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Minimal Surfaces whose Gauss Map Covers Periodically the Pointed Upper Half-Sphere Exactly Once

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Abstract. We identify nonparametric minimal surfaces S which have the property that their Gauss map $\vec{n}$ is periodic and covers the upper half-sphere minus the point $(0,0,1)$ exactly once on each horizontal half-strip of height 2π. This leads us to study periodic harmonic mappings defined on the left half-plane and univalent logharmonic mappings defined on the unit disk.

Keywords. Harmonic mappings, minimal surfaces.

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1. Introduction

Let G be a domain in $\mathbb{C}$ of the form

$$G = \{w = u + iv : -\infty < u < u_0(v),\ v \in \mathbb{R}\}, \tag{1.1}$$

where u_0 satisfies $u_0(v + 2\pi) \equiv u_0(v)$, i.e. u_0 is periodic with period 2π. We are interested in studying the nonparametric regular minimal surfaces over G whose Gauss map $\vec{n}$ is periodic, say $\vec{n}(\zeta + 2\pi i) = \vec{n}(\zeta)$, and have the property that the image of each half strip $D_k = \{\zeta = \xi + i\eta : \xi < 0,\ (-1 + k)\pi < \eta \le (1 + k)\pi\}$ is the open upper half-sphere minus the point $(0,0,1)$ covered exactly once.

If Ω is a simply connected and regulated proper subdomain of $\overline{\mathbb{C}}$ containing the point at infinity, then there exists a nonparametric minimal surface S over Ω whose Gauss map has the property that its image is the upper half-sphere covered exactly once if, and only if, the complement $\mathbb{C} \setminus \Omega$ is a compact convex continuum [BH2]. On the other hand, if Ω is a simply connected, bounded and regulated Jordan domain then the corresponding problem has a solution if, and only if, either $\partial\Omega$ has three or four convexity points where in the latter case $\partial\Omega$ satisfies

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a simple geometric property [BH1]. The problem discussed in this article is of interest, since it lies between the two above mentioned cases, in the sense that in this case infinity is neither interior nor exterior to Ω but belongs to $\partial\Omega$.

In dealing with this problem we are led to study univalent harmonic mappings $F(\zeta) = u(\zeta)+iv(\zeta)$ from the left half-plane $D = \{\zeta : \operatorname{Re}\zeta < 0\}$ onto G satisfying

$$F(\zeta + 2\pi i) \equiv F(\zeta) + 2\pi i, \qquad \zeta \in D,$$

and

$$\operatorname{Re} F(-\infty) = \lim_{\xi\to-\infty} \operatorname{Re} F(\xi + i\eta) = -\infty.$$

Hence, the partial derivatives of F are periodic analytic functions on D with period $2\pi i$. Without loss of generality, we may assume that F is sense-preserving. By [AH2, Lemma 2.2] it follows that F admits the representation

$$F(\zeta) = \zeta + 2\beta\xi + H(\zeta) + \overline{G(\zeta)} \tag{1.2}$$

where

- $\operatorname{Re}\beta > -1/2$,
- H and G are analytic in D,
- $G(-\infty) = \lim_{\xi\to-\infty} G(\xi + i\eta) = 0$,
- $H(-\infty) = \lim_{\xi\to-\infty} H(\xi + i\eta)$ exists and is finite,
- $H(\zeta + 2\pi i) \equiv H(\zeta) + 2\pi i$ and $G(\zeta + 2\pi i) \equiv G(\zeta) + 2\pi i$ on D.

Furthermore, the second dilatation function

$$A(\zeta) = \frac{\overline{\beta} + G'(\zeta)}{1 + \beta + H'(\zeta)}$$

of F satisfies

- $A \in H(D)$ and $|A| < 1$ on D,
- $A(\zeta + 2\pi i) \equiv A(\zeta)$ and
- $A(-\infty) = \lim_{\xi\to-\infty} A(\xi + i\eta)$ exists and is finite.

Observe that β as defined in (1.2) depends only on $A(-\infty)$. Indeed we have

$$A(-\infty) = \frac{\overline{\beta}}{1+\beta}.$$

Consider the mapping $f(z) = \exp F(\log z)$, $\zeta \in U$, or equivalently, $F(\zeta) = \log f(e^\zeta)$, $\zeta \in D$. Observe that f is of the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \qquad z \in U, \tag{1.3}$$

where $\operatorname{Re}\beta > -1/2$ and where h and g are nonvanishing analytic functions on $U = \{z : |z| < 1\}$. Note that F is univalent on D if, and only if, f is univalent on U. Furthermore, F maps D onto a bounded domain Ω and satisfies the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = \frac{a\overline{f}}{f} f_z, \qquad a \in H(U),\ |a| < 1, \tag{1.4}$$

where $a(z) = A(\log z)$ and in particular, $a(0) = A(-\infty) = \overline{\beta}/(1+\beta)$.

We first note that the interplay between f and F will often be used whenever it clarifies the representation of the discussed material. Any nonconstant solution of (1.4) is called a logharmonic mapping. Such mappings have been studied in several papers (e.g. [AB], [AH1], [AH2] and [AH3]). In many cases, it is easier to work with logharmonic mappings of the form (1.3) than with harmonic mappings of the form (1.2) even if the differential equation is nonlinear. For instance, if f is a univalent logharmonic mapping on U satisfying $f(0) = 0$ then f is of the form (1.3). If in addition, $f_z(0) > 0$, then f is an automorphism on U if, and only if, there is a normalized starlike conformal mapping ϕ and two constants $C \neq 0$ and β, $\operatorname{Re}\beta > -1/2$, such that

$$f(z) = C|z|^{2\beta+1}\frac{\phi(z)}{|\phi(z)|}, \qquad z \in U, \tag{1.5}$$

where $1^{2\beta} = 1$ (cf. [AB]). Using the transformation $F(\zeta) = \log f(e^{\zeta})$, we conclude that F is a sense-preserving automorphism on the left half-plane D if, and only if, there is a β, $\operatorname{Re}\beta > -1/2$, and a probability measure μ defined on the Borel σ-algebra over $[0, 2\pi)$ such that

$$F(\zeta) = \zeta + 2\beta\xi + c - 2i\int_0^{2\pi} \arg(1 - e^{it+\zeta})\, d\mu(t)$$

where c stands for a constant.

A regular nonparametric surface S over G with periodic normal directions is a minimal surface if, and only if, there exists a univalent harmonic mapping from D onto G such that the third coordinate s of S satisfies the minimal surface equation ${s_\zeta}^2(\zeta) = -A(\zeta)F_\zeta^2(\zeta)$. The Gauss map of S depends only on $A(\zeta)$ and is

$$\vec{n} = \left(2\operatorname{Im}\sqrt{A}, 2\operatorname{Re}\sqrt{A}, \frac{1-|A|}{1+|A|}\right).$$

Observe that $\vec{n}$ is vertical if, and only if, $A = 0$ and it is horizontal if, and only if, $|A| = 1$. The differential condition for $s(u, v)$ implies that the second dilatation function A is the square of an analytic function on D. Our condition on the Gauss map implies that $\sqrt{A}$ is a conformal mapping from D_k onto the unit disk U minus the origin. Hence, except for a rotation and a translation of the surface, our problem reduces to studying univalent harmonic mappings of the form (1.2) from D onto a domain G with $\beta = 0$ which is a solution of the linear system

$$\overline{F_{\overline{\zeta}}(\zeta)} = \exp(2\zeta)F_\zeta(\zeta). \tag{1.6}$$

Finally, in terms of logharmonic mappings, our problem can be stated as follows:

Problem 1. *Characterize the bounded domains Ω and the univalent solutions f of*

$$\overline{f_{\bar{z}}} = \frac{z^2\overline{f}}{f} f_z, \qquad z \in U. \tag{1.7}$$

which map U onto Ω.

The paper is organized as follows. In Section 2 we introduce some results on the behaviour of the boundary correspondence $f^*(e^{it})$ of a univalent solution of (1.4) where the dilatation function a satisfies $|a| = 1$ on a subinterval I of the unit circle and admits a continuous extension across it.

In Section 3, we give a geometric characterization of the image domain for the case where a is a finite Blaschke product and f is a univalent solution of (1.4). We show that the number of complete resting points is equal to the degree of the Blaschke product. It follows that the image $f(U)$ contains at most N points of logarithmic convexity.

In Section 4, we consider the inverse problem and characterize univalent solutions of (1.4) by means of their boundary correspondence. In special cases, where a is a finite Blaschke product, we also provide a constructive existence proof .

Finally, we solve the given problem in Section 5.

2. Boundary behaviour of logharmonic maps on an interval I of the unit circle where $|a| = 1$

We start with the following Lemma stated in [AB] (cf. [HS1]).

Lemma 2.1. *Let Ω be a simply connected bounded domain of $\mathbb{C}$ whose boundary $\partial\Omega$ is locally connected. Let $a(z)$ be an analytic function on U, $|a| < 1$. Let $f(z)$ be a univalent solution of (1.4) which maps U onto Ω and satisfies $f(0) = 0$. Then the nonrestricted limit $f^*(e^{it})$ of f at e^{it} exists on $\partial U \setminus E$, where E is a countable set. If $e^{it} \in E$, then f^* jumps at e^{it} and the cluster set at e^{it} is a subinterval of a logarithmic spiral.*

Next, we show that the boundary values of f depend strongly on the values of $a(e^{it})$ (compare [BH1]).

Theorem 2.2. *Let Ω be a simply connected bounded domain of $\mathbb{C}$ whose boundary $\partial\Omega$ is locally connected and let $a(z)$ be an analytic function on U, $|a| < 1$. Suppose that the function $a(z)$ has an analytic extension across an open subinterval $I = \{e^{it} : \sigma < t < \tau < \sigma + 2\pi\}$ of the unit circle ∂U such that $|a(z)| \equiv 1$ on I. Let $f(z)$ be a univalent solution of (1.4) which maps U onto Ω and satisfies $f(0) = 0$. Then the following relations hold on I.*

(a) *Let* $\sigma < \tau < \sigma + 2\pi$ *and choose* $\arg f$ *as a continuous function on the the set* $Y := \{z : 1/2 < |z| < 1, \sigma < \arg z < \tau\}$. *Then for* $\sigma < t < t + h < \tau$ *we have*

$$\begin{aligned}\log f^*(e^{i(t+h)}) - \overline{a(e^{i(t+h)}) \log f^*(e^{i(t+h)})} - \log f^*(e^{it}) \\ + \overline{a(e^{it}) \log f^*(e^{it})} + \overline{\int_t^{t+h} \log f^*(e^{i\phi})\, da(e^{i\phi})} \equiv 0.\end{aligned} \tag{2.1}$$

(b) *If* f^* *is continuous at* e^{it}*, then we have*

$$\lim_{h \downarrow 0} \mathrm{Im} \left(\sqrt{a(e^{it})}\, \frac{\frac{f^*(e^{i(t+h)})}{f^*(e^{i(t-h)})} - 1}{h} \right) = 0. \tag{2.2}$$

(c) *If* f^* *jumps at* e^{it}*, which must and can happen only when* $f^*(I)$ *lies on a segment of a logarithmic spiral, then for* $q \in f^*(I)$*, we have*

$$\arg \left(\log \frac{f^*(e^{i(t+0)})}{q} \right) = -\frac{1}{2} \arg a(e^{it}) \mod \pi.$$

(d) *If* f^* *is not constant on a subinterval of* I*, then the right limit*

$$\lim_{h \downarrow 0} \arg \left(\frac{f^*(e^{i(t+h)})}{f^*(e^{i(t+0)})} - 1 \right) = -\frac{1}{2} \arg a(e^{it}) \mod \pi \tag{2.3}$$

exists everywhere on I.

Proof. The relation (2.1) follows from the theory of univalent harmonic mappings. Indeed, $\log f$ is harmonic on Y. Let ψ be a conformal mapping from the unit disk U onto Y. Apply Theorem 2.2 in [BH1] to $\log f \circ \psi$ whose dilatation function is $a \circ \psi$.

Consider a continuous function $w(t)$, $t \geq 0$, $w(t) \neq 0$ and define $\log w(t)$ as a continuous function on $t \geq 0$. Set $w_0 = w(0)$ and $\Delta w(t) = (w(t) - w_0)$. Then, for t close to 0, we have

$$\log \frac{w(t)}{w_0} = \log \left(\frac{w(t)}{w_0} - 1 + 1 \right) = \log \left(\frac{\Delta w(t)}{w_0} + 1 \right) = \frac{\Delta w(t)}{w_0}(1 + o(1)). \tag{2.4}$$

Suppose now that f^* is continuous at e^{it}. Then by (2.1) we have

$$
\begin{aligned}
&-\frac{1}{2h}\overline{\int_{t-h}^{t+h} \log f^*(e^{i\phi})\, da(e^{i\phi})} \\
&\quad = \frac{\log f^*(e^{i(t+h)}) - \log f^*(e^{i(t-h)})}{2h} - a(e^{it})\overline{\frac{\log f^*(e^{i(t+h)}) - \log f^*(e^{i(t-h)})}{2h}} \\
&\qquad - \overline{\frac{a(e^{i(t+h)}) - a(e^{it})}{2h} \log f^*(e^{i(t+h)})} - \overline{\frac{a(e^{it}) - a(e^{i(t-h)})}{2h} \log f^*(e^{i(t-h)})} \\
&\quad = \frac{i}{h}\overline{\sqrt{a(e^{it})}}\,\mathrm{Im}\left(\sqrt{a(e^{it})}[\log f^*(e^{i(t+h)}) - \log f^*(e^{i(t-h)})]\right) \\
&\qquad - \overline{\frac{a(e^{i(t+h)}) - a(e^{it})}{2h} \log f^*(e^{i(t+h)})} - \overline{\frac{a(e^{it}) - a(e^{i(t-h)})}{2h} \log f^*(e^{i(t-h)})}.
\end{aligned}
$$

Letting h tend to zero we get

$$
\lim_{h\downarrow 0} \mathrm{Im}\left(\sqrt{a(e^{it})}\frac{\log f^*(e^{i(t+h)}) - \log f^*(e^{i(t-h)})}{h}\right). \tag{2.5}
$$

Finally, (2.2) follows from (2.4) and (2.5).

Next, suppose that f^* has a jump at e^{it}. Then the corresponding harmonic mapping F^* has a jump at it and its cluster set is a linear segment L. For $Q \in L$, we have

$$
\begin{aligned}
\arg(F^*(i(t+0)) - Q) &= \arg(F^*(i(t+0)) - F^*(i(t-0))) \\
&= -\frac{1}{2}\arg a(e^{it}) \mod \pi
\end{aligned}
$$

which, for $q = \exp(Q)$, implies that

$$
\begin{aligned}
\arg(\log f^*(e^{i(t+0)}) - \log q) &= \arg(\log f^*(e^{i(t+0)}) - \log f^*(e^{i(t-0)})) \\
&= -\frac{1}{2}\arg a(e^{it}) \mod \pi.
\end{aligned}
$$

Finally, (2.3) is an immediate consequence of (2.5) and the statement (c). This concludes the proof of Theorem 2.2. ∎

The relation (2.1) can be expressed in the differential form

$$
a(e^{it})\frac{df^*(e^{it})}{f^*(e^{it})} - \overline{\frac{df^*(e^{it})}{f^*(e^{it})}} = 0 \qquad \text{on } I, \tag{2.6}
$$

or equivalently, by

$$
\lim_{h\downarrow 0} \mathrm{Im}\left(\{\sqrt{a(e^{it})}\frac{df^*(e^{it})}{f^*(e^{it})}\right) = 0 \qquad \text{on } I. \tag{2.7}
$$

Hence, unless $df^*(e^{it}) = 0$, we have that

$$\arg \frac{df^*(e^{it})}{f^*(e^{it})} = -\frac{1}{2} \arg a(e^{it}) \mod \pi \qquad \text{on } I. \tag{2.8}$$

Corollary 2.3. *Let $a(z)$ be a finite Blaschke product. Then, for $I = \partial U$, the relations* (2.1) *to* (2.4) *and* (2.6) *to* (2.8) *are satisfied.*

The following notation is due to C. Pommerenke (see e.g. [P]).

Definition 2.4.

(a) *We call $\theta(\tau)$ a regulated function on the interval $[a, b]$ if the one-sided limits $\theta(\tau + 0)$ and $\theta(\tau - 0)$ exist for all $t \in [a, b]$.*
(b) *Let Ω be a simply connected domain of $\mathbb{C}$ and suppose that its boundary $\partial\Omega$ is locally connected (every prime end is a singleton). Let ϕ be a conformal mapping from U onto Ω. We call Ω a regulated domain if, for each prime end $q = w(\tau) = \phi(e^{i\tau})$ of $\partial\Omega$, the direction angle of the forward (half-)tangent at $w(\tau)$,*

$$\theta(q) = \lim_{s \downarrow \tau} \arg(w(s) - w(\tau)) = \lim_{s \downarrow \tau} \arg(w(s) - q), \tag{2.9}$$

exists and defines a regulated function.

For further details see [P].

As an immediate consequence of (2.3) we have

Theorem 2.5. *Let Ω be a bounded domain whose boundary is locally connected and let $a(z)$ be a finite Blaschke product of degree N. Let f be a univalent solution of* (1.4) *such that $f(0) = 0$ and $f(U) = \Omega$. Then Ω is a regulated domain.*

In what follows, we need the following definition [BH1].

Definition 2.6. *Let Ω be a simply connected regulated domain of $\mathbb{C}$ and let f be a univalent harmonic orientation-preserving mapping from U onto Ω. Let q be a prime end of $\partial\Omega$.*

(a) *If q does not belong to a jump of f^*, we define $\gamma(q)$ and $\delta(q)$ by $(f^*)^{-1}(q) = J(q) = \{e^{it} : \gamma(q) \le t \le \delta(q)\}$.*
(b) *If q is an interior point of a jump of f^* at e^{it}, i.e., if*

$$q = f^*(e^{i(t+0)})^{\lambda} f^*(e^{i(t-0)})^{1-\lambda}, \qquad 0 < \lambda < 1,$$

then define $\gamma(q) = \delta(q) = t$.
(c) *If q is the end point $f^*(e^{i(t-0)})$ of a jump of f^* at e^{it}, then define $\gamma(q)$ as in* (a) *and put $\delta(q) = t$.*
(d) *If q is the end point $f^*(e^{i(t+0)})$ of a jump of f^* at e^{it}, then put $\gamma(q) = t$ and define $\delta(q)$ as in* (a).

Observe that the cluster sets $C(f^*, e^{i\gamma(q)})$ and $C(f^*, e^{i\delta(q)})$ contain q but they may also contain other points if a jump appears. Furthermore, if $J(q) = (f^*)^{-1}(q)$ is a continuum then $|a| \equiv 1$ on $J(q)$. Finally, relation (2.3) implies that

$$\theta(q) = \lim_{h\downarrow 0} \arg\left(\frac{f^*(e^{i(\delta(q)+h)})}{f^*(e^{i(\delta(q)+0)})} - 1\right) = -\frac{1}{2}\arg a(e^{it}) \mod \pi$$

exists at each prime end $q \in f^*(I)$.

Our next result connects between the opening angle $\alpha(q)$ at $q \in \partial\Omega$ and the inverse image $f^{-1}(q)$.

Theorem 2.7. *Let Ω be a simply connected bounded regulated domain of $\mathbb{C}$. Suppose that $|a| < 1$ on U and that a admits an analytic continuation across an open interval $I \subset \partial U$ such that $|a(z)| \equiv 1$ there. Let f be a univalent solution of* (1.4) *such that $f(0) = 0$ and $f(U) = \Omega$. Let $q \in f^*(I)$ and define $J(q) = \{e^{it} : \gamma < t < \delta\}$ as defined in Definition 2.6. Denote by $\alpha(q)$ the opening angle at q as seen from the inside of Ω. Set $A(t) = \arg a(e^{it})$, $e^{it} \in I$, as a continuous function and define $\triangle A(q) = (A(\delta(q)) - A(\gamma(q)))/2$. Then we have the following relation between $\alpha(q)$ and $\triangle A(q)$.*

(a) *If $0 \le \alpha(q) < \pi$, then $\alpha(q) = \triangle A(q)$.*

(b) *If $\pi \le \alpha(q) \le 2\pi$, then either $\alpha(q) = \triangle A(q)$ or $\alpha(q) = \triangle A(q) + \pi$.*

Theorem 2.7 states that the total change of $(\arg a(e^{it}))/2$ over the interval $J(q) = f^{-1}(q)$ is either equal to the opening angle $\alpha(q)$ as seen from the inside of the domain or, if $\pi \le \alpha(q) \le 2\pi$, it can also be $\alpha(q) - \pi$. We shall use the following terminology.

Definition 2.8. *A prime end $q \in \partial\Omega$ is said to be a complete resting point of f^* if $\triangle A(q) = \alpha(q)$.*

Remark 2.9.

(a) If the prime end q is an interior point of a linear segment of $f^*(I)$, then either q is an interior point of a jump of f in which case $\triangle A(q) = 0$ or the inverse image $f^{-1}(q)$ is not a singleton and we have $\triangle A(q) = \pi$.

(b) Each prime end with an opening angle $\alpha(q)$ strictly less than π is a complete resting point of f^*. In particular, if $\alpha(q) = 0$, then $(f^*)^{-1}(q)$ is a singleton yet q is still a complete resting point of f^*. On the other hand, if $\alpha(q) > \pi$, it may happen that $(f^*)^{-1}(q)$ is an interval of $\partial\Delta$ with nonempty interior but q is not a complete resting point.

Proof of Theorem 2.7. One can adopt the proof of Theorem 2.13 in [BH1] while focusing on the following changes. First, one extends f across $J(q)$ logharmonicly and shows that for $t \in J(q)$,

$$Q(t) = \arg \frac{\frac{\partial}{\partial r} f(e^{it})}{f(e^{it})}$$

is a decreasing function in t of a total variation bounded by $\alpha(q)$. Then one observes that for $0 < \alpha(q) < \pi$ the variation is equal to $\alpha(q)$. For $\pi \le \alpha(q) < 2\pi$, $Q(t)$ may have jumps that add up to π. ∎

3. A geometric charaterization of the image domain

From now on we assume that

$$a(z) = e^{i\gamma} \prod_{k=1}^{N} \frac{z - p_k}{1 - \overline{p_k} z} = \sum_{k=0}^{\infty} \alpha_k z^{-k}, \qquad |p_k| < 1,\ 1 \le k \le N, \tag{3.1}$$

is a finite Blaschke product of degree N. Let

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \qquad \operatorname{Re}\beta > -\frac{1}{2},$$

be a univalent solution of the partial differential equation (1.4), i.e.,

$$\overline{f_{\overline{z}}(z)} = a(z)\frac{\overline{f(z)}}{f(z)} f_z(z),$$

where h and g are nonvanishing analytic functions in U. We are interested in characterizing the image domains Ω of such mappings. Observe that

$$\overline{\beta} = a(0)(1 + \beta)$$

so that β depends only on $a(0)$. On the other hand (2.6) relates $\partial\Omega$ to a.

Definition 3.1. *Let Ω be a simply connected bounded regulated domain of $\mathbb{C}$. We say that a prime end $q \in \partial\Omega$ is a point of logarithmic convexity (with respect to Ω) if there exists a neighborhood V of $f^{-1}(q)$ and a line segment L containing* $\log q$ *as an interior point such that $L \setminus \{\log q\}$ lies in the exterior of* $\log f(U \cap V)$. *In other words,* $\log q$ *is point of convexity of $F(it)$, $t \in \mathbb{R}$, where $F(\zeta) = \log f(e^\zeta)$.*

The only bounded regulated domains Ω which have no points of logarithmic convexity are disks. This is in contrast to the fact that any bounded domain of $\mathbb{C}$ has at least three points of convexity. If a is a finite Blaschke product, then every prime end which is a point of logarithmic convexity with opening angle $\alpha(q) < \pi$ is a complete resting point of f^*.

Our next result is a direct consequence of Theorem 2.7 and links the number of complete resting points with the degree of the Blaschke product.

Theorem 3.2. *Let Ω be a simply connected, bounded and regulated domain and let a be a Blaschke product of degree N. Let f be a univalent solution of* (1.4) *with respect to this Blaschke product which maps U onto Ω. Then Ω has exactly N complete resting points and hence, $\partial\Omega$ has at most N points of logarithmic convexity.*

Proof. We shall proceed as in the proof of Theorem 3.3 in [BH1]. Fix $q_0 \in \partial U$. For $\delta(q_0) \le t \le 2\pi + \delta(q_0)$, we define

$$\begin{aligned} B(t) &= \pi \sum_{\alpha(q)=\triangle A(q)} H_q(t) - \pi \sum_{\substack{\alpha(q)=2\pi \\ \delta(q)=\gamma(q)}} H_q(t) - \frac{1}{2}A(t), \\ B(\delta(q_0)) &= \theta(q_0), \end{aligned} \tag{3.2}$$

$H_q(t) = 0$ if $t < \delta(q)$ and $H_q(t) = 1$ if $\delta(q) \le t$. The first sum is taken over the set of all complete resting points and the second sum is taken over all prime ends $q \in f^*(\partial U)$ satisfying $\alpha(q) = 2\pi$ and $\delta(q) = \gamma(q)$. Let us remark that for each prime end $q \in f^*(\partial U)$ we have $B(\delta(q)) = \theta(q)$ and $B(\gamma(q)) = \theta_L(q)$ where $\theta_L(q) - \pi$ is the direction angle of the backward half-tangent of $\partial\Omega$. We want to show that the first sum in (3.2) contains only finitely many terms and that there are no terms in the second sum. We begin by showing that the second sum contains only finitely many terms. Indeed, if not, there is a sequence of points $q_j \in f^*(\partial U)$ which converges from one side to a point $q_1 \in f^*(\partial U)$ for which $\theta(q_j) - \theta_L(q_j)$ converges to $-\pi$ as $j \to \infty$. Therefore, the sequence $\theta(q_j)$ does not converge which contradicts the hypothesis that Ω is a regulated domain.

Next, since $B(\delta(q_0) + 2\pi) - B(\delta(q_0))$ is finite, we conclude that n_R, the number of complete resting points of $f^*(\partial U)$, is finite. Let now $\alpha(q_1) = 2\pi$, $q_1 \in f^*(\partial U)$. If q_1 is not the end of a linear segment pointing inside Ω, then there must be infinitely many complete resting points in each neighborhood of q_1 which contradicts the last conclusion. If q_1 is the end of a linear segment pointing inside Ω, then the case $\delta(q_1) = \gamma(q_1)$ is excluded since q_1 cannot be an interior point of a jump. Therefore $\alpha(q_1) = 2\pi$ is excluded and (3.2) reduces to

$$B(\delta(q_0) + 2\pi) - B(\delta(q_0)) = \pi n_R - \frac{1}{2}\left(A(\delta(q_0) + 2\pi) - A(\delta(q_0))\right).$$

Since B is periodic, we conclude that

$$B(\delta(q_0) + 2\pi) - B(\delta(q_0)) = 0$$

and hence $(A(\delta(q_0 + 2\pi) - A(\delta(q_0)))/2 = N\pi$. This completes the proof of Theorem 3.2. ∎

4. The inverse problem

Suppose that Ω is a bounded regulated domain whose boundary $\partial\Omega$ has n_C points of logarithmic convexity. Assume that a is a finite Blaschke product of degree N, $N \ge n_C$, and that the boundary correspondence f^* satisfies the relation (2.6). Furthermore, let $N - n_C$ additional complete resting points of f^* be given. It is natural to ask if there is a univalent solution f of (1.4) such that $f(0) = 0$, $f(U) = \Omega$ and $f^*(e^{it})$ are the nontangential limits of f a.e. on ∂U. Unfortunately the answer is negative and additional conditions must be satisfied.

Define

$$f_1(z) = \frac{f(z)}{z|z|^{2\beta}}.$$

Then $\log f_1(z)$ is a bounded harmonic function in the unit disk U. Using the Poisson representation formula for $\log f_1$, we have

$$\begin{aligned}
\log f_1(z) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) \log f_1^*(e^{it})\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) (\log f^*(e^{it}) - it) dt \\
&= \frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}(\log f^*(e^{it}) - it)\, dt \\
&\quad + \frac{1}{4\pi}\int_0^{2\pi} \frac{e^{-it}+\overline{z}}{e^{-it}-\overline{z}}(\log f^*(e^{it}) - it)\, dt.
\end{aligned}$$

Differentiating with respect to z, we conclude that

$$(4.1) \qquad \begin{aligned}
\frac{(f_1)_z(z)}{f_1(z)} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}(\log f^*(e^{it}) - it)\, dt, \\
\overline{\frac{(f_1)_{\overline{z}}(z)}{\overline{f_1(z)}}} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}(\overline{\log f^*(e^{it})} + it)\, dt.
\end{aligned}$$

Next, using (4.1) and integration by parts, yields
(4.2)

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{it}-z}\frac{df^*(e^{it})}{f^*(e^{it})} &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{it}-z}\left(\frac{df^*(e^{it})}{f^*(e^{it})} - idt\right) \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{d}{dt}\left(\frac{1}{e^{it}-z}\right)(\log f^*(e^{it}) - it)\, dt \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{-ie^{it}}{(e^{it}-z)^2}(\log f^*(e^{it}) - it)\, dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}(\log f^*(e^{it}) - it)\, dt \\
&= \frac{(f_1)_z(z)}{f_1(z)} = \frac{f_z(z)}{f(z)} - \frac{1+\beta}{z}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{e^{it}-z}\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)} &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{1}{e^{it}-z}\left(\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)}+idt\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}-z)^2}(\overline{\log f^*(e^{it})}+it)\,dt \qquad (4.3)\\
&= \frac{\overline{(f_1)_{\overline{z}}(z)}}{\overline{f_1(z)}} = \frac{\overline{(f)_{\overline{z}}(z)}}{\overline{f(z)}} - \frac{\overline{\beta}}{z}.
\end{aligned}$$

In what follows we need the following property of boundary correspondence.

Definition 4.1. *Let Ω be a simply connected, bounded and regulated domain in $\mathbb{C}$ and let ϕ be a conformal univalent mapping from U onto Ω. We say that a (orientation-preserving) mapping $f^*(e^{it})$ from ∂U into $\partial\Omega$ is a quasihomeomorphism onto $\partial\Omega$ if $f^*(e^{it})$ is the pointwise limit of a sequence of (orientation-preserving) homeomorphisms from ∂U onto $\partial\Omega$ such that the spiral jumps from $f^*(e^{i(t-0)})$ to $f^*(e^{i(t+0)})$ are parts of $\partial\Omega$.*

Theorem 4.2. *Let*

$$a(z) = e^{i\gamma}\prod_{k=1}^{m}\left(\frac{z-p_k}{1-\overline{p_k}z}\right)^{n_k} \qquad (4.4)$$

$n_k > 0$ and $|p_k| < 1$, if $1 \le k \le m$, $p_k \neq p_j$ if $k \neq j$ and $\sum_{k=1}^{m} n_k = N$, be a finite Blaschke product of degree N and let Ω be a regulated bounded domain of $\mathbb{C}$ whose boundary has at most N points of logarithmic convexity. Let $f^*(e^{it})$ be a positively oriented quasihomeomorphism from the unit circle ∂U onto $\partial\Omega$ satisfying (2.6), i.e.,

$$\mathrm{Im}\left(\sqrt{a(e^{it})}\frac{df^*(e^{it})}{f^*(e^{it})}\right) = 0$$

on ∂U. Then the mapping

$$f(z) = z|z|^{2\beta}exp\left(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right)(\log f^*(e^{it})-it)\,dt\right) \qquad (4.5)$$

where $\beta = \overline{a(0)}(1+a(0))/(1-|a(0)|^2)$, is a univalent solution of (1.4),

$$\overline{f_{\overline{z}}(z)} = a(z)\frac{\overline{f(z)}}{f(z)}f_z(z),$$

which maps U onto Ω if, and only if,

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{a(e^{it})-a(z)}{e^{it}-z}\frac{df^*(e^{it})}{f^*(e^{it})} \equiv \frac{1+\overline{a(0)}}{1-|a(0)|^2}\,\frac{a(z)-a(0)}{z} \qquad (4.6)$$

on U.

Proof. Using (2.6) ,(4.2) and (4.3), we have

$$\begin{aligned}
\frac{\overline{(f)_{\overline{z}}(z)}}{\overline{f(z)}} - a(z)\frac{f_z(z)}{f(z)} &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{it}-z}\left(\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)} - a(z)\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)\right) \\
&\quad + \frac{\overline{\beta}}{z} - a(z)\frac{1+\beta}{z} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{a(e^{it})-a(z)}{(e^{it}-z)}\,\frac{df^*(e^{it})}{f^*(e^{it})} + \frac{\overline{\beta}}{z} - a(z)\frac{1+\beta}{z} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{a(e^{it})-a(z)}{(e^{it}-z)}\,\frac{df^*(e^{it})}{f^*(e^{it})} \\
&\quad -(1+\beta)\left(\frac{a(z)-a(0)}{z}\right) \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{a(e^{it})-a(z)}{(e^{it}-z)}\,\frac{df^*(e^{it})}{f^*(e^{it})} \\
&\quad - \frac{1+\overline{a(0)}}{1-|a(0)|^2}\left(\frac{a(z)-a(0)}{z}\right).
\end{aligned}$$

Hence, f is a solution of (1.4) if, and only if, (4.6) holds true. Applying the argument principle, which holds for solutions of (1.4), we conclude that f is univalent on U and that $f(U) = \Omega$. ■

In what follows, we show that the conditions in Theorem 4.2 can be replaced by (2.6) and a system of $[N/2]$ equations($[x]$ denotes the integer part of a positive number x). Let $a(z)$ be a finite Blaschke product of the form (4.4) and define

$$\begin{aligned}
p(z) &= \prod_{k=1}^{m}(z-p_k)^{n_k}, \\
q(z) &= \prod_{k=1}^{m}(1-\overline{p_k}z)^{n_k}, \\
\tau(z) &= \frac{e^{-i\gamma/2}}{2\pi i}zq(z)\int_0^{2\pi} \frac{a(e^{it})-a(z)}{e^{it}-z}\,\frac{df^*(e^{it})}{f^*(e^{it})} \\
&= \frac{e^{-i\gamma/2}zq(z)}{2\pi i}\int_0^{2\pi} \frac{a(e^{it})}{e^{it}-z}\,\frac{df^*(e^{it})}{f^*(e^{it})} - \frac{e^{i\gamma/2}}{2\pi i}\int_0^{2\pi} \frac{zp(z)}{e^{it}-z}\,\frac{df^*(e^{it})}{f^*(e^{it})} \\
&= \frac{e^{-i\gamma/2}zq(z)}{2\pi i}\int_0^{2\pi} \frac{1}{e^{it}-z}\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)} \\
&\quad - \frac{e^{i\gamma/2}zp(z)}{2\pi i}\int_0^{2\pi} \frac{1}{e^{it}-z}\,\frac{df^*(e^{it})}{f^*(e^{it})}.
\end{aligned} \tag{4.7}$$

Observe that τ is a polynomial of degree at most N. Define

$$\sigma(z)=\frac{e^{i\gamma/2}+e^{-i\gamma/2}\overline{p(0)}}{1-|p(0)|^2}\,(p(z)-p(0)q(z)).$$

Then σ is also a polynomial of degree at most N and (4.6) is equivalent to the condition $\tau(z)\equiv\sigma(z)$ on U. Hence, we may replace (4.6) by $N+1$ equations of the form $L_k(\tau-\sigma)=0$, $1\le k\le N+1$, where the L_k's are $N+1$ linearly independent continuous linear functionals defined on the linear space $H(U)$ of analytic functions on U. Since $\tau(0)=\sigma(0)=0$ is automatically satisfied, we can replace $N+1$ by N.

Next, we have

$$\begin{aligned}
p(z) &= z^N\overline{q(\frac{1}{\overline{z}})},\\
q(z) &= z^N\overline{p(\frac{1}{\overline{z}})},\\
\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{it}}{e^{it}-z}\frac{df^*(e^{it})}{f^*(e^{it})} &= 1+\frac{1}{2\pi i}\int_0^{2\pi}\frac{z}{e^{it}-z}\frac{df^*(e^{it})}{f^*(e^{it})},\\
\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{it}}{e^{it}-z}\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)} &= -1+\frac{1}{2\pi i}\int_0^{2\pi}\frac{z}{e^{it}-z}\overline{\left(\frac{df^*(e^{it})}{f^*(e^{it})}\right)}
\end{aligned}$$

so that from (4.7) we conclude that

$$z^N\overline{\tau(\frac{1}{\overline{z}})}=e^{-i\gamma/2}q(z)+e^{i\gamma/2}p(z)-\tau(z).$$

Since we also have

$$z^N\overline{\sigma(\frac{1}{\overline{z}})}=e^{-i\gamma/2}q(z)+e^{i\gamma/2}p(z)-\sigma(z),$$

we deduce that $[N/2]$ equations will do.

Consider the point evaluation at a point p_k defined in the Blaschke product of $a(z)$ in (3.1). Then we have

$$\begin{aligned}
\tau(p_k) &= -e^{-i\gamma/2}p_kq(p_k)\overline{\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{it}}{1-\overline{p_k}e^{it}}\frac{df^*(e^{it})}{f^*(e^{it})}},\\
\sigma(p_k) &= -e^{-i\gamma/2}a(0)q(p_k)\frac{1+\overline{a(0)}}{1-|a(0)|^2}=-e^{-i\gamma/2}\overline{\beta}q(p_k).
\end{aligned}$$

If $p_k\neq 0$, we immediately deduce the condition

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{it}}{1-\overline{p_k}e^{it}}\frac{df^*(e^{it})}{f^*(e^{it})}=\frac{\beta}{\overline{p_k}}. \tag{4.8}$$

Suppose now that $p_k = 0$. We already used the point evaluation functional at the origin in the reduction from $N+1$ conditons to N mentioned above. We therefore consider the derivative functional at the origin. We have $q(0) = 1$ and $p(0) = 0$ which leads us to the condition

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{it}\,\frac{df^*(e^{it})}{f^*(e^{it})} = -\overline{a'(0)}. \tag{4.9}$$

We summarize the above in

Theorem 4.3. *The necessary and sufficient condition* (4.6) *in Theorem 4.2 can be replaced by any linearly independent set of* $[N/2]$ *linear functionals* $L_k(\tau-\sigma) = 0$. *In particular, we may choose them from the relations* (4.8) *and* (4.9).

To illustrate Theorem 4.2 and Theorem 4.3, we consider

$$a(z) = e^{i\gamma}\frac{z-p}{1-\overline{p}z}, \qquad |p| < 1.$$

Since $[N/2] = 0$, we conclude that (2.7) is necessary and sufficient for f^* in order that the function f defined by (4.5), be a univalent mapping from U onto Ω. One easily verifies that conditions (4.6) and (4.8) (with $p_k = p$) are automatically satisfied. By Theorem 3.2, $\partial\Omega$ has either no or one point of logarithmic convexity. In the first case, f^* is constant on $\partial U \setminus \{e^{i\tau}\}$ and the cluster set of f at $e^{i\tau}$ is $\partial\Omega$, a circle centered at the origin. Hence $f(U)$ is a disk centered at the origin. By [AH1] such mappings are of the form

$$f_\beta(z) = Cz|z|^{2\beta}\frac{1-e^{i\tau}\overline{z}}{1-e^{-i\tau}z}, \qquad \operatorname{Re}\beta > -\frac{1}{2},\ C \neq 0. \tag{4.10}$$

The associated dilatation function is

$$a(z) = e^{i\gamma}\frac{z-p}{1-\overline{p}z}$$

where

$$p = e^{i\tau}\frac{\overline{\beta}}{1+\overline{\beta}} \qquad \text{and} \qquad e^{i\gamma} = -e^{-i\tau}\frac{1+\overline{\beta}}{1+\beta}.$$

In this case the function ϕ, as defined in (1.5), is the Koebe mapping $z \mapsto z/(1-e^{-i\tau}z)^2$.

Suppose now that $\partial\Omega$ has one point of logarithmic convexity. Then the boundary values f^* are uniquely determined by (2.7), and (4.5) gives the unique univalent solution which maps U onto Ω. In what follows, we modify the integral formula (4.5) to suit the univalent harmonic mapping $F(\zeta) = \log f(e^\zeta)$. Then, with the notation $A(\zeta) = a(e^\zeta)$, we have

$$\operatorname{Im}\left(\sqrt{A(it)}\,dF^*(it)\right) = 0, \qquad 0 \le t \le 2\pi, \tag{4.11}$$

and the integral representation of F becomes

$$(4.12)\quad F(\zeta)=\zeta+2\beta\xi-i(\alpha+2\arg[1+e^{\zeta-i\alpha}])+\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi}\mathrm{Re}\left(\frac{1+e^{\zeta-it}}{1-e^{\zeta-it}}\right)F(it)\,dt,$$

where $\zeta=\xi+i\eta$ and the argument is to be taken such that $|\arg(1+e^{\zeta-i\alpha})|<\pi$. Integration by parts yields

$$\begin{aligned}\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi}\mathrm{Re}\left(\frac{1+e^{\zeta-it}}{1-e^{\zeta-it}}\right)F(it)\,dt &= i\alpha+i\pi+2i\arg(1+e^{\zeta-i\alpha})+F(i\alpha-i\pi)\\ &\quad -\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi}(t+2\arg(1+e^{\zeta-it}))\,dF(it)\end{aligned}$$

so that (4.12) gets the representation

$$(4.13)\quad F(\zeta)=\zeta+2\beta\xi+i\pi+F(i\alpha-i\pi)-\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi}(t+2\arg(1+e^{\zeta-it}))\,dF(it)$$

where α is an arbitrary real number.

Example 4.4. Let G be the domain

$$G=\{w=u+iv : u<u_0(v)\},$$

where

$$\begin{aligned}u_0(v) &= \begin{cases} v+\pi & \text{if } -\pi\le v\le 0,\\ \pi-v & \text{if } 0\le v\le\pi,\end{cases}\\ u_0(v+2\pi i) &\equiv u_0(v).\end{aligned}$$

Find the univalent solution of

$$\overline{F_{\bar\zeta}(\zeta)}=e^{\zeta}F_{\zeta}(\zeta)$$

which maps D onto G.

Solution. Since $N=1$, $\partial\Omega$ has exactly one complete resting point (Theorem 3.2). Hence, ∂G has in each vertical strip of hight $2\pi i$ exactly one complete resting point. The complete resting points are $q=\pi+2ki\pi$, $k\in\mathbb{Z}$. The condition (4.11) for the boundary values of F is

$$\mathrm{Im}\big(e^{it/2}dF^*(it)\big)=0,\qquad 0\le t\le 2\pi,$$

which implies that F^* have the boundary values

$$\begin{aligned}F^*(it) &= \begin{cases}\pi & \text{if } |t|<\frac{\pi}{2},\\ i\pi & \text{if } \frac{\pi}{2}<t<\frac{3\pi}{2},\end{cases}\\ F^*(it+2\pi i) &\equiv F^*(it)+2\pi i.\end{aligned}$$

Finally, choose $\alpha=\pi$. Since $\beta=0$, (4.13) gives the unique solution

$$F(\zeta)=\zeta+\frac{\pi}{2}-(1+i)\arg(1-ie^{\zeta})+(1-i)\arg(1+ie^{\zeta}).$$

5. Solution of the Problem 1

We want to characterize the domains Ω and the univalent logharmonic mappings f which are solutions of (1.7) and map U onto Ω. The corresponding harmonic function $F(\zeta) = \log f(e^{\zeta})$ is a univalent solution of (1.6), $\overline{F_{\overline{\zeta}}(\zeta)} = e^{2\zeta}F_{\zeta}(\zeta)$, and may be calculated from (4.13) where $\beta = 0$.

By Theorem 3.2, we know that the boundary $\partial\Omega$ of a bounded regulated domain Ω contains exactly two different complete resting points of f^* hence at most two point of logarithmic convexity q_1 and q_2. Condition (2.7) implies that the expression

$$(5.1) \qquad \sqrt{a(e^{it})}\frac{df^*(e^{it})}{f^*(e^{it})} = e^{it}\frac{df^*(e^{it})}{f^*(e^{it})} = e^{it}dF^*(it) = \sqrt{A(it)}dF^*(it),$$

is real on $0 \le t \le 2\pi$ and changes the sign at $t = \delta(q_1)$ and at $t = \delta(q_2)$ (see Definition 2.6). On the other hand, we have $[N/2] = 1$ and condition (4.9) reduces to

$$\int_0^{2\pi} e^{it}\,\frac{df^*(e^{it})}{f^*(e^{it})} = \int_0^{2\pi} \sqrt{A(it)}dF^*(it) = -\overline{a'(0)} = 0.$$

Define $w_k = \log q_k, k = 1,2$. Without loss of generality, we may assume that $0 \le \arg q_1 \le \arg q_2 < 2\pi$ (if not, interchange q_1 with q_2). Since $\log w$ is locally conformal on $\partial\Omega$, we conclude that Theorem 2.7 holds true if Ω is replaced by G and f is replaced by F. In particular, $w = u + iv \in \partial G$, $0 \le v < 2\pi$, is a complete resting point of F^* if, and only if, $q = e^w$ is a complete resting point of f^*. We have

$$\begin{aligned} 0 &= \int_0^{2\pi} \sqrt{A(it)}\, dF^*(it) \\ &= \int_{e^{i\delta(q_1)}}^{e^{i\delta(q_2)}} |dF^*(it)| - \int_{e^{i\delta(q_2)}}^{e^{i\delta(q_1)}+2\pi} |dF^*(it)| = L_1 - L_2 = 0, \end{aligned}$$

where L_k is the Euclidean length of the subarc of ∂G joining w_k to $w_{k+1}, k = 1,2$, where $w_3 = w_1 + 2\pi i$. We thus proved the following theorem.

Theorem 5.1. *Let G be a domain in C of the form* (1.1). *Let $F^*(it)$ be a positively oriented local homeomorphism from ∂D onto ∂G. Then F defined by* (4.13) *is a univalent solution of* (1.6) *mapping D onto G if, and only if, $F^*(it)$ satisfies* (5.1) *and admits exactly two different complete resting points $w_k = u_k + iv_k \in \partial G$, $0 \le v_1 < v_2 \le 2\pi$, such that the Euclidean length L_1 of the subarc of ∂G joining w_1 to w_2 is equal to the Euclidean length L_2 of the subarc of ∂G joining w_2 to $w_1 + 2\pi i$.*

Next, we treat the following three cases separately.

5.1. $\partial\Omega$ has no points of logarithmic convexity. In this case, $\partial\Omega$ is a circle centered at the origin and hence ∂G is a vertical line. Let q_1 and q_2 in $\partial\Omega$ be the complete resting points of f^* and suppose that $w_j = \log q_j$ are the corresponding complete resting points of F^*. The condition $L_1 = L_2$ in Theorem 5.1 implies that $w_1 = w_2 + i\pi$ which implies that $q_1 = -q_2$. Observe that $(f^*)^{-1}(q_k)$ is a circular arc of length π. Hence, we get $f^*(e^{it}) = q_1$ on $\tau - \pi < t < \tau$ and $f^*(e^{it}) = q_2$ on $\tau < t < \tau + \pi$. Using (1.5) and $\beta = 0$, we conclude that such mappings are of the form

$$f(z) = cz\sqrt{\frac{1 - e^{2i\tau}\overline{z}^2}{1 - e^{-i2\tau}z^2}}, \qquad c \neq 0.$$

The associated dilatation function is $a(z) = e^{-2i\tau}z^2 = z^2$ and therefore, $e^{-i2\tau} = 1$. Hence, f reduces to

$$f(z) = cz\sqrt{\frac{1 - \overline{z}^2}{1 - z^2}}, \qquad c \neq 0.$$

Observe that the univalent conformal starlike function ϕ in (1.5) is the mapping $\phi(z) = z/\sqrt{(1 - z^2)}$ and in fact, $f(z) = \sqrt{f_0(z^2)}$, where f_0 is defined in (4.10). Next we apply the transformation $F(\zeta) = \log f(e^\zeta)$ and conclude that

$$F(\zeta) = \zeta - i\arg[1 - e^{2\zeta}] + \log c.$$

The third component of the corresponding minimal surface is given by

$$s(z) = \pm\,\mathrm{Im}\left(\int^\zeta e^\zeta F_\zeta(\zeta)\,d\zeta\right) = \pm\,\mathrm{Im}\left(\int^\zeta \frac{e^\zeta}{1 - e^{2\zeta}}\,d\zeta\right) = \pm\frac{1}{2}\arg\left(\frac{1 + e^\zeta}{1 - e^\zeta}\right).$$

We have shown the following result.

Theorem 5.2. *If ∂G has no point of convexity, then G is the left half-plane and the solution of Problem 1 is the unique minimal surface*

$$\begin{aligned} u &= \xi, \\ v &= \eta - \arg(1 - e^{2\zeta}), \\ s &= \frac{1}{2}\arg\left(\frac{1 + e^\zeta}{1 - e^\zeta}\right), \end{aligned}$$

where $\zeta = \xi + i\eta \in G$.

5.2. $\partial\Omega$ has one point of logarithmic convexity. In this case, the point of logarithmic convexity q_1 is a complete resting point and we have to find a second complete resting point $q_2 \in \partial\Omega$ such that Theorem 5.1 is satisfied. In other words, q_1 and q_2 divide $\partial\Omega$ in two boundary arcs Γ_1 and Γ_2 such that $\log\Gamma_i$, $i = 1, 2$, have equal Euclidean length. The other boundary values of f^* are uniquely determined by (2.7). Finally, (4.5) gives the unique univalent solution of (1.7) which maps U onto Ω.

Example 5.3. Let G be the domain

$$G = \{w = u + iv : u < u_0(v)\},$$

with

$$u_0(v) = \begin{cases} v + \pi & \text{if } -\pi \le v \le 0, \\ \pi - v & \text{if } 0 \le v \le \pi, \end{cases}$$

$$u_0(v + 2\pi i) \equiv u_0(v).$$

Find the univalent solution F of

$$\overline{F_{\overline{\zeta}}(\zeta)} = e^{2\zeta} F_\zeta(\zeta)$$

which maps D onto G.

Solution. The point of convexity is $w_1 = \pi$ and hence the second complete resting point of F^* is $w_2 = i\pi$ (respectively, $q_1 = e^{w_1} = e^\pi$ and $q_2 = -1$ of f^*). The condition (4.11) for the boundary values of F is

$$\operatorname{Im}(e^{it} dF^*(it)) = 0, \qquad 0 \le t \le 2\pi,$$

which implies the boundary values

$$F^*(it) = \begin{cases} \pi & \text{if } |t| < \frac{\pi}{4}, \\ i\pi & \text{if } \frac{\pi}{4} < t < \frac{7\pi}{4}, \end{cases}$$

$$F^*(it + 2\pi i) \equiv F^*(it) + 2\pi i.$$

Finally, $\beta = 0$ in this case and (4.13) gives the unique solution

$$F(\zeta) = \zeta + \frac{\pi}{4} - (1+i)\arg(1 - e^{\zeta + i\pi/4}) + (1-i)\arg(1 - e^{\zeta - i\pi/4}).$$

The corresponding minimal surface equation is

$$\begin{aligned} u &= \xi + \frac{\pi}{4} + \arg(1 - e^{\zeta - i\pi/4}) - \arg(1 - e^{\zeta + i\pi/4}), \\ v &= \eta - \arg(1 - e^{\zeta - i\pi/4}) - \arg(1 - e^{\zeta + i\pi/4}), \\ s &= \pm\frac{1}{\sqrt{2}} \log\left|\frac{1 - e^{\zeta + i\pi/4}}{1 - e^{\zeta - i\pi/4}}\right|, \end{aligned}$$

where $\zeta = \xi + i\eta \in G$.

5.3. $\partial\Omega$ has two points of logarithmic convexity. In this case the two points of logarithmic convexity are denoted by q_1 and q_2. The presentation of the previous subsection applies here too. For instance, $F(z) = F_1(2z)/2$ is an example of such type where F_1 is the mapping given in Example 4.4. The two points of convexity lying in the strip $0 \le \operatorname{Im} w < 2\pi$ are $w_1 = \pi/2$ and $w_2 = \pi/2 + i\pi$.

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