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On some properties of solutions of the biharmonic equation

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Abstract

In the present paper, the properties of the linear complex operator $L(f) = zf_z - \bar{z}f_{\bar{z}}$, which is defined on the class of complex-valued C^1 functions in the plane, are investigated. It is shown that harmonicity and biharmonicity are invariant under the linear operator L. Results concerning starlikeness and convexity of biharmonic functions versus the corresponding harmonic functions are considered. The operator L can be manipulated to express the conditions in the definitions of starlikeness and convexity in a convenient way.

Keywords: Biharmonic; Harmonic; Univalent; Starlike; Jacobian; Orientation preserving

1. Introduction

We say a continuous complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbf{C}$ is biharmonic if the Laplacian of F is harmonic. Note that ΔF is harmonic in D, if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$, where

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

In any simply connected subdomain of D we can write F in the form

$$F = r^2 G + K, \quad z = re^{i\theta}, \tag{1.1}$$

where G and K are harmonic. It is known that H and G can be expressed as

$$\begin{aligned} G &= g_1 + \overline{g_2}, \\ K &= k_1 + \overline{k_2}, \end{aligned} \tag{1.2}$$

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where g_1, g_2, k_1 and k_2 are analytic in D, see [1–3]. Lewy showed, see [2,3], that a harmonic function W is locally univalent if the Jacobian of W, J_W,

$$J_W = |W_z|^2 - |W_{\bar{z}}|^2 \neq 0. \tag{1.3}$$

We say that a function W is orientation preserving if the

$$J_W = |W_z|^2 - |W_{\bar{z}}|^2 > 0. \tag{1.4}$$

Recently, the authors studied univalent biharmonic mappings defined on the unit disc, see [8,9].

Many physical problems are modeled by the biharmonic equation, particularly those arising in fluid dynamics and elasticity problems. Biharmonic functions arise when dealing with transverse displacements of plates and shells. For instance, the biharmonic function can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, such as in Stokes flow problems and in flows through porous media. There is a wealth of applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [5–7]).

In this paper, we consider the differential operator

$$L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}. \tag{1.5}$$

We include the algebraic properties of this operator. It is shown that harmonicity and biharmonicity is invariant under the operator L. We obtain results regarding starlikeness and convexity of univalent biharmonic functions.

We basically show that a biharmonic function of the form $F = r^2G$, where G is harmonic is starlike with $J_F \geqslant 0$ whenever G is starlike.

2. Properties of the operator L

Consider the differential operator

$$L = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}. \tag{2.1}$$

Clearly, L is a linear operator

$$L(af + bg) = aL(f) + bL(g), \tag{2.2}$$

and L satisfies the product rule, namely,

$$L(fg) = fL(g) + gL(f), \tag{2.3}$$

where f and g are C^1 functions.

Consequently L has the following general properties:

Lemma 1. *Let f be a C^1 function then*

(a) $\overline{L(f)} = -L(\bar{f})$,

(b) $\frac{L[|f|^2]}{|f|^2} = 2\mathrm{Im}\frac{L(f)}{f}$.

Proof. To show part (a), from (2.1), we have

$$\overline{L(f)} = \overline{zf_z - \bar{z}f_{\bar{z}}} = \bar{z}\overline{f_z} - z\overline{f_{\bar{z}}} = \bar{z}\bar{f}_{\bar{z}} - z\bar{f}_z = -L(\bar{f}).$$

To prove part (b),

$$L(|f|^2) = L(f \cdot \bar{f}) = fL(\bar{f}) + \bar{f}L(f).$$

Using part (a), we obtain

$$\begin{aligned} L(|f|^2) &= -f\overline{L(f)} + \overline{f}L(f) \\ &= \overline{f}L(f) - \overline{\overline{f}L(f)} \\ &= 2\mathrm{Im}\left[\overline{f}L(f)\right] \\ &= 2|f|^2\mathrm{Im}\left[\frac{L(f)}{f}\right] \end{aligned}$$

$$\frac{L(|f|^2)}{|f|^2} = 2\mathrm{Im}\left[\frac{L(f)}{f}\right]. \qquad \square$$

The following lemma deals with the case when the function is harmonic.

Lemma 2. *Let G be harmonic then we have*:

(1) $L(\log G) = \frac{L(G)}{G}$, *where G is nonvanishing.*
(2) $\frac{L(G_z)}{G_z} = \frac{z\overline{G_z}G_{zz}}{|G_z|^2}$,
(3) $\frac{L(G\overline{z})}{G\overline{z}} = -\frac{\overline{z}\overline{G\overline{z}}G_{\overline{z}\overline{z}}}{|G\overline{z}|^2}$.
(4) $J_{L(G)} = r^2[J_{G_z} + J_{G\overline{z}}] + 2|G_z|^2\mathrm{Re}\left[\frac{L(G_z)}{G_z}\right] + 2|G_{\overline{z}}|^2\mathrm{Re}\left[\frac{L(G_{\overline{z}})}{G\overline{z}}\right] + J_G$.

Proof. Let G be a harmonic function. Then it follows that:

(1) $L(\log G) = z\frac{G_z}{G} - \overline{z}\frac{G_{\overline{z}}}{G} = \frac{L(G)}{G}$.
(2) $L(G_z) = zG_{zz} - \overline{z}G_{\overline{z}\overline{z}} = zG_{zz}$,

$$\frac{L(G_z)}{G_z} = \frac{zG_{zz}}{G_z} = \frac{z\overline{G_z}G_{zz}}{|G_z|^2}.$$

(3) Similar to (2), we can show that $\frac{L(G\overline{z})}{G\overline{z}} = -\frac{\overline{z}\overline{G_{\overline{z}}}G_{\overline{z}\overline{z}}}{|G_{\overline{z}}|^2}$.

To prove (4), we have

$$L_G = zG_z - \overline{z}G_{\overline{z}}.$$

This gives

$$(L_G)_z = zG_{zz} + G_z - \overline{z}G_{\overline{z}z} = zG_{zz} + G_z, \quad \text{since } G_{\overline{z}z} = 0,$$

and then

$$|(L_G)_z|^2 = (L_G)_z\overline{(L_G)_z} = |zG_{zz}|^2 + z\overline{G_z}G_{zz} + \overline{z}G_z\overline{G_{zz}} + |G_z|^2.$$

In a similar fashion, we have the following:

$$(L_G)_{\overline{z}} = zG_{z\overline{z}} - G_{\overline{z}} - \overline{z}G_{\overline{z}\overline{z}} = -(G_{\overline{z}} + \overline{z}G_{\overline{z}\overline{z}}),$$

and

$$|(L_G)_{\overline{z}}|^2 = |G_{\overline{z}}|^2 + zG_{\overline{z}}\overline{G_{\overline{z}\overline{z}}} + \overline{z}\overline{G_{\overline{z}}}G_{\overline{z}\overline{z}} + r^2|G_{\overline{z}\overline{z}}|^2.$$

Thus,

$$\begin{aligned} J_{L(G)} &= |(L_G)_z|^2 - |(L_G)_{\overline{z}}|^2 = r^2(|G_{zz}|^2 - |G_{\overline{z}\overline{z}}|^2) + 2\mathrm{Re}[\overline{z}G_z\overline{G_{zz}} - zG_{\overline{z}}\overline{G_{\overline{z}\overline{z}}}] + J(G) \\ &= r^2[J_{G_z} + J_{G_{\overline{z}}}] + 2|G_z|^2\mathrm{Re}\left[\frac{L(G_z)}{G_z}\right] + 2|G_{\overline{z}}|^2\mathrm{Re}\left[\frac{L(G_{\overline{z}})}{G_{\overline{z}}}\right] + J_G. \qquad \square \end{aligned}$$

The following lemma shows that biharmonicity is invariant under the operator L.

Lemma 3

(*a*) *L preserves harmonicity*,
(*b*) *L preserves biharmonicity.*

Proof. (a) is clear.
To show (b), we observe that

$$L(r^2) = \bar{z}(r^2)_{\bar{z}} - z(r^2)_z = \bar{z}z - z\bar{z} = 0.$$

This and (2.2) implies that

$$L(r^2G + K) = r^2L(G) + L(K),$$

which gives the result. □

Straight forward calculations yield the following lemma:

Lemma 4. *Let F be a biharmonic map of the form $F = r^2G$, where G is harmonic. Then*

(1) $\dfrac{L(F)}{F} = \dfrac{L(G)}{G}$,
(2) $\dfrac{L^2(F)}{L(F)} = \dfrac{L^2(G)}{L(G)}$.

We shall need the following two definitions (see [4]):

Definition 1. We say that a univalent biharmonic (harmonic) function, F with $F(0) = 0$, is starlike if the curve: $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, F is starlike if $\frac{\partial \arg F(re^{it})}{\partial t} = \mathrm{Re}\,\frac{zF_z - \bar{z}F_{\bar{z}}}{F} > 0$, for $z \neq 0$. The starlikeness measure is defined by

$$\mathrm{Re}\,\frac{zF_z - \bar{z}F_{\bar{z}}}{F} = F_{\mathrm{st}}, \tag{2.4}$$

and the Jacobian of F is given by

$$|F_z|^2 - |F_{\bar{z}}|^2 = J_F.$$

Definition 2. A univalent biharmonic (harmonic) function, F with $F(0) = 0$ and $\frac{\partial}{\partial t}F(re^{it}) \neq 0$ whenever $0 < r < 1$, is said to be convex if the curve: $F(re^{it})$ is convex for each $0 < r < 1$. In other words, F is convex if $\frac{\partial \arg \frac{\partial}{\partial t} F(re^{it})}{\partial t} > 0$, for $z \neq 0$.

In the following lemma, the starlikeness and convex measures are expressed in terms of the operator L.

Lemma 5

(a) $\mathrm{Re}\,\frac{zF_z - \bar{z}F_{\bar{z}}}{F} = \mathrm{Re}\,\frac{L(F)}{F}$,
(b) $\frac{\partial \arg \frac{\partial}{\partial t} F(re^{it})}{\partial t} = \mathrm{Re}\,\frac{L^2(F)}{L(F)}$.

Proof. (2.1) implies (a).
To show (b):

$$\frac{\partial \arg \frac{\partial}{\partial t} F(re^{it})}{\partial t} = \mathrm{Im}\,\frac{\frac{\partial^2}{\partial t^2} F(re^{it})}{\frac{\partial}{\partial t} F(re^{it})} = \mathrm{Im}\,\frac{\frac{\partial}{\partial t} L(F)}{L(F)} = \mathrm{Im}\,\frac{L(\frac{\partial}{\partial t} F)}{L(F)} = \mathrm{Im}\,\frac{L(\mathrm{i}L(F))}{L(F)} = \mathrm{Re}\,\frac{L^2(F)}{L(F)}.$$

This implies part (b). □

As an immediate consequence of Lemma 4, we have the following corollary:

Corollary 1. *Let F be a univalent biharmonic function. Then*

(1) *F is starlike if and only if* $\mathrm{Re}\frac{L(F)}{F} \geqslant 0$.
(2) *If in addition,* $L(F) \neq 0$ *for* $z \neq 0$, *F is convex if and only if* $\mathrm{Re}\frac{L^2(F)}{L(F)} \geqslant 0$.
(3) *If F is convex and L(F) is univalent then L(F) is starlike.*

Lemma 6. *Let F be a biharmonic mapping in the unit disc U. If F is of the form* $F = r^2G$, *where G is harmonic in U then*

$$J_F = r^4 J_G + 2r^2|G|^2 G_{\mathrm{st}}.$$

Proof. Since $F = r^2G$ this implies that

$$F_z = \bar{z}G + r^2 G_z,$$

and

$$F_{\bar{z}} = zG + r^2 G_{\bar{z}}.$$

Hence, it follows that

$$|F_z|^2 = (\bar{z}G + r^2G_z)(z\overline{G} + r^2\overline{G_z}) = r^2|G|^2 + r^2\bar{z}\overline{G_z}G + r^2 zG_z\overline{G} + r^4|G_z|^2,$$

and

$$|F_{\bar{z}}|^2 = (zG + r^2G_{\bar{z}})(\bar{z}\overline{G} + r^2\overline{G_{\bar{z}}}) = r^2|G|^2 + r^2 z\overline{G_{\bar{z}}}G + r^2\bar{z}G_{\bar{z}}\overline{G} + r^4|G_{\bar{z}}|^2.$$

Thus, the Jacobian of F is given by

$$\begin{aligned} J_F(z) &= |F_z|^2 - |F_{\bar{z}}|^2 = r^2G(\bar{z}\overline{G_z} - z\overline{G_{\bar{z}}}) + r^2\overline{G}(zG_z - \bar{z}G_{\bar{z}}) + r^4(|G_z|^2 - |G_{\bar{z}}|^2) \\ &= r^2[\bar{z}G\overline{G_z} + z\overline{G}G_z - zG\overline{G_{\bar{z}}} - \bar{z}\overline{G}G_{\bar{z}}] + r^4(|G_z|^2 - |G_{\bar{z}}|^2) \\ &= r^2[2\mathrm{Re}(z\overline{G}G_z) - 2\mathrm{Re}(\bar{z}\overline{G}G_{\bar{z}})] + r^4(|G_z|^2 - |G_{\bar{z}}|^2) = 2r^2|G|^2\mathrm{Re}\left(\frac{zG_z - \bar{z}G_{\bar{z}}}{G}\right) + r^4(|G_z|^2 - |G_{\bar{z}}|^2) \\ &= r^4J_G + 2r^2|G|^2G_{\mathrm{st}}. \qquad \square \end{aligned}$$

Our main result is the following theorem:

Theorem 1. *Let F be a biharmonic mapping in the unit disc U. If F is of the form* $F = r^2G$, *where G is harmonic, orientation preserving and starlike in U then* $J_F(z) > 0$, *when* $0 < |z| < 1$, $J_F(0) = 0$ *and therefore, F is starlike univalent.*

Proof. That $J_F(z) > 0$, when $0 < |z| < 1$, $J_F(0) = 0$ is a consequence of Lemma 6.

Since G is starlike, it follows that G and F are zero only at $z = 0$ and in addition, by using Lemma 4, $\mathrm{Re}\frac{L(F)}{F} = \mathrm{Re}\frac{L(G)}{G} > 0$, when $z \neq 0$. Hence F is an open map.

Let $E_r = \{G(re^{it})\colon 0 \leqslant t < 2\pi\}$ and $C_r = \{F(re^{it})\colon 0 \leqslant t < 2\pi\}$, when $0 < r < 1$. It follows that both are Jordan starlike. As G is univalent E_r rotates around 0 once. But $C_r = r^2E_r$ and $J_F > 0$, hence C_r makes one round around 0. This and the degree principle implies that F is univalent in $|z| \leqslant r$, for all $0 < r < 1$. Consequently, F univalent in U. This and Definition 1 imply the conclusion. $\square$

Theorem 2. *Let D be a simply connected domain bounded by a starlike curve. Let* Ψ *be a positively oriented one to one continuous function from the boundary of U,* ∂U, *onto the boundary of D,* ∂D. *Let G be the solution of the Dirichlet problem with respect to* Ψ *and F be the corresponding biharmonic functions, given by* $F = r^2G$. *If G maps U onto D, orientation preserving and satisfies* $\mathrm{Re}\frac{L(G)}{G} > 0$ *then G and F are starlike and onto.*

Proof. Since G is harmonic and orientation preserving, $J_G > 0$. As Ψ is positively oriented, the degree principle implies that G is univalent in U. Thus, by the condition $\mathrm{Re}\frac{L(G)}{G} > 0$, It follows that G is starlike. Then Theorem 1 implies that $J_F(z) > 0$, for $z \neq 0$ and F is starlike. □

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