JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **213**, 106–116 (1997)
ARTICLE NO. AY975464

Cauchy Transforms and Multipliers

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Submitted by William F. Ames

Received September 20, 1996

In this paper we prove a number of results on Cauchy transforms and multipliers. For example, we prove that if $f = IF \in \mathcal{F}_0$, where I is an inner function satisfying a certain condition and F is outer, then $F \in \mathcal{F}_0$.

1. INTRODUCTION

Let U be the open unit disk and let $T = \{z\colon |z| = 1\}$. Let $\mathcal{M}$ denote the set of complex valued Borel measures on T. For each $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T \frac{1}{(1 - \bar{\zeta} z)^\alpha} \, d\mu(\zeta), \tag{1}$$

for $|z| < 1$. For $\alpha = 0$, let $\mathcal{F}_0$ denote the family of functions f having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T \log \frac{1}{1 - \bar{\zeta} z} \, d\mu(\zeta) + f(0). \tag{2}$$

$\mathcal{F}_\alpha$ is a Banach space with respect to the norm

$$\begin{cases} \|f\|_{\mathcal{F}_\alpha} = \inf \|\mu\|, & \text{for } \alpha > 0 \\ \|f\|_{\mathcal{F}_0} = \inf \|\mu\| + |f(0)|, & \end{cases} \tag{3}$$

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0022-247X/97 $25.00

where μ varies over all measures in M for which (1) or (2) hold and where $\|\mu\|$ denotes the total variation norm of μ [3, 6].

A function f analytic in U is called a multiplier of $\mathcal{F}_\alpha$ provided that $fg \in \mathcal{F}_\alpha$ for all $g \in \mathcal{F}_a$. Let M_α denote the set of all multipliers of $\mathcal{F}_\alpha$. Let

$$\|f\|_{M_\alpha} = \sup\left\{\|fg\|_{\mathcal{F}_\alpha} : g \in \mathcal{F}_a, \|q\|_{\mathcal{F}_\alpha} \leq 1\right\}. \tag{4}$$

Note that

$$\|fg\|_{M_\alpha} \leq \|f\|_{M_\alpha} \cdot \|g\|_{M_\alpha}. \tag{5}$$

Hence M_α with this norm is a Banach algebra [2, p. 182; 7; 8; 6].

In Section 2 we show for w analytic in U and $f \in M_\alpha$, with $\|f\|_{M_\alpha} < 1$ that $w \circ f \in M_\alpha$. This implies that $f \in H^\infty$, for all $f \in M_\alpha$ and provides many examples of functions in M_α.

In [6] it was proved that if $f' \in H^1$ then $f \in M_\alpha$ for all $\alpha > 0$. In [4] it was proved that if $f' \in H^p$ for some $p > 1$ then $f \in M_0$. Both statements are sufficient but not necessary. In [5] an example was given of a function $f \in M_0$ which is not continuous on $\overline{U}$. In Section 3, we describe the boundary behavior of $f \in M_\alpha$. To be specific, we show that at each point of the boundary of U, the limit exists from the inside of each orocycle at the same point.

It is known [1, 2] that if $f = IF \in H^p$, where I is inner and F is outer, then $\|f\|_p = \|F\|_p$. In [8] it was shown that if $f = IF \in \mathcal{F}_1$, where I is inner and F is outer, then $\|F\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$. Recently, D. Hallenbeck and K. Samotij conjectured (private communication) that if $f = IF \in \mathcal{F}_\alpha$, where $0 \leq \alpha \leq 1$, I is inner, and F is outer then $F \in \mathcal{F}_\alpha$.

In Section 4, we prove that if $f = IF \in \mathcal{F}_0$ and I satisfies condition (12) then $F \in \mathcal{F}_0$.

2. M_α AS A BANACH ALGEBRA

We can easily show that M_α, $\alpha \geq 0$, is a complex Banach algebra with norm as in (4) and (5), see [2, p. 182]. Let M_α^* denote the dual space of M_α and $\mathcal{A}_\alpha$ denote the subset of the unit ball of M_α^* consisting of elements that are also multiplicative, see [2, p. 184]. Define the Gelfand transform of $f \in M_\alpha$, by

$$\hat{f}(m) = m(f),$$

for all $m \in \mathcal{A}_\alpha$ and

$$\|\hat{f}\| = \sup_{m \in \mathcal{A}_\alpha} |\hat{f}(m)|.$$

As

$$\|f\|_{M_\alpha} = \sup_{L \in M_\alpha^*, \|L\|=1} |L(t)|,$$

$\|\hat{f}\| \leq \|f\|_{M_\alpha}$. We say that $\hat{f}$ is an isometry when $\|\hat{f}\| = \|f\|_{M_\alpha}$. This is equivalent to the condition $\|f^2\|_{M_\alpha} = \|f\|_{M_\alpha}^2$ [2, p. 185]. Unfortunately not every $\hat{f}$ is an isometry, because it was shown that

$$n^{1-\alpha} \leq \|z^n\|_{M_\alpha} \leq cn^{1-\alpha},$$

for all $0 \leq \alpha \leq 1$, see [3]. Certainly, $\|z\|_{M_\alpha} > 1$. Hence $\|z^n\|_{M_\alpha} < \|z\|_{M_\alpha}^n$, for all $n \geq 1$ and consequently, $\hat{z}$ is not an isometry. We conjecture that the constant functions are the only isometries in M_α.

Now we recall the following general theorem [2, p. 182]

THEOREM A. *Let A be a Banach algebra with norm $\|\cdot\|$ and let m be a complex homomorphism of A. Then m is a continuous linear functional with norm*

$$\|m\| = \sup_{\|f\| \leq 1} |m(f)| \leq 1.$$

Remark. Theorem A implies that every complex homomorphism of M_α is an element of $\mathcal{A}_\alpha$.

This theorem immediately gives another proof for the following known result [3, 7].

THEOREM B. $|f(z)| \leq \|f\|_{M_\alpha}$, *for all $z \in U$, $f \in M_\alpha$, and where* $\alpha \geq 0$.

Proof. Fix $z \in U$ and let $m(f) = f(z)$. Then m is a complex homomorphism on M_α. This and Theorem A give the result.

The following theorems concern subordination:

THEOREM 1. *Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in U and $f \in M_\alpha$, where $\alpha \geq 0$. If $\|f\|_{M_\alpha} < 1$ then $\phi \circ f \in M_\alpha$ and $\sum_{n=0}^{\infty} a_n f^n$ converges to $\phi \circ f$ in M_α.*

Proof. Let ϕ_k denote the kth partial sum of the Taylor series of ϕ. Use (5) to get

$$\|f^n\|_{M_\alpha} \leq \|f\|_{M_\alpha}^n.$$

Hence

$$\|\phi_k \circ f\|_{M_\alpha} \leq \sum_{n=0}^{\infty} |a_n| \, \|f\|_{M_\alpha}^n < \infty,$$

for all k. This is because $\|f\|_{M_\alpha} < 1$ and ϕ analytic. Since $\|\phi_k \circ f\|_{M_\alpha}$ is uniformly bounded and $\phi_k \circ f \to \phi \circ f$ locally uniformly we conclude that $\phi \circ f \in M_\alpha$ and the series converges to $\phi \circ f$ in M_α, see [3].

THEOREM 2. *Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in U, with $\sum_0^\infty |a_n| < \infty$. Let $f \in M_\alpha$, where $\alpha \geq 0$. Then each of the following follows*

(a) *If $\|f\|_{M_\alpha} \leq 1$ then $\phi \circ f \in M_\alpha$ and $\sum_{n=0}^{\infty} a_n f^n$ converges to $\phi \circ f$ in M_α.*

(b) *If $\|f^n\|_{M_\alpha}$ is uniformly bounded for all $n \geq 1$ then $\phi \circ f \in M_\alpha$.*

Proof. The proof is similar to the proof of Theorem 1.

It is not easy to deal with the M_α-norm. To be specific, we could not express $\|\phi \circ f\|_{M_\alpha}$ in terms of $\|f\|_{M_\alpha}$. However, if we use the Gelfand-norm then we have:

THEOREM 3. *Let ϕ be as in the statement of Theorem 2 and let $f \in M_\alpha$, with $\|f\|_{M_\alpha} \leq 1$. Then*

$$\|\phi \circ \hat{f}\| = \|\phi(\|\hat{f}\| z)\|_\infty.$$

Proof. Let $m \in \mathcal{A}_\alpha$. Then by theorem 2, as $\sum_0^\infty |a_n| < \infty$, $\phi \circ f \in M_\alpha$ and so

$$m(\phi \circ f) = \sum_0^\infty a_n (m(f))^n. \tag{6}$$

But, as $\|\hat{f}(m)\| \leq 1$,

$$\{e^{i\theta} m(f) : m \in \mathcal{A}_\alpha\} = \{z : |z| \leq \|\hat{f}\|\}.$$

This and (6) imply the result.

3. BOUNDARY BEHAVIOR OF $f \in M_\alpha$

It was shown in [6] that if $f' \in H^1$ the $f \in M_\alpha$ for all $\alpha > 0$. In [4] it was proved that $f' \in H^p$ for some $p > 1$ implies $f \in M_0$ while $f' \in H^1$ does not imply $f \in M_0$. In [5] an example was constructed of a function in M_0 that is not continuous on $\overline{U}$. In the following theorem we show that functions in M_0 are almost continuous on $\overline{U}$. Let $D_t(\theta)$, $1 > t > 0$, be the region inside the orocycle $C_t(\theta)$ given by

$$C_t(\theta) = \left\{ z: \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = t \right\}$$

and inside the circle $\{z: |z - e^{i\theta}| < 1/2\}$. Then we have the following theorem

THEOREM 4. *Let $f \in M_0$. Then*

$$\lim_{z \to e^{i\theta}} f(z) \text{ exists}, \qquad z \in \overline{D_t(\theta)}.$$

Proof. Choose $e^{i\theta} = 1$ and assume that $f(0) = 0$. Then

$$f(z) \log \frac{1}{1 - z} \in F_0.$$

Hence

$$f(z) \log \frac{1}{1 - z} = \int_T \log \frac{1}{1 - \bar{x}z} \, d\mu(x)$$

and

$$f(z) = \int_T \frac{\log(1 - \bar{x}z)}{\log(1 - z)} \, d\mu(x). \tag{7}$$

Choose $z \in \overline{D_t(1)}$. Then

$$\frac{1 - |z|^2}{|1 - z|^2} \geq t$$

and $|1 - z| < 1/2$. Take logarithms to conclude that

$$\frac{\log(1 - |z|^2)}{\log|1 - z|} \leq \frac{\log t}{\log |1 - z|} + 2. \tag{8}$$

If we assume that the branches of all logarithms in (7) are principal then the arguments are all bounded by $\pi/2$ and hence

$$\left| \frac{\log(1 - \bar{x}z)}{\log(1 - z)} \right| \leq \frac{\log(1 - |z|)}{\log|1 - z|} + \frac{\log 2 + \pi/2}{|\log|1 - z||}$$

$$\leq \frac{|\log t| + \log 2 + \pi/2}{|\log|1 - z||} + 2.$$

The last inequality follows from (8). This implies that the integrand in (7), as $z \to 1$, converges to the function

$$F(x) = \begin{cases} 0, & \text{for } x \neq 1 \\ 1, & \text{for } x = 1. \end{cases}$$

Hence we conclude, using the Lebesgue dominated convergence theorem, that

$$\lim_{z \to 1} f(z) = \mu(\{1\}).$$

Remark. The limit of Theorem 4 remains the same when we replace the orocycle by a path that is more tangential. Let $(1+z)/(1-z) = u + iv$ and let

$$L_t = \{z : uv = t, |1 - z| < 1/2\}.$$

Then $L_t \to 1$ as $u \to 0$.

Assume that $t > 0$ and let $z \in L_t$. Then

$$\frac{u^4 + t^2}{u^2} = u^2 + \frac{t^2}{u^2} = \left|\frac{1+z}{1-z}\right|^2. \tag{9}$$

Take logarithms in (9), to get

$$\log|u^4 + t^2| = 2\log u + 2\log\left|\frac{1+z}{1-z}\right|. \tag{10}$$

Assume that $u < 1$. Then $\log t^2 < \log|{}^4 + t^2| < \log(t^2 + 1)$. Write $\log u = \log((1 - |z|^2)/|1 - z|^2)$. Then (10) gives

$$\frac{\log(1 - r^2)}{\log|1 - z|} \leq c,$$

for some constant c. This is similar to (8). Continue as in the proof of Theorem 4 to conclude that

$$\lim_{z \to 1} f(z) = \mu(\{1\}),$$

along L_t.

4. OUTER PART AND INNER PART OF $f \in \mathcal{F}_0$

It was shown in [8] that if $f \in \mathcal{F}_1$ and

$$f = IF, \tag{11}$$

where F is outer and I is inner then $\|F\|_{\mathcal{F}_1} \leq \|f\|_{\mathcal{F}_1}$.

In this section we prove:

THEOREM 5. *If $f \in \mathcal{F}_0$ is as in* (11) *and I satisfies*

$$\int_T \int_0^1 |I'(tx)|\, dt |d\mu(x)| < C\|zf\|_{\mathcal{F}_0}, \tag{12}$$

for all μ that give zf as in (2), *then $\|F\|_{\mathcal{F}_0} \leq C_1 \|zf\|_{\mathcal{F}_0}$, where C, C_1 are uniform constants.*

This is a consequence of the following more general proposition.

PROPOSITION 1. *If $f = IF \in \mathcal{F}_0$, where I is an inner function satisfying* (12) *then $\|F\|_{\mathcal{F}_0} \leq C_1 \|zf\|_{\mathcal{F}_0}$.*

Proof. The Frostman Theorem [2] implies that there is a sequence of Blaschke products converging, in the H^∞-norm, to I of the form

$$I_\zeta = \frac{I - \zeta}{1 - \bar{\zeta} I}.$$

To be specific, $\|I_\zeta - I\|_\infty < \epsilon$ when $|\zeta| < \delta$. We may choose a sequence of ζ converging to 0 so that each I_ζ has only simple zeros. Put

$$I_\zeta = B(z) = \prod_{j=1}^{\infty} \frac{z - a_j}{1 - \bar{a}_j z}$$

and let B_n be the subproduct from $j = 1$ to $j = n$. Since it is known [4] that $f \in \mathcal{F}_0$ is equivalent to $zf \in \mathcal{F}_0$, we put

$$zf(z) = \int_T \int_0^1 \frac{\bar{x} z}{1 - t\bar{x}z}\, dt\, d\mu(x), \tag{13}$$

where $0 \leq t < 1$ and $\mu \in M$ given zf as in (2). Then

$$f(\rho z) = \int_T \int_0^\rho \frac{\bar{x}}{1 - t\bar{x}z} dt\, d\mu(x), \tag{14}$$

where $0 \leq t \leq \rho < 1$. Put

$$\frac{\bar{x}}{(1 - t\bar{x}z)} \frac{1}{B_n(z)} = \frac{\bar{x}/B_n(1/t\bar{x})}{1 - t\bar{x}z} + \sum_{j=1}^{n} \frac{\left(1 - |a_j|^2\right)\bar{x}}{B_j(a_j)(z - a_j)\left(1 - t\bar{x}a_j\right)}, \tag{15}$$

where $B_j(a_j) = \prod_{1 \leq k \neq j \leq n}((a_k - a_j)/(1 - \bar{a}_k a_j))$. Note that

$$\frac{1}{B_n(1/t\bar{x})} = \overline{B_n(tx)}$$

and, from (14), that

$$\int_T \int_0^\rho \frac{\bar{x}}{1 - t\bar{x}a_j}\, dt\, d\mu(x) = I(\rho a_j) F(\rho a_j),$$

for all j. Integrate (15) with respect to dt, $d\mu$ and then take limits to get

$$\begin{aligned} \frac{f(\rho z)}{I(z)} &= \lim_{\zeta \to 0} \lim_{n \to \infty} \frac{f(\rho z)}{B_n(z)} \\ &= \lim_{\zeta \to 0} \lim_{n \to \infty} \int_T \int_0^\rho \frac{\bar{x}\overline{B_n(tx)}}{1 - t\bar{x}z}\, dt\, d\mu(x) \\ &\quad + \lim_{\zeta \to 0} \lim_{n \to \infty} \sum_{j=1}^{n} \frac{\left(1 - |a_j|^2\right) I(\rho a_j) F(\rho a_j)}{B_j(a_j)(z - a_j)} \\ &= A_1 + A_2. \end{aligned} \tag{16}$$

We first consider A_1. As $\rho < 1$, (16) implies that

$$A_1 = \int_T \int_0^\rho \frac{\bar{x}\overline{I(tx)}}{1 - t\bar{x}z}\, dt\, d\mu(x). \tag{17}$$

Integrate (17), using integration by parts with respect to dt, to get

$$\begin{aligned} zA_1 &= \int_T \log \frac{1}{1 - \bar{x}\rho z} \left(\overline{I(\rho x)}\, d\mu(x)\right) \\ &\quad - \int_0^\rho \int_T \bar{x} \log \frac{1}{1 - t\bar{x}z} \overline{I'(tx)}\, d\mu(x)\, dt. \end{aligned} \tag{18}$$

For each $t < 1$,

$$\frac{1}{1 - t\bar{x}z} = \int_0^{2\pi} \frac{1}{1 - \bar{y}z} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}}\, d\theta/2\pi, \tag{19}$$

for $z \in U$ and $y = e^{i\theta}$. Integrate (19) with respect to z to get

$$\log \frac{1}{1 - t\bar{x}z} = \int_0^{2\pi} \frac{t\bar{x}}{\bar{y}} \log \frac{1}{1 - \bar{y}z} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}}\, d\theta/2\pi. \tag{20}$$

Let

$$\Phi_\rho(\theta) = \int_T \int_0^\rho \frac{t\bar{x}^2}{\bar{y}} \Re \frac{1 + txe^{-i\theta}}{1 - txe^{-i\theta}} \overline{I'(tx)}\, dt\, d\mu(x). \tag{21}$$

Then

$$\int_0^{\rho}\int_T \bar{x}\log\frac{1}{1-t\bar{x}z}\overline{I'(tx)}\,d\mu(x)\,dt$$
$$= \int_0^{2\pi}\log\frac{1}{1-\bar{y}z}\Phi_\rho(\theta)\,d\theta/2\pi. \tag{22}$$

Now we show that the L^1-norm of Φ_ρ is uniformly bounded. Note that

$$\int_0^{2\pi}\Re\frac{1+txe^{-i\theta}}{1-txe^{-i\theta}}e^{i\theta}\,d\theta/2\pi = tx.$$

Hence it follows from (21), by using (12), that

$$\int_0^{2\pi}|\Phi_\rho(\theta)|\,d\theta/2\pi \le \int_T\int_0^{\rho}\left|\overline{I'(tx)}\right|dt|d\mu(x)| \le C\|zf\|_{F_0}.$$

This implies, by using the Helly selection theorem, that there is a subsequence of $\rho \to 1$ so that $\Phi_\rho\,d\theta$ converges weak star to some $\nu_1 \in M$.

Now replace t by ρ in (20) and let

$$\Psi_\rho(\theta) = \int_T \frac{\rho\bar{x}^2}{\bar{y}}\Re\frac{1+\rho xe^{-i\theta}}{1-\rho xe^{-i\theta}}\overline{I(\rho x)}\,d\mu(x). \tag{23}$$

Clearly, the L_1-norm of Ψ_ρ is uniformly bounded and so Ψ_ρ converges weak star to some measure $\nu_2 \in M$. Hence

$$\int_T\log\frac{1}{1-\bar{x}\rho z}\left(\overline{I(\rho x)}\,d\mu(x)\right) = \int_0^{2\pi}\log\frac{1}{1-\bar{y}z}\Psi_\rho(\theta)\,d\theta/2\pi \tag{24}$$

and then (18) becomes, by using (22) and (24)

$$zA_1 = \int_T\log\frac{1}{1-\bar{x}z}\Psi_\rho(x)\,d\theta - \int_T\log\frac{1}{1-\bar{x}z}\Phi_\rho(\theta)\,d\theta.$$

Hence

$$\lim_{\rho\to 1} zA_1 = \int_T\log\frac{1}{1-\bar{x}z}\,d(\nu_1(x)-\nu_2(x)). \tag{25}$$

Second, we find A_2. It is not hard to notice that, see [1],

$$\frac{1}{2\pi}\int_T\frac{f(\rho w)\overline{B_n(w)}}{w-z}\,dw = \sum_{j=1}^{n}\frac{\left(1-|a_j|^2\right)f(\rho a_j)}{B_j(a_j)(a_j-z)} + \frac{f(\rho z)}{B_n(z)}. \tag{26}$$

Since $|f(\rho w)\overline{B_n(w)}| \leq |f(\rho w)|$ and since $f \in H^1$, we conclude using the Lebesgue dominated convergence theorem that the limit, as $n \to \infty$, $\zeta \to 0$, and $\rho \to 1$, of the left side of (26) is equal to $F(z)$. Hence, from (26) and (16)

$$\lim_{\rho \to 1} A_2 = F(z) - F(z) = 0. \tag{27}$$

Therefore (27), (26), and (16) imply that there is a measure $\nu \in \mathcal{M}$ so that

$$F(z) = \int_T \log \frac{1}{1 - \bar{x}z} \, d\nu(x).$$

Hence $F \in \mathcal{F}_0$.

Remark. (1) The condition (12) is satisfied by any inner function in M_α for some $\alpha \geq 0$. This is because for any $f \in M_\alpha$

$$\int_0^1 |f'(re^{i\theta})| \, dr \leq \|f\|_{M_\alpha},$$

for all θ [6, 8].

(2) We believe that condition (12) in Proposition 1 can be removed.

Finally we state the following corollary

COROLLARY. *If $f = IF \in M_0$, where I is an inner function satisfying the condition*

$$\int_T \int_0^1 |I'(tx)| \, dt |d\mu(x)| \leq C\|zf\|_{M_0},$$

for all μ that give zf as in (2) *then*

$$\|F\|_{M_0} \leq C_1 \|zf\|_{M_0}.$$

REFERENCES

1. P. L. Duren, "Theory of H^P Spaces," Academic Press, New York, 1970.
2. J. Garnett, "Bounded Analytic Functions," Academic Press, New York, 1980.
3. D. J. Hallenbeck, T. H. MacGregor, and K. Samotij, Fractional Cauchy transforms, inner functions and multipliers, *Proc. London Math. Soc.* (3) **72** (1996), 157–187.

4. D. J. Hallenbeck and K. Samotij, On Cauchy integrals of logarithmic potentials and their multipliers, *J. Math. Anal. Appl.* **174** (1993), 614–634.
5. D. J. Hallenbeck and K. Samotij, The growth of derivatives of multipliers of Cauchy transforms of logarithmic potentials, manuscript.
6. R. A. Hilbschweiler and T. H. MacGregor, Multipliers of families of Cauchy-Stieltjes transforms, *Trans. Amer. Math. Soc.* **331** (1992), 377–394.
7. S. V. Hruscev and S. A. Vinogradov, Inner functions and multipliers of Cauchy type integrals, *Ark. Mat.* **19** (1981), 23–42.
8. S. A. Vinogradov, Properties of multipliers of Cauchy–Stieltjes integrals and some factorization problems for analytic functions, *Amer. Math. Soc. Transl.* 2 **115** (1980), 1–32.