Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 338 (2008) 705–709
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Landau's theorem for biharmonic mappings

Z. Abdulhadi, Y. Abu Muhanna *

Department of Mathematics, American University of Sharjah, Sharjah, Box 26666, United Arab Emirates

Received 22 December 2006
Available online 5 June 2007
Submitted by D. Khavinson

Abstract

In this paper, we show the existence of Landau constant for biharmonic mappings of the form $F(z)=|z|^2G(z)+K(z)$, $|z|<1$, where G and K are harmonic.

Keywords: Biharmonic; Harmonic; Univalent; Jacobian; Orientation-preserving

1. Introduction

A four times continuously differentiable complex-valued function $F=u+iv$ in a domain $D\subseteq \mathbf{C}$ is biharmonic if the Laplacian of F is harmonic. Note that ΔF is harmonic in D, if F satisfies the biharmonic equation $\Delta(\Delta F)=0$, where

$$\Delta = 4\frac{\partial^2}{\partial z\partial \bar{z}}.$$

Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them.

It is easy to show that a mapping F is biharmonic in a simply connected domain D if and only if F has the following representation:

$$F=r^2G+K, \quad re^{i\theta}\in D, \tag{1.1}$$

where G and K are complex-valued harmonic functions in D (for details see [1]). It is known that H and G can be expressed as

$$\begin{aligned} G &= g_1+\overline{g_2},\\ K &= k_1+\overline{k_2}, \end{aligned} \tag{1.2}$$

* Corresponding author.
E-mail addresses: zahadi@aus.edu (Z. Abdulhadi), ymuhanna@aus.edu (Y. Abu Muhanna).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.05.065

where g_1, g_2, k_1 and k_2 are analytic in D (for details see [2,4,6]).

Denote the Jacobian of W by J_W, then

$$J_W = |W_z|^2 - |W_{\bar{z}}|^2. \tag{1.3}$$

Denote

$$\lambda_W = |W_z| - |W_{\bar{z}}|,$$
$$\Lambda_W = |W_z| + |W_{\bar{z}}|, \tag{1.4}$$

then $J_W = \lambda_W.\Lambda_W$.

Lewy [4,6], showed that a complex-valued harmonic function W is locally univalent in a domain $D \subset C$ if and only if $J_W \neq 0$.

The classical Landau theorem [3,5] for bounded analytic functions states that if f is analytic in the unit disk U with $f(0) = 0$, $f'(0) = 1$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $U_{\rho_0} = \{z\colon |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}$$

and $f(U_{\rho_0})$ contains a disk U_{R_0} with $R_0 = M\rho_0^2$. This result is sharp.

Chen, Gauthier and Hengartner [3] obtained a version of the Landau theorem for bounded harmonic mappings of the unit disk. Unfortunately their result is not sharp. Better estimates were given in [5] and later in [7].

In specific, it was shown in [7] that if f is harmonic in the unit disk U with $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $U_{\rho_1} = \{z\colon |z| < \rho_1\}$ with

$$\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}$$

and $f(U_{\rho_1})$ contains a disk U_{R_1} with $R_1 = \frac{\pi\rho_1}{4M} - 2M\frac{\rho_1^2}{1-\rho_1}$. This result is the best known but not sharp.

We now quote the Schwarz lemma for harmonic mappings which will be used in proving the coming theorems:

Lemma 1 *(Schwarz lemma). Let f be a harmonic mapping of the unit disk U with $f(0) = 0$ and $f(U) \subset U$. Then*

$$\left|f(z)\right| \leqslant \frac{4}{\pi}\arctan|z| \leqslant \frac{4}{\pi}|z|,$$
$$\Lambda_f(0) \leqslant \frac{4}{\pi}. \tag{1.5}$$

In Theorems 1 and 2, we show that Landau's theorem extends to bounded biharmonic mappings of the unit disk.

In Theorem 1, we show that if F is a biharmonic mapping in U, as in (1.1), such that $G(0) = K(0) = 0$, $J_F(0) = 1$ and $|G(z)|$ and $|K(z)|$ are both bounded by M for $z \in U$ then there is a constant $0 < \rho_2 < 1$ so that F is univalent in $|z| < \rho_2$. In specific ρ_2 satisfies

$$\frac{\pi}{4M} - 2\rho_2 M - 2M\left(\frac{\rho_2^2}{(1-\rho_2)^2} + \frac{1}{(1-\rho_2)^2} - 1\right) = 0$$

and $F(U_{\rho_2})$ contains a disk U_{R_2}, where

$$R_2 = \frac{\pi}{4M}\rho_2 - 2M\frac{\rho_2^3 + \rho_2^2}{1-\rho_2}.$$

This result is not sharp.

In Theorem 2, we show that if G be harmonic in U such that $G(0) = 0$, $J_G(0) = 1$ and $|G(z)| < M$ for $z \in U$ then there is a constant $0 < \rho_3 < 1$ so that $F = r^2 G$ is univalent in the disk $|z| < \rho_3$, where ρ_3 is the solution of the equation

$$\frac{\pi}{4M} = 4M\frac{\rho_3}{1-\rho_3} + 2M\left(\frac{1}{(1-\rho_3)^2} - 1\right)$$

and $f(U_{\rho_3})$ contains a disk U_{R_3} with

$$R_3 = \frac{\pi}{4M}\rho_3^3 - 2M\frac{\rho_3^4}{1-\rho_3}.$$

This result is not sharp.

2. The case $F = r^2G + K$

We first give a Landau theorem for general biharmonic mappings:

Theorem 1. *Let $F = r^2G + K$, $z = re^{i\theta}$ be a biharmonic mapping of the unit disk U, as in* (1.1)*, such that $F(0) = K(0) = 0$, $J_F(0) = 1$ and $|G|$ and $|K|$ are both bounded by M. Then there is a constant $0 < \rho_2 < 1$ so that F is univalent in $|z| < \rho_2$. In specific ρ_2 satisfies*

$$\frac{\pi}{4M} - 2\rho_2 M - 2M\left(\frac{\rho_2^2}{(1-\rho_2)^2} + \frac{1}{(1-\rho_2)^2} - 1\right) = 0$$

and $F(U_{\rho_2})$ contains a disk U_{R_2}, where

$$R_2 = \frac{\pi}{4M}\rho_2 - 2M\frac{\rho_2^3+\rho_2^2}{1-\rho_2}.$$

Proof. Let $G(z) = \sum_0^\infty a_n z^n + \overline{\sum_0^\infty b_n z^n}$ and $K(z) = \sum_1^\infty c_n z^n + \overline{\sum_1^\infty d_n z^n}$. For fixed $0 < \rho < 1$, choose z_1, z_2 with $z_1 \neq z_2$, $|z_1| < \rho$ and $|z_2| < \rho$. Then

$$F(z_1) - F(z_2) = \int_{[z_1,z_2]} F_z(z)\,dz + F_{\bar z}(z)\,d\bar z = \int_{[z_1,z_2]} \left(\bar z G + r^2 g_1' + K_z\right)dz + \left(zG + r^2\overline{g_2'} + K_{\bar z}\right)d\bar z,$$

where $[z_1, z_2]$ is the line-segment from z_1 to z_2 and g_1, g_2 are as in (1.2).

Note that

$$J_F(0) = \left|K_z(0)\right|^2 - \left|K_{\bar z}(0)\right|^2 = J_K(0) = 1 \tag{2.1}$$

and hence by Lemma 1

$$\lambda_K(0) = \frac{1}{\Lambda_K(0)} \geqslant \frac{\pi}{4M}.$$

Then

$$\begin{aligned}
\left|F(z_1) - F(z_2)\right| &\geqslant \left|\int_{[z_1,z_2]} \left(K_z(0)\,dz + K_{\bar z}(0)\,d\bar z\right)\right| \\
&\quad - \left|\int_{[z_1,z_2]} G(z)(\bar z\,dz + z\,d\bar z) + \int_{[z_1,z_2]} r^2\left(g_1'(z)\,dz + \overline{g_2'(z)}\,d\bar z\right)\right. \\
&\quad \left. + \int_{[z_1,z_2]} \left(K_z(z) - K_z(0)\right)dz + \left(K_{\bar z}(z) - K_{\bar z}(0)\right)d\bar z\right| \\
&\geqslant |z_2 - z_1|\left(\lambda_K(0) - 2\rho M - \sum_1^\infty\left(|a_n| + |b_n|\right)n\rho^{n+1} - \sum_2^\infty\left(|c_n| + |d_n|\right)n\rho^{n-1}\right) \\
&\geqslant |z_2 - z_1|\left(\frac{\pi}{4M} - 2\rho M - 2M\sum_1^\infty n\rho^{n+1} - 2M\sum_2^\infty n\rho^{n-1}\right) \\
&= |z_2 - z_1|\left(\frac{\pi}{4M} - 2\rho M - 2M\left(\frac{\rho^2}{(1-\rho)^2} + \frac{1}{(1-\rho)^2} - 1\right)\right).
\end{aligned}$$

Clearly there is a ρ so that $|F(z_1)-F(z_2)|>0$. Let ρ_2 be the largest such ρ. In other words, choose $\rho_2>0$ so that

$$\frac{\pi}{4M}-2\rho_2 M-2M\left(\frac{\rho_2^2}{(1-\rho_2)^2}+\frac{1}{(1-\rho_2)^2}-1\right)=0.$$

For $|z|=\rho_2$,

$$\left|F(z)\right|\geqslant|c_1z+d_1\bar z|-\rho_2^2\left|\sum_1^\infty\left(a_nz^n+b_n\bar z^n\right)\right|-\left|\sum_2^\infty\left(c_nz^n+d_n\bar z^n\right)\right|\geqslant\frac{\pi}{4M}\rho_2-2M\frac{\rho_2^3+\rho_2^2}{1-\rho_2}=R_2. \qquad \square$$

Remark 1. Using a proof similar to the proof of Theorem 1, one can obtain larger ρ_2 and R_2 for the case when the condition $J_F(0)=1$ is replaced by the conditions $K_z(0)=1$ and $K_{\bar z}(0)=0$.

The next theorem is different. Because, when $K=0$, the Jacobian $J_F(0)=0$ and hence we assume that $J_G(0)=1$ instead.

Theorem 2. *Let G be harmonic in U such that $G(0)=0$, $J_G(0)=1$ and $|G(z)|<M$ for $z\in U$. Then there is a constant $0<\rho_3<1$ so that $F=r^2G$ is univalent in the disk $|z|<\rho_3$. ρ_3 is the solution of the equation $\frac{\pi}{4M}=4M\frac{\rho_3}{1-\rho_3}+2M(\frac{1}{(1-\rho_3)^2}-1)$ and $f(U_{\rho_3})$ contains a disk U_{R_3} with $R_3=\frac{\pi}{4M}\rho_3^3-2M\frac{\rho_3^4}{1-\rho_3}$. This result is not sharp.*

Proof. Fix $0<\rho<1$ and choose z_1,z_2 with $z_1\neq z_2$, $|z_1|<\rho$ and $|z_2|<\rho$. Then

$$F(z_1)-F(z_2)=\int_{[z_1,z_2]}F_z(z)\,dz+F_{\bar z}(z)\,d\bar z=\int_{[z_1,z_2]}\left(\bar zG+r^2h'\right)dz+\left(zG+r^2\overline{g'}\right)d\bar z,$$

where $[z_1,z_2]$ is the line-segment from z_1 to z_2, $z=tz_2+(1-t)z_1$ and $0\leqslant t\leqslant1$, and $G=h+\bar g$.
Let $h(z)=\sum_{n=1}^\infty a_nz^n$ and $g(z)=\sum_{n=1}^\infty b_nz^n$, $|a_1|^2-|b_1|^2=1$. Then by using Lemma 1

$$\begin{aligned}
\left|F(z_1)-F(z_2)\right|&=\left|\int_{[z_1,z_2]}\left(\bar zG+r^2h'\right)dz+\left(zG+r^2\overline{g'}\right)d\bar z\right|\\
&=\left|\int_{[z_1,z_2]}G(z)(\bar z\,dz+z\,d\bar z)+\int_{[z_1,z_2]}r^2(h'\,dz+\overline{g'}\,d\bar z)\right|\\
&=\Bigg|\int_{[z_1,z_2]}\left(h'(0)z+\overline{g'(0)z}\right)(\bar z\,dz+z\,d\bar z)\\
&\quad+\int_{[z_1,z_2]}\left[\left(h(z)-h'(0)z\right)+\overline{\left(g(z)-g'(0)z\right)}\right](\bar z\,dz+z\,d\bar z)\\
&\quad+\int_{[z_1,z_2]}r^2\left(h'(0)\,dz+\overline{g'(0)}\,d\bar z\right)+\int_{[z_1,z_2]}r^2\left(\left(h'(z)-h'(0)\right)dz+\left(\overline{g'(z)}-\overline{g'(0)}\right)d\bar z\right)\Bigg|\\
&=\Bigg|h'(0)\int_{[z_1,z_2]}\left(2r^2\,dz+z^2\,d\bar z\right)+\overline{g'(0)}\int_{[z_1,z_2]}\left(2r^2\,d\bar z+\bar z^2\,dz\right)\\
&\quad+\int_{[z_1,z_2]}\left[\left(h(z)-h'(0)z\right)+\overline{\left(g(z)-g'(0)z\right)}\right](\bar z\,dz+z\,d\bar z)\\
&\quad+\int_{[z_1,z_2]}r^2\left(\left(h'(z)-h'(0)\right)dz+\left(\overline{g'(z)}-\overline{g'(0)}\right)d\bar z\right)\Bigg|
\end{aligned}$$

$$\geqslant \left| \int\limits_{[z_1,z_2]} \left(2r^2\,dz + z^2\,d\bar{z}\right) \right| \left(\left|h'(0)\right| - \left|g'(0)\right|\right) - 2|z_2 - z_1| \sum_2^\infty \left(|a_n| + |b_n|\right) \int_0^1 r^{n+1}\,dt$$

$$- |z_2 - z_1| \sum_2^\infty \left(|a_n| + |b_n|\right) n \int_0^1 r^{n+1}\,dt$$

$$\geqslant |z_2 - z_1| \left(\left| \int_0^1 2r^2\,dt \right| - \left| \int_0^1 z^2\,dt \right| \right) \left(\left|h'(0)\right| - \left|g'(0)\right|\right)$$

$$- 2|z_2 - z_1| \sum_2^\infty \left(|a_n| + |b_n|\right) \int_0^1 r^{n+1}\,dt - |z_2 - z_1| \sum_2^\infty \left(|a_n| + |b_n|\right) n \int_0^1 r^{n+1}\,dt$$

$$\geqslant |z_2 - z_1| \left[\left| \int_0^1 r^2\,dt \right| \left(\left|h'(0)\right| - \left|g'(0)\right|\right) - 4M \sum_2^\infty \rho^{n-1} \int_0^1 r^2\,dt - 2M \sum_2^\infty n\rho^{n-1} \int_0^1 r^2\,dt \right]$$

$$\geqslant |z_2 - z_1| \left(\int_0^1 r^2\,dt \right) \left[\left(\left|h'(0)\right| - \left|g'(0)\right|\right) - 4M \sum_2^\infty \rho^{n-1} - 2M \sum_2^\infty n\rho^{n-1} \right]$$

$$\geqslant |z_2 - z_1| \left(\int_0^1 r^2\,dt \right) \left[\frac{\pi}{4M} - 4M \frac{\rho}{1-\rho} - 2M \left(\frac{1}{(1-\rho)^2} - 1 \right) \right].$$

Choose ρ_3 so that $\frac{\pi}{4M} - 4M\frac{\rho_3}{1-\rho_3} - 2M(\frac{1}{(1-\rho_3)^2} - 1) = 0$.
Then F is univalent in $|z| < \rho_3$ and furthermore, we have for $|z| = \rho_3$,

$$\left|F(z)\right| = \rho_3^2 \left| \sum_1^\infty a_n z^n + b_n \bar{z}^n \right| \geqslant \rho_3^2 |a_1 z + b_1 \bar{z}| - \rho_3^2 \left| \sum_2^\infty a_n z^n + b_n \bar{z}^n \right| \geqslant \frac{\pi}{4M}\rho_3^3 - 2M\frac{\rho_3^4}{1-\rho_3} = R_3. \qquad \square$$

Acknowledgment

The authors would like to thank the referee for the careful reading of the article and for the helpful suggestions.

References

[1] Z. Abdulhadi, Y. Abu Muhanna, S. Khoury, On univalent solutions of the biharmonic equations, J. Inequal. Appl. (5) (2005) 469–478.
[2] Y. Abu Muhanna, G. Schober, Harmonic mappings onto convex mapping domains, Canad. J. Math. XXXIX (6) (1987) 1489–1530.
[3] H. Chen, P. Gauthier, W. Hengartner, Bloch constants for planar harmonic mappings, Proc. Amer. Math. Soc. 128 (2000) 3231–3240.
[4] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Math. Ser. A 9 (1984) 3–25.
[5] M. Dorff, M. Nowark, Landau's theorem for planar harmonic mappings, Comput. Methods Funct. Theory 4 (1) (2004) 151–158.
[6] P. Duren, Harmonic Mappings in the Plane, Cambridge Univ. Press, 2004.
[7] A. Grigoryan, Landau and Bloch theorems for harmonic mappings, Complex Var. Theory Appl. 51 (1) (2006) 81–87.