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COMPACT COMPOSITION OPERATORS ON SPACES OF EXPONENTIAL CAUCHY KERNELS

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Abstract We study the action of the composition operator on the analytic function spaces whose kernels are of exponential Cauchy type. These function spaces become Banach spaces when the kernels are integrated with respect to the complex Borel measures of the unit circle. Necessary and sufficient conditions for the composition operator to be compact are found.

Keywords Exponential Cauchy transform · Convex space · Composition operators · Laguerre polynomial

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1 Cauchy type analytic function spaces

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbf{C}$. Let $\mathbf{T} = \partial\mathbf{D}$ be the boundary of $\mathbf{D}$ and let $H(\mathbf{D})$ denote the class of holomorphic functions on $\mathbf{D}$. $H(\mathbf{D})$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\mathbf{D}$. We denote by $\mathbf{M}$ the set of all complex-valued Borel measures on $\mathbf{T}$ and $\mathbf{M}^*$ the subset of $\mathbf{M}$ consisting of probability measures. An analytic function f is subordinate to g in $\mathbf{D}$, written $f(z) \prec g(z)$, if there exists an analytic self-map φ in $\mathbf{D}$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $f(z) = g[\varphi(z)]$. If in particular g is also univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$.

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Let $z \in \mathbf{D}$ and let $k \in H(\mathbf{D})$ be one of the Cauchy type kernels. We define X to be the subspace of $H(\mathbf{D})$ consisting of functions for which there exists a measure $\mu \in \mathbf{M}$ such that

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)\, d\mu(x), \tag{1}$$

where $x = e^{it} \in \mathbf{T}$. The norm on X defined by

$$\|f_\mu\|_X = \inf_{\mu \in \mathbf{M}} \left\{ \|\mu\| : f_\mu(z) = \int_{\mathbf{T}} k(xz)\, d\mu(x) \right\} \tag{2}$$

makes X into a Banach space. If the series expansion of the kernel function k is given by

$$k(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then the series of the function is given by

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)\, d\mu(x) = \sum_{n=0}^{\infty} a_n \mu_n z^n, \tag{3}$$

where

$$\mu_n = \int_{\mathbf{T}} x^n\, d\mu(x) = \int_{-\pi}^{+\pi} e^{int}\, d\mu\big(e^{it}\big).$$

According to the Lebesgue decomposition theorem $\mathbf{M} = \mathbf{M}_a \oplus \mathbf{M}_s$, where $\mathbf{M}_a := \{\mu_a \in \mathbf{M} : \mu_a \ll m\}$ with m being the normalized Lebesgue measure on the unit circle, and $\mathbf{M}_s := \{\mu_s \in \mathbf{M} : \mu_s \perp m\}$ the singular measures. Thus any $\mu \in \mathbf{M}$ can be written as $\mu = \mu_a + \mu_s$ where $\mu_a \in \mathbf{M}_a$, $\mu_s \in \mathbf{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Consequently, the Banach space X may be written as $X = (X)_a \oplus (X)_s$, where $(X)_a$ is isomorphic to $L^1/\overline{H_0^1}$ the closed subspace of $\mathbf{M}$ of absolutely continuous measures, and $(X)_s$ is isomorphic to $\mathbf{M}_s$ the subspace of $\mathbf{M}$ of singular measures. If $f \in X_a$ then the singular part is null and the measure μ for which the integral in (1) holds reduces to $d\mu(e^{it}) = g(e^{it})\, dt$, where $g(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\mathbf{T}$. In this case, the functions in $(X)_a$ may then be written as

$$f_\mu(z) = \int_{-\pi}^{\pi} k\big(e^{it} z\big) g\big(e^{it}\big)\, dt,$$

where if $g(e^{it})$ is nonnegative then $\|f\|_X = \|g(e^{it})\|_{L^1}$.

If the kernel function in (1) is replaced by $K(z) = (1-z)^{-1}$, $K^\alpha(z) = (1-z)^{-\alpha}$ or $K_e(z) = \exp[K(z)]$, respectively, then the corresponding analytic function spaces are the classical Cauchy transform space $\mathbf{K}$ [4], the fractional Cauchy transform spaces F_α [5] and the exponential Cauchy transform space $\mathbf{K}_e$ introduced in [6], thus using (1) and replacing a_n by appropriate coefficients we get the following:

$$\mathbf{K} = \left\{ f_\mu \in H(\mathbf{D}) : f_\mu(z) = \int_{\mathbf{T}} K(xz)\, d\mu(x) = \sum_{n=0}^{\infty} \mu_n z^n \right\},$$

$$\mathbf{K}_\alpha = \left\{ f_\mu \in H(\mathbf{D}) : f_\mu(z) = \int_{\mathbf{T}} K^\alpha(xz)\, d\mu(x) = \sum_{n=0}^{\infty} A_n(\alpha) \mu_n z^n \right\}, \tag{4}$$

$$\mathbf{K}_e = \left\{ f_\mu \in H(\mathbf{D}) : f_\mu(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x) = \sum_{n=0}^{\infty} e A_n \mu_n z^n \right\},$$

where

$$\mu_n = \int_{\mathbf{T}} x^n\, d\mu(x) = \int_{\mathbf{T}} e^{int}\, d\mu\big(e^{it}\big),$$

$$A_n(\alpha) = (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n}, \tag{5}$$

$$A_n = L_n^{(-1)}(-1) = \sum_{i=0}^{n} \frac{1}{i!} \binom{n-1}{n-i}.$$

Clearly $\mathbf{K}$ is a special case of $\mathbf{K}_\alpha$ when $\alpha = 1$. It is known that $\mathbf{K}_\alpha \subset \mathbf{K}_\beta$ for $0 < \alpha < \beta$. It was also shown in [6] that $\mathbf{K} \subset (\mathbf{K}_e)_a$ and if $f \in \mathbf{K}$ then $\|f\|_{\mathbf{K}_e} < \|h\|_{L^1} \|f\|_{\mathbf{K}}$, where $h \in L^1$.

The next result gives us examples of elements of $\mathbf{K}_e$.

Lemma 1 *Suppose that* $|w| \le 1$ *and let* $f_w(z) = K_e(wz) = \exp[(1 - wz)^{-1}]$ *for* $|z| < 1$. *Then* $f_w(z) \in \mathbf{K}_e$ *and there exists a probability measure* $\mu \in \mathbf{M}^*$ *such that*

$$f_w(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \quad \textit{and} \quad \|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1.$$

Proof For $|w| \le 1$ and $|z| < 1$ we have $\operatorname{Re}\{K(wz)\} = \operatorname{Re}\{(1 - wz)^{-1}\} > \frac{1}{2}$. The Riesz–Herglotz formula implies that there exists a probability measure $\mu = \mu_w \in \mathbf{M}^*$ such that

$$K(wz) = (1 - wz)^{-1} = \int_{\mathbf{T}} K(xz)\, d\mu(x) = \int_{\mathbf{T}} (1 - xz)^{-1}\, d\mu(x). \tag{6}$$

The left-hand-side of the above equality is $(1 - wz)^{-1} = \sum_{n=0}^{\infty} w^n z^n$ and the right-hand-side is $\sum_{n=0}^{\infty} \mu_n z^n$. Equating coefficients of the power series of both sides of (6) we get $w^n = \mu_n = \int_{\mathbf{T}} x^n\, d\mu(x)$ for $n = 0, 1, 2, \dots$ and thus

$$\begin{aligned} f_w(z) = K_e(wz) &= e \sum_{n=0}^{\infty} A_n w^n z^n \\ &= e \sum_{n=0}^{\infty} A_n \left(\int_{\mathbf{T}} x^n\, d\mu(x) \right) z^n \\ &= \int_{\mathbf{T}} \left(e \sum_{n=0}^{\infty} A_n x^n z^n \right) d\mu(x) \\ &= \int_{\mathbf{T}} K_e(xz)\, d\mu(x). \end{aligned}$$

Hence $f_w \in \mathbf{K}_e$ and since $\mu \in \mathbf{M}^*$, we have $\|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1$. □

Corollary 1 *A special case of the previous result is*

$$K_e(xz) \in \mathbf{K}_e \quad \textit{for all } x \in \mathbf{T} \quad \textit{and} \quad \big\|K_e(xz)\big\|_{\mathbf{K}_e} = 1.$$

Lemma 2 *Suppose* $\{f_{\mu_n}\}_{n=1}^{\infty}$ *is a sequence of functions in* $\mathbf{K}_e$ *such that there is a constant* A *for which* $\|f_{\mu_n}\|_{\mathbf{K}_e} \leq A$ *for* $n = 1, 2, \ldots$. *If* $f_\mu(z) = \lim_{n\to\infty} f_{\mu_n}(z)$ *exists for* $|z| < 1$, *then* $f \in \mathbf{K}_e$ *and* $\|f\|_{\mathbf{K}_e} \leq A$.

Proof Let $z \in \mathbf{D}$ and suppose that $f_{\mu_n} \in \mathbf{K}_e$ for $n = 1, 2, \ldots$. Then by definition we have

$$f_{\mu_n}(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_n(x) \quad \text{and} \quad \mu_n \in \mathbf{M}, \left\| f_{\mu_n}(z) \right\|_{\mathbf{K}_e} = \|\mu_n\| \leq A.$$

The Banach–Alaoglu theorem yields a subsequence $\{\mu_{n_k}\}$ for $k = 1, 2, \ldots$, $\|\mu_{n_k}\| \leq A$ and $\mu \in \mathbf{M}, \|\mu\| \leq A$ such that $\mu_{n_k} \to \mu \in \mathbf{M}$ as $k \to \infty$ in the weak* topology. Hence we get

$$\int_{\mathbf{T}} K_e(xz)\, d\mu_{n_k}(x) \longrightarrow \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \quad \text{as } k \to \infty.$$

Since $f_\mu(z) = \lim_{k\to\infty} f_{\mu_{n_k}}(z)$, we have

$$f_\mu(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \in \mathbf{K}_e \quad \text{and} \quad \|f_\mu\| \leq A.$$

□

2 The composition operator on $\mathbf{K}_e$

If φ is an analytic self-map of the unit disc $\mathbf{D}$, we say that φ induces a bounded composition operator C_φ on X if there exists a positive constant A such that for all $f \in X$, $\|C_\varphi(f)\|_X = \|(f \circ \varphi)\|_X \leq A\|f\|_X$. A bounded operator C_φ will be a compact operator if the image of every bounded set of X is relatively compact (i.e. has compact closure) in X. Equivalently, C_φ is a compact operator on X if and only if for every bounded sequence $\{f_n\}$ of X, $\{C_\varphi(f_n)\}$ has a convergent subsequence in X.

The composition operator C_φ has been thoroughly studied on the Cauchy space $\mathbf{K}$ (for instance in [3] and [4]) and on the fractional Cauchy spaces $\mathbf{K}_\alpha$ (for instance in [2] and [5]). In particular, it is known that:

1. If $\alpha > 0$ and φ is a conformal automorphism of $\mathbf{D}$, then $C_\varphi(f) = f \circ \varphi \in \mathbf{K}_\alpha$ for every $f \in \mathbf{K}_\alpha$.
2. If $\alpha \geq 1$ and φ is an analytic self-map of the unit disc $\mathbf{D}$, then $C_\varphi(f) = f \circ \varphi \in \mathbf{K}_\alpha$ for every $f \in \mathbf{K}_\alpha$.
3. Let G_α denote the set of functions that are subordinate to $K^\alpha(z) = (1 - z)^{-\alpha}$ in $\mathbf{D}$. If $\alpha \geq 1$ then a function f belongs to the closed convex hull of G_α if and only if there is a probability measure $\mu \in \mathbf{M}^*$ such that $f(z) = \int_{\mathbf{T}} K^\alpha(xz)\, d\mu(x)$.
4. C_φ is compact on $\mathbf{K}$ if and only if $C_\varphi(\mathbf{K}) \subset (\mathbf{K})_a$.
5. If $\alpha \geq 1$ then C_φ is compact on $\mathbf{K}_\alpha$ if and only if $C_\varphi[K^\alpha(xz)] \in (\mathbf{K}_\alpha)_a$ for all $|x| = 1$.

The results 1–3 are in [5], the result 4 is known from [3] and was extended to the result 5 in [2]. The operator C_φ is also bounded and Möbius invariant on $\mathbf{K}_e$. There is no loss of generality in assuming that $\varphi(0) = 0$, and we will assume so throughout the article. Our focus then is only on when the composition operator is compact on $\mathbf{K}_e$.

Our main result extends the one in [3] to the case of Exponential Cauchy Transforms.

Main Theorem *If* C_φ *is the composition operator then we have the following*:

1. *The operator* C_φ *is compact in* $\mathbf{K}_e$ *if and only if* $C_\varphi(\mathbf{K}_e) \subset (\mathbf{K}_e)_a$.

2. *Let* $\varphi \in H(\mathbf{D})$, *with* $\|\varphi\|_\infty < 1$. *Then* C_φ *is compact on* $\mathbf{K}_e$.

We need the following interesting two results due to Y. Abu Muhanna and D. Hallenbeck in [1].

Theorem 1 *Let* Δ *be a bounded convex body, with* $0 \in \Delta$ *and let* H *be a covering function mapping the unit disk onto the exterior of the bounded convex body* $\Omega = c\Delta$. *Suppose that* $\log H$ *is univalent and also maps the unit disk onto the complement of a convex set. Then any analytic function* f *subordinate to* H *can be expressed as*

$$f(z) = \int_{\mathbf{T}} H(xz)\, d\mu(x) \tag{7}$$

for some positive Borel measure μ *on the unit circle with* $\|\mu\| = 1$.

The previous theorem includes the following special case.

Theorem 2 *If* φ *is an analytic self-map of the unit disc* $\mathbf{D}$, *with* $\varphi(0) = 0$ *then there exist probability measures* $\mu, \nu \in \mathbf{M}^*$ *such that*

$$C_\varphi\big[K_e(z)\big] = K_e\big(\varphi(z)\big) = \int \exp\big(K(xz)\big)\, d\mu(x) = \exp\bigg(\int_{\mathbf{T}} K(xz)\, d\nu(x)\bigg).$$

Then λK_e, with $|\lambda| = 1$ are all of the universal coverings of $c\mathbf{D}$.

Lemma 3 *Suppose* $g_x(e^{it})$ *is a nonnegative* L^1*-continuous function of* x *and let* $\{\mu_n\}$ *be a sequence of nonnegative Borel measures that are weak* convergent to* μ. *Define*

$$w_n(t) = \int_{\mathbf{T}} g_x\big(e^{it}\big)\, d\mu_n(x) \quad \textit{and} \quad w(t) = \int_{\mathbf{T}} g_x\big(e^{it}\big)\, d\mu(x).$$

Then $\|w_n - w\|_{L^1} \longrightarrow 0$.

Proof Suppose $g_x(e^{it})$ is a nonnegative L^1-continuous function of x and for $z \in \mathbf{D}$ let

$$g_x(z) = \int \mathrm{Re}\bigg(\frac{1 + e^{it} z}{1 - e^{it} z}\bigg) g_x\big(e^{it}\big)\, d(t),$$

$$w_n(z) = \int g_x(z)\, d\mu_n(x) \quad \text{and}$$

$$w(z) = \int g_x(z)\, d\mu(x).$$

Notice that all functions are positive and harmonic in $\mathbf{D}$ and that the radial limits of $w_n(z)$ and $w(z)$ are $w_n(t)$ and $w(t)$, respectively. Then, for $|z| \le \rho < 1$,

$$\big|g_x(z) - g_y(z)\big| \le \frac{1}{1 - \rho} \big\|g_x\big(e^{it}\big) - g_y\big(e^{it}\big)\big\|_{L^1}.$$

The continuity condition implies that $g_x(z)$ is uniformly continuous in x for all $|z| \le \rho < 1$. Weak star convergence implies that $w_n(z) \to w(z)$ uniformly on $|z| \le \rho < 1$ and consequently the convergence is locally uniform on **D**. In addition, we have

$$\left\| w_n\left(\rho e^{it}\right) \right\|_{L^1} \to \left\| w\left(\rho e^{it}\right) \right\|_{L^1}.$$

Hence we conclude that

$$\left\| w_n\left(\rho e^{it}\right) - w\left(\rho e^{it}\right) \right\|_{L^1} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Now using Fatou's Lemma we conclude that

$$\left\| w_n\left(e^{it}\right) - w\left(e^{it}\right) \right\|_{L^1} \longrightarrow 0. \qquad \square$$

Lemma 4 *Let $g_x(e^{it})$ be a nonnegative L^1-continuous function of x such that $\|g_x\|_{L^1} \le a < \infty$ and $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$.*

Let $f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x)$, and let L be the operator given by

$$L\left[f(z)\right] = \iint g_x\left(e^{it}\right) K_e\left(e^{it} z\right) dt\, d\mu(x).$$

Then L is a compact operator on $\mathbf{K}_e$.

Proof First note that the condition that $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$ implies that the operator L is a well defined function on $\mathbf{K}_\alpha$. Let $\{f_n(z)\}$ be a bounded sequence in $\mathbf{K}_\alpha$ and let $\{\mu_n\}$ be the corresponding norm bounded sequence of measures in **M**. Since every norm bounded sequence of measures in **M** has a weak star convergent subsequence, without loss of generality we can assume that $\{\mu_n\}$ is convergent to $\mu \in \mathbf{M}$. We want to show that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_\alpha$. First, let us assume that $d\mu_n(x) \gg 0$ for all n, and let $w_n(t) = \int g_x(e^{it})\, d\mu_n(x)$ and $w(t) = \int g_x(e^{it})\, d\mu(x)$, then we know from Lemma 3 that $w_n(t), w(t) \in L^1$ for all n, and $w_n(t) \to w(t)$ in L^1. Now, since $g_x(e^{it})$ is a nonnegative continuous function in x and $\{\mu_n\}$ is weak star convergent to μ, we have

$$L\left(f_n(z)\right) = \iint K_e\left(e^{it} z\right) g_x\left(e^{it}\right) d(t)\, d\mu_n(x) = \int K_e\left(e^{it} z\right) w_n(t)\, dt,$$

$$L\left(f(z)\right) = \iint K_e\left(e^{it} z\right) g_x\left(e^{it}\right) d(t)\, d\mu(x) = \int K_e\left(e^{it} z\right) w(t)\, dt.$$

Furthermore, because $w_n(t)$ is nonnegative we have

$$\left\| L(f_n) \right\|_{\mathbf{K}_e} = \|w_n\|_{L^1},$$

$$\left\| L(f) \right\|_{\mathbf{K}_e} = \|w\|_{L^1}.$$

Now, since $\|w_n - w\|_{L^1} \to 0$ we get $\|L(f_n) - L(f)\|_{\mathbf{K}_e} \to 0$ which shows that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_e$ and thus L is a compact operator for the case where μ is a positive measure.

In the case where μ is a complex measure we write

$$d\mu_n(x) = \left(d\mu_n^1(x) - d\mu_n^2(x)\right) + i\left(d\mu_n^3(x) - d\mu_n^4(x)\right),$$

where each $d\mu_n^j(x) \gg 0$ and define $w_n^j(t) = \int g_x(e^{it})\, d\mu_n^j(x)$ then

$$w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x) = \left(w_n^1(t) - w_n^2(t)\right) + i\left(w_n^3(t) - w_n^4(t)\right).$$

Using an argument similar to the one above we get

$$w_n^j(t), w^j(t) \in L^1, \quad \text{and} \quad \left\| w_n^j - w^j \right\|_{L^1} \longrightarrow 0.$$

Hence $\|w_n - w\|_{L^1} \longrightarrow 0$, where

$$w(t) = \left(w^1(t) - w^2(t)\right) + i\left(w^3(t) - w^4(t)\right) = \int g_x\left(e^{it}\right) d\mu(x).$$

So,

$$\left\| L(f_n) - L(f) \right\|_{F_\alpha} \le \|w_n - w\|_{L^1} \longrightarrow 0.$$

Finally, we conclude that the operator L is compact. □

Now we are ready to prove our main theorem which characterizes compact composition operators on $\mathbf{K}_e$.

Theorem 3 *If φ is an analytic self-map of the unit disc $\mathbf{D}$, with $\varphi(0) = 0$ then the operator C_φ is compact on $\mathbf{K}_e$ if and only if $C_\varphi[K_e(xz)] \in (\mathbf{K}_e)_a$ for all x such that $|x| = 1$.*

Proof Assume that C_φ is compact on $\mathbf{K}_e$ and let $\{f_j(z)\}_{j=1}^{\infty}$ be the bounded sequence of functions defined as

$$f_j(z) = K_e(\rho_j xz) = \exp\left(\frac{1}{1 - \rho_j xz}\right) = \exp\left[K(\rho_j xz)\right],$$

where $0 < \rho_j < 1$ and $\lim_{j\to\infty} \rho_j = 1$. Clearly, $f_j \in H^\infty \cap \mathbf{K}_e$ and there exist probability measures $\mu_j \in \mathbf{M}^*$ such that

$$f_j(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_j(x),$$

where $\|f_j\|_{\mathbf{K}_e} = \|\mu_j\| = 1$. Since C_φ is compact on $\mathbf{K}_e$, then $C_\varphi(f_j) \in \mathbf{K}_e$ and $\|C_\varphi(f_j)\| \le \|C_\varphi\| \|f_j\|_{\mathbf{K}_e} = \|C_\varphi\|$ for all j. Furthermore, $C_\varphi(f_j) \in H^\infty \cap \mathbf{K}_e \subset (\mathbf{K}_e)_a$ for every j and thus there exists a nonnegative L^1-function $g_j(x)$ such that $d\mu_j(x) = g_j(x)\, dt$ and

$$C_\varphi\left[f_j(z)\right] = \int_{\mathbf{T}} K_e(xz) g_j(x)\, dt.$$

Since the operator C_φ is compact, the sequence $\{C_\varphi(f_j)\}_{j=1}^{\infty}$ has a convergent subsequence that converges to $C_\varphi[K_e(z)] \in (\mathbf{K}_e)_a$ because of Lemma 2 and the fact that $(\mathbf{K}_e)_a$ is a closed subspace of $\mathbf{K}_e$.

For the converse, let $f \in \mathbf{K}_e$. There exists a measure in $\mathbf{M}$ such that

$$f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x).$$

Then

$$(f\circ\varphi)(z)=C_\varphi\big[f(z)\big]=\int_{\mathbf{T}}K_e\big[x\varphi(z)\big]\,d\mu(x)=\int_{\mathbf{T}}C_\varphi\big[K_e(xz)\big]\,d\mu(x),$$

where by the assumption that $C_\varphi[K_e(xz)]\in(\mathbf{K}_e)_a$ and thus we have

$$C_\varphi\big[K_e(xz)\big]=\int_{\mathbf{T}}g_x\big(e^{it}\big)K_e\big(e^{it}z\big)\,dt,$$

where $g_x(e^{it})$ is a positive L^1-continuous function of x. Hence

$$\begin{aligned}C_\varphi(f)(z)&=\int_{\mathbf{T}}C_\varphi\big[K_e(xz)\big]\,d\mu(x)\\&=\int_{\mathbf{T}}\int_{\mathbf{T}}g_x\big(e^{it}\big)K_e\big(e^{it}z\big)\,dt\,d\mu(x)\end{aligned}$$

which was proven to be compact on $\mathbf{K}_e$ in Lemma 4. □

Now we are ready to give the proof of the main theorem.

Main Theorem *We have the following.*

1. *The operator C_φ is compact on $\mathbf{K}_e$ if and only if $C_\varphi(\mathbf{K}_e)\subset(\mathbf{K}_e)_a$.*
2. *Let $\varphi\in H(\mathbf{D})$ with $\|\varphi\|_\infty<1$. Then C_φ is compact on $\mathbf{K}_e$.*

Proof Let C_φ be the composition operator.

(1) $C_\varphi[K_e(xz)]=K_e[x\varphi(z)]\in H^\infty\cap\mathbf{K}_e\subset(\mathbf{K}_e)_a$ and is subordinate to $K_e(z)$ hence

$$C_\varphi\big[K_e(xz)\big]=\int_{\mathbf{T}}K_e\big(e^{it}z\big)g_x\big(e^{it}\big)\,dt\in(\mathbf{K}_e)_a,$$

where $g_x(e^{it})$ is a nonnegative L^1-function.

(2) If $\varphi\in H(\mathbf{D})$ with $\|\varphi\|_\infty<1$ then $C_\varphi[K_e(xz)]=K_e[x\varphi(z)]\in H^\infty\cap\mathbf{K}_e\subset(\mathbf{K}_e)_a$ for all x such that $|x|=1$ which implies that C_φ is compact on $\mathbf{K}_e$ from the previous theorem. Furthermore, there exists $g_x(e^{it})\ge 0$ such that

$$C_\varphi\big[K_e(xz)\big]=\int_{\mathbf{T}}K_e\big(e^{it}z\big)g_x\big(e^{it}\big)\,dt.$$

Since

$$1=C_\varphi\big[K_e(0)\big]=\int_{\mathbf{T}}K_e(0)g_x\big(e^{it}\big)\,dt=\int_{\mathbf{T}}g_x\big(e^{it}\big)\,dt,$$

we get $\|g_x(e^{it})\|_1=1$. □

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