3i} dz \\ &= 2\pi i \frac{\cos(3i)}{3i(3i+3i)} \\ \int_{C_3} \frac{\cos(z)}{z^3+9z} dz &= 2\pi i \frac{\cos(-3i)}{-3i(-3i-3i)} \\ \int_{C_2} \frac{\cos(z)}{z^3+9z} &= 2\pi i/9\end{aligned}$$

For a particular contour, we retrieve Gauss's mean value theorem, as follows.

LEMMA Suppose that $f$ is analytic on and in a circle of radius $R$ around $z_0$. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))dt.$$

REMARK This is a continuous analogue of something we did for homework, for polynomials.

PROOF Let $C$ be a contour which wraps around the circle of radius $R$ around $z_0$ exactly once in the counterclockwise direction. On one hand, we have:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}dz$$

On the other hand, this is

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_0^{2\pi} f(z(t))z'(t)dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + R\exp(it))}{(z_0 + R\exp(it)) - z_0}(iR\exp(it))dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))dt.
\end{aligned}$$

□

As a consequence of Cauchy's theorem, we can prove the following expression for the derivative of $f$.

THEOREM Suppose $f$ is analytic in a domain containing $z$. Let $C$ be a simple closed contour in $D$ which goes around $z$ exactly once. Then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw.$$

PROOF Once you know what you're trying to prove, this isn't too hard. Let $M$ be the maximum value of $|f(w)|$ on $C$. Let $m^*$ be the minimum distance from $z$ to a point on $C$, and let $m = m^*/2$. Suppose that $|\Delta z| < m$. Then by the Cauchy integral theorem, we have

$$\begin{aligned}
\frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw &= \frac{1}{\Delta z 2\pi i}\int_C \frac{f(w)}{(w-(z+\Delta z))}dw - \frac{1}{\Delta z 2\pi i}\int_C \frac{f(w)}{(w-z)}dw - \\
&\quad \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw \\
&= \frac{1}{2\pi i}\int_C \frac{\Delta z f(w)}{(w-(z+\Delta z))(w-z)^2}dw
\end{aligned}$$

Then take absolute values, etc. For each $w \in C$, $|w-z| \geq m^* > m$ and $|w-(z+\Delta z)| > m$, so

$$\begin{aligned}
\left|\frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw\right| &\leq \frac{|\Delta z|}{|2\pi|}\int_C \frac{M}{m^3}|dw| \\
&= \frac{\Delta z M}{2\pi m^3}.
\end{aligned}$$

Now, start taking $\Delta z$ ever-smaller... □ Lecture 2

In fact, we can use this to prove that derivatives of *all orders* exist, as follows.

THEOREM Suppose that $f$ is continuous on and inside the simple closed contour $C$. Then each function

$$F_n(z) = \int_C \frac{f(w)}{(w-z)^n} dw$$

is analytic in $D$, with derivative

$$F_n'(z) = nF_{n+1}(z).$$

PROOF We omit the proof that $F_1$ is continuous. At least if $f$ is analytic, we have already shown that $F_1$ is analytic, and its derivative is $F_2$.

We now proceed by induction, and suppose that the theorem is true for all functions $f$ and all orders less than $n$. Fix a value $z$, and construct the function

$$g(w) = g_z(w) = \frac{f(w)}{(w-z)}.$$

For any complex number $\alpha$, let

$$\begin{aligned} G_n(\alpha) &= \int_C \frac{g(w)}{(w-\alpha)^n} dw \\ &= \int_C \frac{f(w)}{(w-\alpha)^n (w-z)}. \end{aligned}$$

Suppose that $N_r(z)$ is any neighborhood of $z$ whose closure is contained inside $C$. If $w \in C$, and if $z + \Delta z$ is inside $N_r(z)$, then $|w - z|$ is bounded away from zero. Since $f$ is continuous, this shows that $G_n(z + \Delta z)$ is bounded on $|\Delta z| < r$.

Then we have identities

$$\begin{aligned} F_n(z) &= G_{n-1}(z) \\ F_n(z+\Delta z) - G_{n-1}(z+\Delta z) &= \int_C f(w) \left( \frac{1}{(w-(z+\Delta z))^n} - \frac{1}{(w-(z+\Delta z))^{n-1}(w-z)} \right) dw \\ &= \int_C \frac{f(w)(\Delta z)}{(w-z)(w-(z+\Delta z))^n} dw \\ &= \Delta z G_n(z+\Delta z). \end{aligned}$$

Moreover, since $g(w)$ is continuous on and inside the contour $C$ (away from $z$), we may assume that $G_{n-1}$ is differentiable, with derivative $(n-1)G_n(z)$.

We now show that $F_n$ is continuous at $z$, and then differentiable. For the former, we have

$$F_n(z+\Delta z) - F_n(z) = G_{n-1}(z+\Delta z) - G_{n-1}(z) + \Delta z G_n(z+\Delta z)$$

By the continuity of $G_{n-1}$,

$$\lim_{\Delta z \to 0}(F_n(z+\Delta z) - F_n(z)) = \lim_{\Delta z \to 0}(G_{n-1}(z+\Delta z) - G_{n-1}(z)) + \lim_{\Delta z \to 0} \Delta z G_n(z+\Delta z)$$

But $G_n$ is bounded, so

$$= 0,$$

as desired. To show differentiability, use:

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{F_n(z+\Delta z) - F_n(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{G_{n-1}(z+\Delta z) - G_{n-1}(z)}{\Delta z} + G_n(z+\Delta z) \\ &= (n-1)G_n(z) + G_n(z) \end{aligned}$$

since $G_n$ is now continuous!

$$\begin{aligned} &= nG_n(z) \\ &= nF_{n+1}(z). \end{aligned}$$

□ Lecture 2

In summary, we have the following representation of all derivatives of an analytic function:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw.$$

THEOREM [Morera] Suppose $f$ is *continuous* on $D$. If for every closed contour $C$ in $D$ we have $\int_C f(z)dz = 0$, then $f$ is analytic in $D$.

PROOF By hypothesis and the Cauchy-Goursat theorem, $f$ admits an antiderivative, $F$. But if $F$ has one derivative (namely, $f$), then it has derivatives of all orders. In particular $f = F'$ must have a derivative. □

## 34 Boundedness and the Maximum Modulus Principle

Can we ever fit the entire complex plane into a finite circle? If we're willing to use a function which is merely continuous, this is no problem; use

$$f(x) = \frac{z}{1+|z|}.$$

In this section we'll prove Liouville's theorem, which says that this is impossible if $f$ is analytic.