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Landau's theorem for functions with logharmonic Laplacian

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ABSTRACT

In this paper, we show the existence of Landau constant for functions with logharmonic Laplacian of the form $F(z) = |z|^2L(z) + K(z)$, $|z| < 1$, where L is logharmonic and K is harmonic. Moreover, the problem of minimizing the area is solved

1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined on the unit disk $U = \{z : |z| < 1\}$. A logharmonic function is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\bar{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. Suppose that f is univalent logharmonic function with respect to a with $a(0) = 0$. If $f(0) = 0$ then f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{1.2}$$

where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. In this case, $F(\zeta) = \log f(e^{\zeta})$ is univalent and harmonic in the half-plane $\{\zeta; \mathrm{Re}(\zeta) < 0\}$, such functions play an important role in the theory of minimal surfaces having periodic Gauss map (for details study of harmonic functions and logharmonic functions to be found in [1–5,7,8,10]). If $0 \notin f(U)$, then $\log(f(z))$ is univalent and harmonic, and the representation of f as in (1.2) with h and g are nonvanishing analytic functions in U.

We consider the class of all continuous complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbf{C}$ such that the Laplacian of F is logharmonic. Note that $\log(\triangle F)$ is harmonic in D, if it satisfies the Laplace's equation $\triangle(\log(\triangle F)) = 0$, where

$$\triangle = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

In any simply connected domain D we can write

$$F = r^2 L + H, \quad z = re^{i\theta}, \tag{1.3}$$

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where L is logharmonic and H is harmonic in D. It is known that L and H can be expressed as,

$$L = h_1\overline{g_1}, \qquad (1.4)$$
$$H = h_2 + \overline{g_2},$$

where h_1, g_1, h_2 and g_2 are analytic in D. Denote by $L_{Lh}(U)$ the set of all functions of the form (1.3), which are defined on the unit disk U (for details see [1]).

Denote the Jacobian of W by J_W, then

$$J_W = |W_z|^2 - |W_{\bar{z}}|^2. \qquad (1.5)$$

Denote

$$\lambda_W = |W_z| - |W_{\bar{z}}|,$$
$$\Lambda_W = |W_z| + |W_{\bar{z}}|, \qquad (1.6)$$

then $J_W = \lambda_W \cdot \Lambda_W$.

Lewy [7,10], showed that a harmonic function W is locally univalent if Jacobian of W, J_W,

$$J_W \neq 0. \qquad (1.7)$$

The classical Landau theorem states that if f is analytic in the unit disk U with $f(0) = 0$, $f'(0) = 1$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $U_{\rho_0} = \{z : |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}$$

and $f(U_{\rho_0})$ contains a disk U_{R_0} with $R_0 = M\rho_0^2$. This result is sharp, with the external function $f(z) = Mz\frac{(1-Mz)}{(M-z)}$ (see [12]).

Chen et al. [6] obtained a version of the Landau theorem for bounded harmonic mappings of the unit disk. Unfortunately their result is not sharp. Better estimates were given in [9] and later in [11].

In specific, it was shown in [11] that if f is harmonic in the unit disk U with $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $U_{\rho_1} = \{z : |z| < \rho_1\}$ with

$$\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}$$

and $f(U_{\rho_1})$ contains a disk U_{R_1} with $R_1 = \frac{\pi}{4M} - 2M\frac{\rho_1^2}{1-\rho_1}$. This result is the best known but not sharp.

We now quote the Schwarz lemma for harmonic mappings which will be used in proving the coming theorems:

Lemma 1 (*Schwarz lemma*). *Let f be a harmonic mapping of the unit disk U with $f(0) = 0$ and $f(U) \subset U$. Then*

$$|f(z)| \leqslant \frac{4}{\pi}\arctan|z| \leqslant \frac{4}{\pi}|z|,$$
$$\Lambda_f(0) \leqslant \frac{4}{\pi}. \qquad (1.8)$$

In Theorem 1, we consider the problem of minimizing the area for the case $F(z) = r^2L(z)$. In Theorems 2 and 3, we show that Landau's theorem extends to bounded functions with logharmonic Laplacian.

In Theorem 2, we show that if L be logharmonic in U such that $L(0) = 0$, $J_L(0) = 1$ and $|L(z)| < M$ for $z \in U$ then there is a constant $0 < \rho_2 < 1$ so that $F = r^2L$ is univalent in the disk $|z| < \rho_1$, where ρ_1 is the solution of the equation

$$1 = 2\rho_2 M\frac{1}{1-\rho_2^2} - 2M\frac{\rho_2}{\left(1-\rho_2^2\right)^2}$$

and $f(U_{\rho_2})$ contains a disk U_{R_2} with

$$R_2 = \rho_2^3 - 2M\frac{\rho_2^4}{1-\rho_2^2}.$$

This result is not sharp.

In Theorem 3, we show that if F is in the class $L_{Lh}(U)$, such that $L(0) = K(0) = 0$, $J_F(0) = 1$ and $|L(z)|$ and $|K(z)|$ are both bounded by M for $z \in U$ then there is a constant $0 < \rho_3 < 1$ so that F is univalent in $|z| < \rho_3$. In specific, ρ_3 satisfies

$$\frac{\pi}{4M} - 2\rho_3 M - 2M\left(\frac{\rho_3^3}{\left(1-\rho_3^2\right)^2} + \frac{1}{\left(1-\rho_3\right)^2} - 1\right) = 0$$

and $F(U_{\rho_3})$ contains a disk U_{R_3}, where

$$R_3 = \frac{\pi}{4M}\rho_3 - \rho_3^2 M \frac{1}{1-\rho_3^2} - 2M\frac{\rho_3^2}{1-\rho_3}.$$

This result is not sharp.

2. The Case $F = r^2 G$

First we establish a lower bound for the area of the range of $F(z) = r^2 L(z)$.

Theorem 1. *Let $F(z) = r^2 L(z)$, where $L = h\bar{g}$ is starlike logharmonic in U. If $g(0) = 1$ and $h'(0) = 1$. Let $A(r,F)$ denotes the area of $F(U_r)$, where $U_r = \{z : |z| < r\}$, for $r < 1$. Then,*

$$A(r,F) \geqslant 2\pi\left[-2r + r^2 - \frac{2r^3}{3} + \frac{r^4}{2} - \frac{r^5}{5} + \frac{r^6}{6} - \frac{r^8}{8} + 2\ln(1+r)\right].$$

Equality holds if and only if $L_0(z) = r^2 \frac{z\left(1+\frac{\bar{z}}{2}\right)}{\left(1+\frac{z}{2}\right)}$ or one of its rotations.

Proof. Let $F(z) = r^2 L(z)$, where $L(z) = h(z)\overline{g(z)}$ be a logharmonic mapping defined on the unit disc . Then L satisfies (1.1) for some $a \in H(U)$ such that $|a(z)| < 1$ and $a(0) = 0$. Hence,

$$A(r,F) = \int\int_{U_r} J_F dA = \int\int_{U_r} (|F_z|^2 - |F_{\bar{z}}|^2) r dr d\theta \geqslant \int_0^r \int_0^{2\pi} 2|L|^2|z|^2 \mathrm{Re}\left[\frac{zL_z - zL_{\bar{z}}}{L}\right] + r^4[|L_z|^2 - |L_{\bar{z}}|^2]\rho d\theta d\rho \tag{2.1}$$

By Schwarz lemma, we have

$$[|L_z|^2 - |L_{\bar{z}}|^2] = |L_z|^2[1 - |a|^2] \geqslant |L_z|^2[1 - |\rho|^2]. \tag{2.2}$$

Since L is starlike logharmonic mapping, it follows from [3] that $\psi(z) = \frac{zh}{g}$ is starlike. Therefore, we have

$$\mathrm{Re}\frac{zL_z - zL_{\bar{z}}}{L} = \mathrm{Re}\frac{z\psi'(z)}{\psi(z)} \geqslant \frac{1-\rho}{1+\rho}. \tag{2.3}$$

Substituting (2.2) and (2.3) in (2.1) we obtain that

$$A(r,F) \geqslant \int_0^r 2\rho^2\frac{1-\rho}{1+\rho}\int_0^{2\pi}|L|^2 d\theta d\rho + \int_0^r \rho^5(1-\rho^2)\int_0^{2\pi}|L_z|^2 d\theta d\rho. \tag{2.4}$$

Writing $hg = z[1 + \sum_{n=1}^{\infty} c_n z^n]$,
we get

$$\int_0^{2\pi}|L|^2 d\theta = 2\pi\rho^2\left[1 + \sum_{n=1}^{\infty}|c_n|^2\rho^{2n}\right]. \tag{2.5}$$

Also, writing $h'g = [1 + \sum_{n=1}^{\infty} d_n z^n]$,
we obtain

$$\int_0^{2\pi}|L_z|^2 d\theta = 2\pi\left[1 + \sum_{n=1}^{\infty}|d_n|^2\rho^{2n}\right]. \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we deduce that $A(r,F) \geqslant 2\pi\int_0^r\left[\rho^4\left(\frac{1-\rho}{1+\rho}\right) + \rho^5(1-\rho^2)\right]d\rho = 2\pi\Big[k - 2r + r^2 - \frac{2r^3}{3} + \frac{r^4}{2} - \frac{r^5}{5} + \frac{r^6}{6} - \frac{r^8}{8} + 2\ln(1+r)\Big]$. □

In the next theorem we give a Landau's theorem for functions with logharmonic Laplacian of the form $F = r^2 L(z)$.

Theorem 2. *Let L be logharmonic in U such that $L(0) = 0$, $J_L(0) = 1$ and $|L(z)| < M$ for $z \in U$. Then there is a constant $0 < \rho_1 < 1$ so that $F = r^2 L$ is univalent in the disk $|z| < \rho_2$, ρ_2 is the solution of the equation $1 = 2\rho M\frac{1}{1-\rho^2} - 2M\frac{\rho}{(1-\rho^2)^2}$ and $f(U_{\rho_2})$ contains a disk U_{R_2} with $R_2 = \rho_2^2 - 2M\frac{\rho_2^4}{1-\rho_2^2}$. This result is not sharp.*

Proof. Fix $0 < \rho < 1$ and choose z_1, z_2 with $z_1 \neq z_2$, $|z_1| < \rho$ and $|z_2| < \rho$. Then we have

$$F(z_1) - F(z_2) = \int_{[z_1,z_2]} F_z(z)dz + F_{\bar{z}}(z)d\bar{z} = \int_{[z_1,z_2]} (\bar{z}L + r^2 h'\bar{g})dz + (zG + r^2 h\overline{g'})d\bar{z},$$

where $[z_1, z_2]$ is the line-segment from z_1 to z_2, $z = tz_2 + (1-t)z_1$ and $0 \leqslant t \leqslant 1$. Hence

$$\begin{aligned}|F(z_1)-F(z_2)| &= \left|\int_{[z_1,z_2]}(\bar{z}L + r^2 h'\bar{g})dz + (zL + r^2 h\overline{g'})d\bar{z}\right| = \left|\int_{[z_1,z_2]} L(z)(\bar{z}dz + zd\bar{z}) + \int_{[z_1,z_2]} r^2 h'\bar{g}dz + \int_{[z_1,z_2]} r^2 h\overline{g'}d\bar{z}\right| \\ &= \left|\int_{[z_1,z_2]} r^2 dz + \int_{[z_1,z_2]} L(z)(\bar{z}dz + zd\bar{z}) + \int_{[z_1,z_2]} r^2(h'\bar{g}-1)dz + \int_{[z_1,z_2]} r^2 h\overline{g'}d\bar{z}\right| \\ &\geqslant \left|\int_{[z_1,z_2]} r^2 dz\right| - 2|z_2 - z_1|\sum_{n=1}^{\infty}(|a_n||b_n|)\left|\int_0^1 r^{2n}dt\right| - 2|z_2 - z_1|\sum_{n=1}^{\infty}(|a_n||b_n|)n\left|\int_0^1 r^{2n+1}dt\right| \\ &\geqslant |z_2 - z_1|\left[\left|\int_0^1 r^2 dt\right| - 2\rho M\sum_{n=1}^{\infty}\rho^{2n-2}\left|\int_0^1 r^2 dt\right| - 2M\sum_{n=1}^{\infty} n\rho^{2n-1}\left|\int_0^1 r^2 dt\right|\right] \\ &\geqslant |z_2 - z_1|\left|\int_0^1 r^2 dt\right|\left[1 - 2\rho M\frac{1}{1-\rho^2} - 2M\frac{\rho}{(1-\rho^2)^2}\right].\end{aligned}$$

Choose ρ_2 so that $1 - 2\rho M\frac{1}{1-\rho^2} - 2M\frac{\rho}{(1-\rho^2)^2} = 0$.

Then F is univalent in $|z| < \rho_2$ and furthermore, we have for $|z| = \rho_2$,

$$|F(z)| = \rho_2^3\left|\sum_{n=1}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n\right| \geqslant \rho_2^3 - \rho_2^3 M\sum_{n=1}^{\infty}\rho^{2n-1} = \rho_2^3 - 2M\frac{\rho_2^4}{1-\rho_2^2} = R_2. \quad \square$$

3. The general case $F = r^2 L + K$

Next we give a Landau theorem for functions of logharmonic Laplacian of the form $F = r^2 L + K$:

Theorem 3. *Let $F = r^2 L + K$, $z = re^{i\theta}$ be in $L_{Lh}(U)$, where L is logharmonic and K is harmonic in the unit disc U such that $L(0) = K(0) = 0$, $J_F(0) = 1$ and $|L|$ and $|K|$ are both bounded by M. Then There is a constant $0 < \rho_3 < 1$ so that F is univalent in $|z| < \rho_3$. In specific, ρ_3 satisfies*

$$\frac{\pi}{4M} - 2\rho_3 M - 2M\left(\frac{\rho_3^3}{(1-\rho_3^2)^2} + \frac{1}{(1-\rho_3)^2} - 1\right) = 0$$

and $F(U_{\rho_3})$ contains a disk U_{R_3}, where

$$R_3 = \frac{\pi}{4M}\rho_3 - \rho_3^2 M\frac{1}{1-\rho_3^2} - 2M\frac{\rho_3^2}{1-\rho_3}.$$

Proof. Let $L(z) = h(z)\overline{g(z)} = \left(z + \sum_{n=2}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$ and $K(z) = \sum_0^{\infty} c_n z^n + \overline{\sum_0^{\infty} d_n z^n}$. Fix $0 < \rho < 1$ and choose z_1, z_2 with $z_1 \neq z_2, |z_1| < \rho$ and $|z_2| < \rho$. Then

$$F(z_1) - F(z_2) = \int_{[z_1,z_2]} F_z(z)dz + F_{\bar{z}}(z)d\bar{z} = \int_{[z_1,z_2]}(\bar{z}L + r^2 h'\bar{g} + K_z)dz + \left(zL + r^2 h\overline{g'} + K_{\bar{z}}\right)d\bar{z},$$

where $[z_1, z_2]$ is the line-segment from z_1 to z_2.

Note that

$$J_F(0) = |K_z(0)|^2 - |K_{\bar{z}}(0)|^2 = J_K(0) = 1 \tag{3.1}$$

and hence

$$\lambda_K(0) = \frac{1}{\Lambda_K(0)} \geqslant \frac{\pi}{4M}.$$

Then

$$\begin{aligned}|F(z_1)-F(z_2)| &\geqslant \left|\int_{[z_1,z_2]}(K_z(0)dz + K_{\bar{z}}(0)d\bar{z})\right| - \left|\int_{[z_1,z_2]} L(z)(\bar{z}dz + zd\bar{z}) + \int_{[z_1,z_2]} r^2(h'(z)\overline{g(z)}dz + h(z)\overline{g'(z)}d\bar{z})\right. \\ &\quad \left. + \int_{[z_1,z_2]}(K_z(z) - K_z(0))dz + (K_{\bar{z}}(z) - K_{\bar{z}}(0))d\bar{z}\right| \\ &\geqslant |z_2 - z_1|\left(\lambda_K(0) - 2\rho M - 2\sum_{n=1}^{\infty}(|a_n||b_n|)n\rho^{2n+1} - \sum_{n=2}^{\infty}(|c_n| + |d_n|)n\rho^{n-1}\right) \\ &\geqslant |z_2 - z_1|\left(\frac{\pi}{4M} - 2\rho M - 2M\sum_{n=1}^{\infty} n\rho^{2n+1} - 2M\sum_{n=2}^{\infty} n\rho^{n-1}\right) \\ &= |z_2 - z_1|\left(\frac{\pi}{4M} - 2\rho M - 2M\left(\frac{\rho^3}{(1-\rho^2)^2} + \frac{1}{(1-\rho)^2} - 1\right)\right).\end{aligned}$$

Clearly there is a ρ so that $|F(z_1) - F(z_2)| > 0$. Let ρ_3 be the largest such ρ. In other words, choose $\rho_3 > 0$ so that

$$\frac{\pi}{4M} - 2\rho_3 M - 2M\left(\frac{\rho_3^3}{(1-\rho_3^2)^2} + \frac{1}{(1-\rho_3)^2} - 1\right) = 0.$$

For $|z| = \rho_3$,

$$|F(z)| \geqslant |c_1 z + d_1 \bar{z}| - \rho_3^2 \left| \left(z + \sum_{n=2}^{\infty} a_n z^n\right)\left(\overline{\sum_{n=0}^{\infty} b_n z^n}\right)\right| - \left|\sum_{n=2}^{\infty} c_n z^n + d_n \bar{z}^n\right| \geqslant \frac{\pi}{4M}\rho_3 - \rho_3^2 M \sum_{n=0}^{\infty} \rho_3^{2n} - 2M \sum_{n=2}^{\infty} \rho_3^n$$
$$\geqslant \frac{\pi}{4M}\rho_3 - \rho_3^2 M \frac{1}{1-\rho_3^2} - 2M\frac{\rho_3^2}{1-\rho_3} = R_3. \quad \square$$

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