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On the univalence of the log-biharmonic mappings

Z. Abdulhadi, Y. Abu Muhanna,* and S. Khuri

Department of Mathematics, American University of Sharjah, Sharjah, Box 26666 United Arab Emirates

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Abstract

In this paper, we establish the univalence and starlikeness connection between log-biharmonic mappings and logharmonic mappings.

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1. Introduction

A continuous complex-valued function $f = u + iv$ in a domain $D \subseteq \mathbf{C}$ is log-biharmonic if $\log f$ is biharmonic, i.e., the Laplacian of $\log f$ is harmonic. In any simply connected domain D , it can be shown that f has the form

$$f(z) = H(z)G(z)|z|^2, \quad (1.1)$$

where H and G are nonvanishing logharmonic mappings in D such that $H(0) \neq 0$ and $G(0) = 1$. A function f is said to be logharmonic in D , if there is an analytic function a , and f is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_z}}{f} = a \frac{f_z}{f}, \quad (1.2)$$

* Corresponding author.

E-mail addresses: zahadi@aus.ac.ae (Z. Abdulhadi), ymuhanna@aus.ac.ae (Y. Abu Muhanna), skhoury@aus.ac.ae (S. Khuri).

It has been shown that if f is nonvanishing logharmonic mapping, then f can be expressed as

$$f = h\bar{g}, \quad (1.3)$$

where h and g are analytic functions in D . On the other hand, if f vanishes at $z = 0$ but is not identically zero, and $|a(z)| < 1$ for every $z \in D$, then the logharmonic function f admits the following representation

$$f = z|z|^{2\beta}h\bar{g}, \quad (1.4)$$

where $\operatorname{Re} \beta > -1/2$, and h and g are analytic functions in D such that $h(0)g(0) \neq 0$ (see [1]). Univalent logharmonic mappings has been studied extensively (for details see [1–3]).

Note that the composition $f \circ \phi$ of a logharmonic function f with analytic function ϕ is logharmonic, while this is not true when f is log-biharmonic. Without loss of generality, we consider the class of log-biharmonic mappings defined on the unit disk U . Observe that if f is log-biharmonic, then $\log f$ is a biharmonic mapping.

Many physical problems are modeled by log-biharmonic functions, particularly those arising in fluid flow theory and elasticity. The log-biharmonic functions are closely associated with the biharmonic functions, which appear in Stokes flow problems (i.e., low-Reynolds-number flows). There is an enormous number of problems involving Stokes which arise in engineering and biological transport phenomena (for details see [6–8]). Many applications of the complex variable theory were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading, i.e., in cases when the solutions are biharmonic functions or functions associated with them.

In Section 2, we consider the case when $f = G^{|z|^2}$. We prove that when G maps ∂U univalently onto a bounded Jordan curve, and $\log f$ maps U onto the inside of the Jordan curve then $\log G$ is univalent inside U . In addition, we prove that $\log f$ is starlike whenever $\log G$ is starlike.

In Section 3, we give sufficient conditions that make $f = HG^{|z|^2}$ locally univalent. In addition, some examples are given.

2. The case $f = G^{|z|^2}$

First, we deal with the solution of the Dirichlet problem for logharmonic mappings of the form (1.4).

Lemma 1. *Let f^* be a nonvanishing continuous complex-valued function defined on ∂U . Then there exists h and g analytic function on U which are independent of β such that*

$$f(z) = z|z|^{2\beta}h\bar{g}, \quad \operatorname{Re}(\beta) > -1/2,$$

is a logharmonic solution of the Dirichlet problem (i.e., $f(e^{it}) = f^(e^{it})$). Furthermore, if $g(0) = 1$, then h and g are uniquely determined.*

Proof. Define $\psi(z) = \log(f(z)/z|z|^{2\beta})$ as the Poisson integral of the continuous boundary function $\log f^*(e^{it}) - it$. Then ψ is harmonic on U , continuous on \bar{U} and is uniquely determined. Therefore $\psi = h_1 + \bar{g}_1$, and the decomposition is unique if $g_1(0) = 0$. Define $h = e^{h_1}$ and $g = e^{g_1}$. Lemma 1 follows. \square

Next, we show that the solution function is univalent.

Theorem 1. Let D be a simply connected domain bounded by a Jordan curve. Let f^* be an orientation-preserving homeomorphism of ∂U onto ∂D . Let $f(z) = z|z|^{2\beta}h\bar{g}$, $\text{Re}(\beta) > -1/2$, be the logharmonic solution of the Dirichlet problem with respect to f^* . Then

- (a) f is univalent on U , and $f(U) = D$ if and only if $|a| < 1$ on U .
- (b) The corresponding log-biharmonic functions will be given by $f_0 = (f/z|z|^{2\beta})|z|^2$. If $\log f_0$ is onto then $\log(f/z|z|^{2\beta})$ is one to one and onto.

Proof. (a) Let f be univalent on U . Then $|a(z)| \neq 1$ for all $z \in U$. Hence, either $|a| > 1$ or $|a| < 1$ on U . The first case excluded since f^* is orientation-preserving. Assume now that $|a| < 1$ on U . Then, we can apply the argument principle to a $w \in C \setminus \partial D$. In other words, for $\rho \in (0, 1)$,

$$K_\rho = \frac{1}{2\pi} \int_{|z|=\rho} d \arg(f - w)$$

is nondecreasing in ρ . Since the winding number of $f(\partial U)$ with respect to w is 1 if $w \in D$, and 0 if $w \in C \setminus \bar{D}$, we conclude that f is univalent on U and $f(U) = D$.

(b) Let $a \in U$ and $b = \log f_0(a)$. Let J denote the line segment in D , containing b , intersecting ∂D exactly twice and given by $\alpha u + \beta v = \gamma$. Let $I = (\log f_0)^{-1}(J)$. Note that $a \in I$ and I intersects ∂U exactly twice. If we let $\log(f(z)/z|z|^{2\beta}) = u_1(z) + i v_1(z)$, then $\log(f/z|z|^{2\beta})(I)$ is the curve $J_1 = \{(u_1, v_1) : \alpha u_1 + \beta v_1 = \gamma/|z|^2\}$ and hence

$$\left(\log \frac{f}{z|z|^{2\beta}} \right)^{-1} (J_1) = I.$$

Since $\log f_0 = r^2 \log(f/z|z|^{2\beta})$, J_1 is a simple curve that intersect ∂D twice. Hence, as $\log(f/z|z|^{2\beta})$ is harmonic, I cannot contain an open set in U . Because if this happens then $\log(f/z|z|^{2\beta})$ maps an open set to a point and consequently, $\log(f/z|z|^{2\beta})$ would be constant which is impossible. Hence I is a curve that divides U into two components. In one, $\alpha u_1 + \beta v_1 > \gamma/|z|^2$, and in the other, $\alpha u_1 + \beta v_1 < \gamma/|z|^2$. Let

$$w(z) = \alpha u_1(z) + \beta v_1(z) + iV(z),$$

where V is the harmonic conjugate. Then $w(z)$ is analytic in U and, for $z \in I$,

$$\text{Re } w(z) = \gamma/|z|^2.$$

Then $w(z) \neq 0$ at $z = a$. Hence

$$\alpha u_{1x}(a) + \beta v_{1x}(a) \neq 0 \tag{2.1}$$

or

$$\alpha u_{1y}(a) + \beta v_{1y}(a) \neq 0. \quad (2.2)$$

Choose $\alpha = v_{1y}(a)$ and $\beta = -u_{1y}(a)$. This makes (2.2) = 0, hence (2.1) $\neq 0$, and hence the determinant (the Jacobian of $\log(f/z|z|^{2\beta})$) of the system is not zero. Since $\log(f/z|z|^{2\beta})$ is one to one on ∂U , the degree principle implies that $\log(f/z|z|^{2\beta})$ is one to one in U and, in addition, $\log(f/z|z|^{2\beta})$ is onto (see [4]). \square

The following example shows that if $\log(f/z|z|^{2\beta})$ is one to one, it does not follow that $r^2 \log(f/z|z|^{2\beta})$ is one to one:

Example 1. Let $\log(f(z)/z|z|^{2\beta}) = ze^z + 1 + i$ and $F(z) = r^2 \log(f(z)/z|z|^{2\beta})$. By drawing contours, using the $f(z)$ software, we showed that in $|z| < 1$, $\log(f(z)/z|z|^{2\beta})$ is one to one while F is not. (See Scheme 1.)

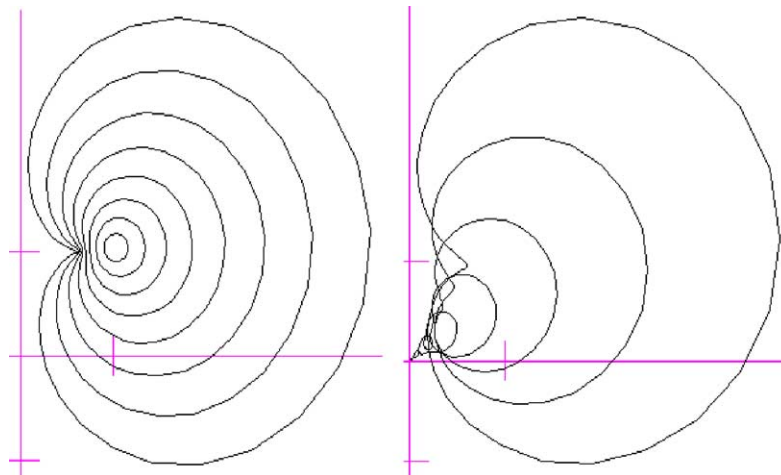
Next, we show that when $\log G(z)$ is starlike then $\log f(z)$ is starlike and univalent.

Definition 1. We say that a biharmonic (harmonic) function is starlike on U if it is orientation-preserving, $F(0) = 0$, $F(z) \neq 0$ when $z \neq 0$, and the curve $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words,

$$\frac{\partial \arg F(re^{it})}{\partial t} = \operatorname{Re} \frac{zF_z - \bar{z}F_{\bar{z}}}{F} > 0$$

(see [5]).

Remark 1. Note that starlike functions are univalent in U . The starlikeness condition implies that, for each $0 < r < 1$, f is univalent on the circle $|z| = r$, and as $f(0) = 0$,



Scheme 1.

the orientation-preserving condition and the degree principle imply that it is univalent on $|z| \leq r$ and maps $|z| < r$ onto the inside of the curve $f(re^{it})$.

If we denote the measure of starlikeness of F by

$$F_{st} = \operatorname{Re} \frac{zF_z - \bar{z}F_{\bar{z}}}{F} \tag{2.3}$$

then we have the following lemma.

Lemma 2. *Let f be a log-biharmonic mapping in the unit disk U . Suppose that f is of the form $f(z) = G(z)^{|z|^2}$, where G is a nonvanishing logharmonic mapping in U and $G(0) = 1$. If $\log G(z)$ is starlike mapping, then $\log f(z)$ is starlike univalent in U .*

Proof. Let $f(z) = G(z)^{|z|^2}$, where G is a logharmonic mapping in U and $G(0) = 1$. Simple Calculations give

$$\frac{f_z}{f} = \bar{z} \log G + |z|^2 \frac{G_z}{G}$$

and

$$\frac{f_{\bar{z}}}{f} = z \log G + |z|^2 \frac{G_{\bar{z}}}{G}.$$

Hence, it follows that

$$\begin{aligned} \left| \frac{f_z}{f} \right|^2 &= \left(\bar{z} \log G + |z|^2 \frac{G_z}{G} \right) \left(z \overline{\log G} + |z|^2 \frac{\overline{G_z}}{G} \right) \\ &= |z|^2 |\log G|^2 + |z|^2 \bar{z} \frac{\overline{G_z}}{G} \log G + |z|^2 z \frac{G_z}{G} \overline{\log G} + |z|^4 \left| \frac{G_z}{G} \right|^2, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{f_{\bar{z}}}{f} \right|^2 &= \left(z \log G + |z|^2 \frac{G_{\bar{z}}}{G} \right) \left(\bar{z} \overline{\log G} + |z|^2 \frac{\overline{G_{\bar{z}}}}{G} \right) \\ &= |z|^2 |\log G|^2 + |z|^2 z \frac{G_{\bar{z}}}{G} \overline{\log G} + |z|^2 \bar{z} \frac{\overline{G_{\bar{z}}}}{G} \log G + |z|^4 \left| \frac{G_{\bar{z}}}{G} \right|^2. \end{aligned}$$

Thus the Jacobian of f is given by

$$\begin{aligned} J_f(z) &= |f_z|^2 - |f_{\bar{z}}|^2 \\ &= |f| \left[|z|^2 \log G \left(\bar{z} \frac{\overline{G_z}}{G} - z \frac{\overline{G_{\bar{z}}}}{G} \right) + |z|^2 \overline{\log G} \left(z \frac{G_z}{G} - \bar{z} \frac{G_{\bar{z}}}{G} \right) \right. \\ &\quad \left. + |z|^4 \left(\left| \frac{G_z}{G} \right|^2 - \left| \frac{G_{\bar{z}}}{G} \right|^2 \right) \right] \\ &= |f| \left[|z|^2 \left(\bar{z} \log G \frac{\overline{G_z}}{G} + z \overline{\log G} \frac{G_z}{G} - z \log G \frac{\overline{G_{\bar{z}}}}{G} - \bar{z} \overline{\log G} \frac{G_{\bar{z}}}{G} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + |z|^4 \left(\left| \frac{G_z}{G} \right|^2 - \left| \frac{G_{\bar{z}}}{G} \right|^2 \right) \\
= & |f| \left[|z|^2 \left(2 \operatorname{Re} \left(z \overline{\log G} \frac{G_z}{G} \right) - 2 \operatorname{Re} \left(\bar{z} \overline{\log G} \frac{G_{\bar{z}}}{G} \right) \right) \right. \\
& \left. + |z|^4 \left(\left| \frac{G_z}{G} \right|^2 - \left| \frac{G_{\bar{z}}}{G} \right|^2 \right) \right] \\
= & |f| \left[2|z|^2 |\log G|^2 \operatorname{Re} \left(\frac{z G_z - \bar{z} G_{\bar{z}}}{G \log G} \right) + |z|^4 \left(\left| \frac{G_z}{G} \right|^2 - \left| \frac{G_{\bar{z}}}{G} \right|^2 \right) \right]. \quad (2.4)
\end{aligned}$$

□

Since G is logharmonic orientation-preserving mapping and $\log G$ is starlike, we deduce that

$$\left| \frac{f_z}{f} \right|^2 - \left| \frac{f_{\bar{z}}}{f} \right|^2 > 0.$$

Hence, f is orientation-preserving and locally univalent. Direct calculations yields

$$\frac{\partial \arg \log f(re^{it})}{\partial t} = \operatorname{Re} \frac{z f_z - \bar{z} f_{\bar{z}}}{f \log f} = \operatorname{Re} \frac{z G_z - \bar{z} G_{\bar{z}}}{G \log G}.$$

Hence $\log f$ is starlike since $\log G$ does.

Corollary 1. $r^2 \log f(z)$ is starlike for all conformal starlike function $\log f(z)$.

Example 2. It is known that the harmonic Koebe function

$$\log k_0(z) = h(z) + \overline{g(z)},$$

where

$$\begin{aligned}
h(z) &= \left(z - \frac{1}{2}z^2 + \frac{1}{6}z^3 \right) (1-z)^{-3}, \\
g(z) &= \left(\frac{1}{2}z^2 + \frac{1}{6}z^3 \right) (1-z)^{-3},
\end{aligned}$$

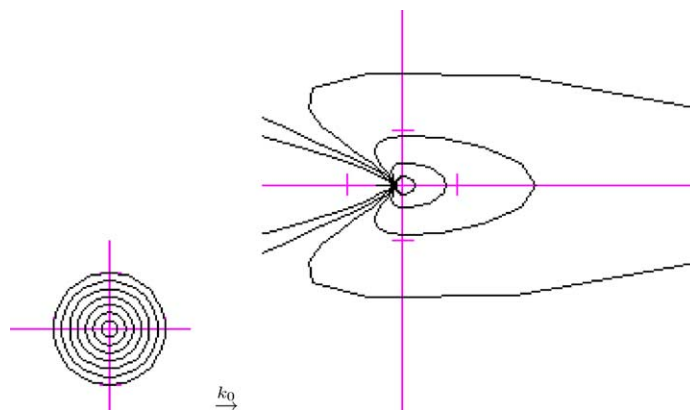
is starlike and hence the function $r^2 \log k_0$ is also starlike. $\log k_0$ maps the unit disk univalently onto C minus the slit $-\infty < t < -1/6$. (See Scheme 2.)

Example 3. The following harmonic function

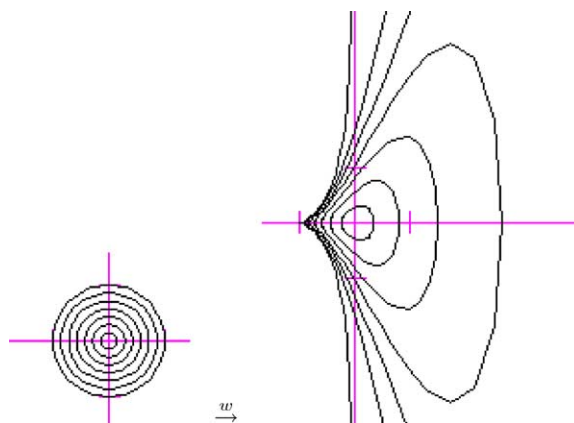
$$\log w(z) = \operatorname{Re}(1+z)/(1-z) - 1 + i \operatorname{Im}(z/(1-z)^2)$$

is starlike and maps onto the half plane $\{z: \operatorname{Re} z > -1\}$. Hence the function $r^2 \log w$ is also starlike. (See Scheme 3.)

Remark 2. We do not know whether the converse of the theorem is true.



Scheme 2.



Scheme 3.

3. The general case

Theorem 2. Let $f = HG^{|z|^2}$ be a log-biharmonic mapping in the unit disk U , where H and G are logharmonic mapping such that H and G are nonvanishing and $G(0) = 1$. Suppose that $\log G$ is starlike harmonic and H is orientation preserving. If

$$\operatorname{Re}[(\log H)_{\bar{z}} \overline{|z|^2 (\log G)_{\bar{z}}}] < \operatorname{Re}[(\log H)_z \overline{|z|^2 (\log G)_z}], \tag{3.1}$$

then f is locally univalent.

Proof. Let $f = HG^{|z|^2}$ be a log-biharmonic mapping in the unit disk U , where H and G are logharmonic mapping such that H and G are nonvanishing and $G(0) = 1$. Since $\log G$ is starlike harmonic, it follows that

$$\operatorname{Re} \frac{zG_z - \bar{z}G_{\bar{z}}}{G \log G} > 0.$$

Hence, we have

$$\frac{f_{\bar{z}}}{f} = z \log G + |z|^2 \frac{G_{\bar{z}}}{G} + \frac{H_{\bar{z}}}{H}$$

and

$$\frac{f_z}{f} = \bar{z} \log G + r^2 \frac{G_z}{G} + \frac{H_z}{H}.$$

Therefore,

$$\left| \frac{f_{\bar{z}}}{f_z} \right| = \left| \frac{z \log G + |z|^2 \frac{G_{\bar{z}}}{G} + \frac{H_{\bar{z}}}{H}}{\bar{z} \log G + |z|^2 \frac{G_z}{G} + \frac{H_z}{H}} \right| = \left| \frac{1 + \frac{\bar{z}G_{\bar{z}}}{G \log G} + \frac{H_{\bar{z}}}{H \log G}}{1 + \frac{zG_z}{G \log G} + \frac{H_z}{\bar{z}H \log G}} \right|.$$

Let

$$a = \frac{\bar{z}G_{\bar{z}}}{G \log G}, \quad b = \frac{H_{\bar{z}}}{zH \log G}, \quad c = \frac{zG_z}{G \log G}, \quad \text{and} \quad d = \frac{H_z}{\bar{z}H \log G}.$$

Note that,

$$\begin{aligned} |1 + a + b|^2 &= (1 + a + b)(1 + \bar{a} + \bar{b}) \\ &= 1 + 2 \operatorname{Re}(a) + 2 \operatorname{Re}(b) + 2 \operatorname{Re}(a\bar{b}) + |a|^2 + |b|^2. \end{aligned}$$

Since $\log G$ is starlike, it follows that

$$\operatorname{Re} a = \operatorname{Re} \left(\frac{\bar{z}G_{\bar{z}}}{G \log G} \right) < \operatorname{Re} \left(\frac{zG_z}{G \log G} \right) = \operatorname{Re} c.$$

Also, H and G are orientation-preserving, and since $|G_{\bar{z}}/G| < |G_z/G|$ and $|H_{\bar{z}}/H| < |H_z/H|$, we obtain

$$\begin{aligned} |a| &= \left| \frac{\bar{z}G_{\bar{z}}}{G \log G} \right| < \left| \frac{zG_z}{G \log G} \right| = |c|, \\ |b| &= \left| \frac{H_{\bar{z}}}{zH \log G} \right| \leq \left| \frac{H_z}{\bar{z}H \log G} \right| = |d|. \end{aligned}$$

We have

$$\begin{aligned} 2 \operatorname{Re} b + 2 \operatorname{Re}(a\bar{b}) &= 2 \operatorname{Re} b + 2 \operatorname{Re}(\bar{a}b) = 2 \operatorname{Re} b(1 + \bar{a}), \\ b(1 + \bar{a}) &= \frac{H_{\bar{z}}}{zH \log G} \left(1 + \frac{\bar{z}G_{\bar{z}}}{G \log G} \right), \end{aligned}$$

and

$$d(1 + \bar{c}) = \frac{H_z}{\bar{z}H \log G} \left(1 + \frac{\bar{z}G_z}{G \log G} \right).$$

Therefore,

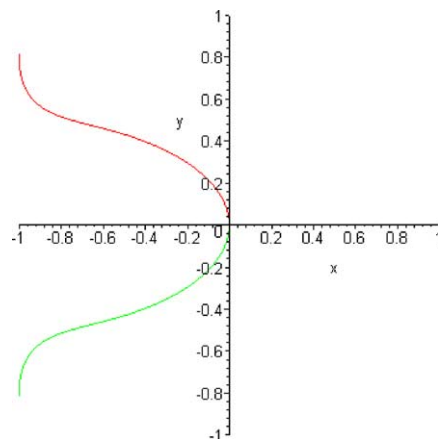


Fig. 1.

$$\begin{aligned}
 b(1 + \bar{a}) - d(1 + \bar{c}) &= \frac{H_z}{zH \log G} \left(1 + \frac{z\bar{G}_z}{\bar{G} \log G} \right) - \frac{H_z}{\bar{z}H \log G} \left(1 + \frac{\bar{z}\bar{G}_z}{\bar{G} \log G} \right) \\
 &= \frac{H_z}{zH \log G} - \frac{H_z}{\bar{z}H \log G} + \frac{H_z \bar{G}_z}{H\bar{G} |\log G|^2} - \frac{H_z \bar{G}_z}{H\bar{G} |\log G|^2} \\
 &= \frac{(\log H)_{\bar{z}} [\bar{z} \log \bar{G} + |z|^2 (\log G)_{\bar{z}}] - (\log H)_z [z \log G + |z|^2 (\log G)_z]}{|z|^2 |\log G|^2} \\
 &= \frac{(\log H)_{\bar{z}} (|z|^2 \log G)_{\bar{z}} - (\log H)_z (|z|^2 \log G)_z}{|z|^2 |\log G|^2}.
 \end{aligned}$$

Hence, it follows from condition (3.1) that $\text{Re}[b(1 + \bar{a}) - d(1 + \bar{c})] < 0$. \square

Remark 3. The theorem may not be true when $\log G$ is not starlike. Here is an example: Choose

$$\log G(z) = 0.3\bar{z}^2 + z + 1 \quad \text{and} \quad F(z) = r^2 \log G(z).$$

$\log G$ is orientation preserving and univalent in U . Suppose to the contrary that there are z_1 and z_2 in U , $z_1 \neq z_2$, and $\log G(z_1) = \log G(z_2)$. Then it follows that

$$|z_1 + z_2| = \frac{1}{0.3} \left| \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} \right| > 3.3,$$

impossible. However, it was shown, using Maple, that the equation

$$J_F(z) = |1.3\bar{z}^3 + 2r^2|^2 - |0.9r^2\bar{z} + z^2 + z|^2 = 0$$

has infinitely many zeros with graph as shown in Fig. 1.

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