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Inequalities for a Polynomial and Its Derivative

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1. INTRODUCTION AND STATEMENT OF RESULTS

If $P(z)$ is a polynomial of degree n, then

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |P(z)| \tag{1}$$

and

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leqslant R^n \operatorname*{Max}_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [6]). Inequality (2) is a simple deduction from the maximum modulus principle (see [5, 346] or [4, Vol I, 137, Problem 269]).

In both (1), (2) equality holds only for $P(z) = me^{i\alpha}z^n$, that is, when $P(z)$ has all its zeros at the origin. It was conjectured by P. Erdös and later proved by Lax [3] (see also [1]) that if $P(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|. \tag{3}$$

On the other hand, Turán [7] showed that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\operatorname*{Max}_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|. \tag{4}$$

Thus in (3) as well as in (4) equality holds for those polynomials of degree

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n which have all their zeros on $|z|=1$. Ankeny and Rivlin [2] used (3) to prove that if $P(z)$ does not vanish in $|z|<1$, then

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) \operatorname*{Max}_{|z|=1} |P(z)|, \tag{5}$$

which is much better than (2). Besides, equality in (5) holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$.

In this paper, we shall first obtain a result concerning the minimum modulus of a polynomial $P(z)$ and its derivative $P'(z)$ analogous to (2) and (1), when there is a restriction on the zeros of $P(z)$. We prove

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant 1$, then*

$$\operatorname*{Min}_{|z|=1} |P'(z)| \geqslant n \operatorname*{Min}_{|z|=1} |P(z)| \tag{6}$$

and

$$\operatorname*{Min}_{|z|=R>1} |P(z)| \geqslant R^n \operatorname*{Min}_{|z|=1} |P(z)|. \tag{7}$$

Both the estimates are sharp with equality for $P(z)=me^{i\alpha}z^n$, $m>0$.

Next we prove the following interesting generalization of (3).

THEOREM 2. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z|<1$, then*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left\{\operatorname*{Max}_{|z|=1} |P(z)| - \operatorname*{Min}_{|z|=1} |P(z)|\right\}. \tag{8}$$

The result is best possible and equality in (8) holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\beta|\geqslant|\alpha|$.

As an application of Theorem 2, we also obtain the following generalization of the inequality (5).

THEOREM 3. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z|<1$, then*

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) \operatorname*{Max}_{|z|=1} |P(z)| - \left(\frac{R^n-1}{2}\right) \operatorname*{Min}_{|z|=1} |P(z)|. \tag{9}$$

The result is best possible and equality in (9) holds for $P(z)=\alpha z^n+\beta$, where $|\beta|\geqslant|\alpha|$.

Finally we present a generalization of the inequality (4).

THEOREM 4. *If $P(z)$ is a polynomial of degree n which has all its zeros in $|z| \leqslant 1$, then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \{\max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)|\}. \tag{10}$$

The result is best possible and equality in (10) holds for $P(z) = \alpha z^n + \beta$, where $|\beta| \leqslant |\alpha|$.

2. PROOFS OF THE THEOREMS

Proof of Theorem 1. If $P(z)$ has a zero on $|z| = 1$, then inequalities (6) and (7) are trivial. So we suppose that $P(z)$ has all its zeros in $|z| < 1$. If $m = \text{Min}_{|z|=1} |P(z)|$, then $m > 0$ and $m \leqslant |P(z)|$ for $|z| = 1$. Therefore, if α is a complex number such that $|\alpha| < 1$, then it follows by Rouche's theorem that the polynomial $F(z) = P(z) - \alpha m z^n$ of degree n has all its zeros in $|z| < 1$. By the Gauss–Lucas theorem, the polynomial

$$F'(z) = P'(z) - n\alpha m z^{n-1}$$

has all its zeros in $|z| < 1$ for every complex number α with $|\alpha| < 1$. This implies that

$$nm\,|z|^{n-1} \leqslant |P'(z)| \qquad \text{for} \quad |z| \geqslant 1.$$

If this is not true, then there is a point $z = z_0$, $|z_0| \geqslant 1$, such that

$$|nmz_0^{n-1}| > |P'(z_0)|.$$

We can, therefore, take $\alpha = P'(z_0)/nmz_0^{n-1}$, then $|\alpha| < 1$ and $F'(z_0) = 0$. But this contradicts the fact that $F'(z) \neq 0$ for $|z| \geqslant 1$. Hence

$$|P'(z)| \geqslant nm\,|z|^{n-1} \qquad \text{for} \quad |z| \geqslant 1. \tag{11}$$

In particular, (11) gives

$$\min_{|z|=1} |P'(z)| \geqslant nm = n \min_{|z|=1} |P(z)|.$$

This proves inequlaity (6). To prove inequality (7), we observe that if $Q(z) = z^n \overline{P(1/\bar{z})}$, then $Q(z)$ has all its zeros in $|z| > 1$ and $m \leqslant |P(z)| = |Q(z)|$ for $|z| = 1$. Therefore, the function $m/Q(z)$ is analytic in $|z| \leqslant 1$ and $|m/Q(z)| \leqslant 1$ for $|z| = 1$. Hence by the maximum modulus principle it follows that $m \leqslant |Q(z)|$ for $|z| \leqslant 1$. Replacing z by $1/\bar{z}$ and noting that

$z^n\overline{Q(1/\bar{z})} = P(z)$, we conclude that $m\,|z|^n \leqslant |P(z)|$ for $|z| \geqslant 1$. Taking in particular $z = Re^{i\theta}$, $0 \leqslant \theta < 2\pi$, $R \geqslant 1$, we get

$$|P(Re^{i\theta})| \geqslant mR^n,$$

which gives

$$\operatorname*{Min}_{|z|=R>1} |P(z)| = \operatorname*{Min}_{|z|=1} |P(Rz)| \geqslant R^n \operatorname*{Min}_{|z|=1} |P(z)|.$$

This proves the inequality (7) and Theorem 1 is completely proved.

Proof of Theorem 2. If $m = \mathrm{Min}_{|z|=1} |P(z)|$, then $m \leqslant |P(z)|$ for $|z| = 1$. Since all the zeros of $P(z)$ lie in $|z| \geqslant 1$, therefore, for every complex number α such that $|\alpha| < 1$, it follows (by Rouche's theorem for $m > 0$) that the polynomial $F(z) = P(z) - \alpha m$ does not vanish in $|z| < 1$. Thus if $z_1, z_2, \ldots, z_n$ are the zeros of $F(z)$, then $|z_j| \geqslant 1$, $j = 1, 2, \ldots, n$, and

$$\frac{zF'(z)}{F(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j},$$

so that

$$Re\,\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})} = \sum_{j=1}^{n} Re\,\frac{e^{i\theta}}{e^{i\theta} - z_j} \leqslant \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2},$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $F(z)$. This implies

$$|e^{i\theta}F'(e^{i\theta})| \leqslant |nF(e^{i\theta}) - e^{i\theta}F'(e^{i\theta})|$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $F(z)$. Since this inequality is trivially true for points $e^{i\theta}$ which are the zeros of $F(z)$, it follows that

$$|F'(z)| \leqslant |nF(z) - zF'(z)| \qquad \text{for} \quad |z| = 1. \tag{12}$$

If we define $Q(z) = z^n\overline{P(1/\bar{z})}$ and $G(z) = z^n\overline{F(1/\bar{z})}$, then we have $G(z) = Q(z) - \bar{\alpha}mz^n$ and it can be easily seen that

$$|G'(z)| = |nF(z) - zF'(z)| \qquad \text{for} \quad |z| = 1.$$

Hence from (12) we get

$$|P'(z)| = |F'(z)| \leqslant |G'(z)| = |Q'(z) - \bar{\alpha}nmz^{n-1}| \tag{13}$$

for $|z| = 1$ and for every α with $|\alpha| < 1$. Since all the zeros of $Q(z)$ lie in $|z| \leqslant 1$, therefore, by Theorem 1, we have for $|z| = 1$

$$|Q'(z)| \geqslant \operatorname*{Min}_{|z|=1} |Q(z)| = n \operatorname*{Min}_{|n|=1} |P(z)| = nm.$$

Hence we can choose argument of α in (13) such that

$$|Q'(z)-\bar{\alpha}nmz^{n-1}|=|Q'(z)|-|\alpha|\,nm \qquad \text{fpr} \quad |z|=1.$$

Using this in (13) and letting $|\alpha|\to 1$, we obtain

$$|P'(z)|\leqslant |Q'(z)|-nm \qquad \text{for} \quad |z|=1. \tag{14}$$

If $P(z)$ is a polynomial of degree n, then [1, Lemma 2]

$$|P'(z)|+|nP(z)-zP'(z)|\leqslant n\operatorname*{Max}_{|z|=1}|P(z)| \qquad \text{for} \quad |z|=1. \tag{15}$$

Since

$$|Q'(z)|=|nP(z)-zP'(z)| \qquad \text{for} \quad |z|=1,$$

it follows from (15) that

$$|P'(z)|+|Q'(z)|\leqslant n\operatorname*{Max}_{|z|=1}|P(z)| \qquad \text{for} \quad |z|=1. \tag{16}$$

Inequality (14) gives with the help of inequality (16) that

$$\begin{aligned} 2\,|P'(z)| &\leqslant |P'(z)|+|Q'(z)|-nm \\ &\leqslant n(\operatorname*{Max}_{|z|=1}|P(z)|-\operatorname*{Min}_{|z|=1}|P(z)|) \qquad \text{for} \quad |z|=1, \end{aligned}$$

which immediately gives (8) and Theorem 2 is proved.

Proof of Theorem 3. Let $M=\operatorname{Max}_{|z|=1}|P(z)|$ and $m=\operatorname{Min}_{|z=1}|P(z)|$. Since $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, therefore, by Theorem 2 we have

$$|P'(z)|\leqslant (n/2)(M-m) \qquad \text{for} \quad |z|=1.$$

Now $P'(z)$ is a polynomial of degree $n-1$; therefore, it follows by (2) that for all $r\geqslant 1$ and $0\leqslant\theta<2\pi$

$$|P'(re^{i\theta})|\leqslant (n/2)\,r^{n-1}(M-m).$$

Also for each θ, $0\leqslant\theta<2\pi$ and $R>1$, we have

$$P(Re^{i\theta})-P(e^{i\theta})=\int_1^R e^{i\theta}P'(te^{i\theta})\,dt.$$

This gives

$$|P(Re^{i\theta})-P(e^{i\theta})| \leqslant \int_1^R |P'(te^{i\theta})|\,dt$$
$$\leqslant \frac{(M-m)}{2}\int_1^R nt^{n-1}\,dt$$
$$=\frac{1}{2}(R^n-1)(M-m),$$

for each θ, $0 \leqslant \theta < 2\pi$ and $R>1$. Hence

$$|P(Re^{i\theta})| \leqslant |P(e^{i\theta})| + \tfrac{1}{2}(R^n-1)(M-m)$$
$$\leqslant M + \tfrac{1}{2}(R^n-1)(M-m), \qquad (17)$$

for each, θ, $0<\theta<2\pi$ and $R>1$. From (17) we conclude that

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) M - \left(\frac{R^n-1}{2}\right) m.$$

This proves the desired result.

Proof of Theorem 4. Let $m = \text{Min}_{|z|=1} |P(z)|$, then $m \leqslant |P(z)|$ for $|z|=1$. Since all the zeros of $P(z)$ lie in $|z| \leqslant 1$, therefore, for every complex number α, such that $|\alpha|<1$, it follows (by Rouche's theorem for $m>0$) that the polynomial $F(z)=P(z)-m\alpha$ has all its zeros in $|z| \leqslant 1$. Hence if $z_1, z_2, ..., z_n$ are the zeros of $F(z)$, then $|z_j| \leqslant 1$, $j=1, 2, ..., n$, and

$$Re\,\frac{e^{i\theta}F'(e^{i\theta})}{F(e^{i\theta})} = \sum_{j=1}^{n} Re\,\frac{e^{i\theta}}{e^{i\theta}-z_j} \geqslant \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2},$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $F(z)$. This gives

$$|F'(e^{i\theta})/F(e^{i\theta})| \geqslant Re(e^{i\theta}F'(e^{i\theta}))/F(e^{i\theta}) \geqslant \frac{n}{2},$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $F(z)$. This further implies

$$|F'(e^{i\theta})| \geqslant (n/2)|F(e^{i\theta})|$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$. Hence

$$|P'(z)| = |F'(z)| \geqslant (n/2)|F(z)| = (n/2)|P(z)-\alpha m| \qquad \text{for} \quad |z|=1$$

and for every α, with $|\alpha| < 1$. Choosing argument of α suitably and letting $|\alpha| \to 1$, we get

$$|P'(z)| \geqslant (n/2)(|P(z) + m) \qquad \text{for} \quad |z| = 1,$$

which gives

$$\operatorname*{Max}_{|z|=1} |P'(z)| \geqslant \frac{n}{2} (\operatorname*{Max}_{|z|=1} |P(z)| + \operatorname*{Min}_{|z|=1} |P(z)|).$$

This completes the proof of Theorem 4.

3. Some Remarks

Remark 1. Let $P(z)$ be a polynomial of degree n which has all its zeros in $|z| \leqslant 1$. If $Q(z) = z^n \overline{P(1/\bar{z})}$, then the polynomial $Q(z)$ does not vanish in $|z| < 1$ and $|P(z)| = |Q(z)|$ for $|z| = 1$, so that

$$\operatorname*{Min}_{|z|=1} |Q(z)| = \operatorname*{Min}_{|z|=1} |P(z)|.$$

Applying (14) to the polynomial $Q(z)$ and noting that $z^n \overline{Q(1/\bar{z})} = P(z)$, it follows that

$$|P'(z)| - |Q'(z)| \geqslant n \operatorname*{Min}_{|z|=1} |P(z)| \qquad \text{for} \quad |z| = 1. \tag{18}$$

We also note that for $|z| = 1$

$$|Q'(z)| = |zP'(z) - nP(z)| \geqslant |P'(z)| - n|P(z)|,$$

and therefore,

$$|P'(z)| - |Q'(z)| \leqslant n\, |P(z)| \qquad \text{for} \quad |z| = 1. \tag{19}$$

From (18) and (19) we obtain

$$\operatorname*{Min}_{|z|=1} (|P'(z)| - |Q'(z)|) = n \operatorname*{Min}_{|z|=1} |P(z)|, \tag{20}$$

for every polynomial $P(z)$ having all its zeros in $|z| \leqslant 1$. Moreover, the minimums of both sides in (20) are attained at the same point $|z_0| = 1$. This follows from the fact that if $|P(z_0)| = \operatorname{Min}_{|z|=1} |P(z)|$ and $|z_0| = 1$, then (from (18) and (19)) we get $|P'(z_0)| - |Q'(z_0)| = n\, |P(z_0)|$.

Remark 2. In (7) equality holds only for $P(z)=me^{i\alpha}z^n$. For if $m=\text{Min}_{|z|=1} |P(z)|$ and $P(z)$ does not have the form $me^{i\alpha}z^n$, then $Q(z)=z^n\overline{P(1/\bar{z})}$ is not a constant. From the proof of the inequality (7), it follows that $m<|Q(z)|$ for $|z|<1$ and therefore, $m|z|^n<|P(z)|$ for $|z|>1$. This implies $\text{Min}_{|z|=R>1} |P(z)|>mR^n=R^n \text{Min}_{|z|=1} |P(z)|$. If $P(z)=me^{i\alpha}z^n$, then we have clearly equality in (7).

Remark 3. If in Theorem 3, $M = \text{Max}_{|z|=1} |P(z)|$ and $m = \text{Min}_{|z|=1} |P(z)|$, then equality in (9) holds only for $P(z)=(\alpha(M-m)/2)z^n+(\beta(M+m)/2)$, where $|\alpha|=|\beta|=1$. This follows from the fact that if $P(z)$ does not have the form $(\alpha(M-m)/2)z^n+(\beta(M+m)/2)$, $|\alpha|=|\beta|=1$, then in the proof of Theorem 3, by virtue of (2), we have the strict inequality

$$|P'(re^{i\theta})|<(n/2)\,r^{n-1}(M-m), \qquad \text{for all } r>1 \text{ and } 0\leqslant\theta<2\pi.$$

Hence we also have the strict inequality in (17) for all $R>1$ and $0\leqslant\theta<2\pi$, which gives

$$\underset{|z|=R>1}{\text{Max}} |P(z)|<\left(\frac{R^n+1}{2}\right)M-\left(\frac{R^n-1}{2}\right)m.$$

Finally, if $P(z)=(\alpha(M-m)/2)z^n+(\beta(M+m)/2)$, $|\alpha|=|\beta|=1$, then $\text{Max}_{|z|=R>1} |P(z)|=((R^n+1)/2)M-((R^n-1)/2)\,m$.

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